Recent techniques and results on the Erdös-Pósa property

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Abstract

Several min-max relations in graph theory can be expressed in the framework of the Erdős–Pósa property. Typically, this property reveals a connection between packing and covering problems on graphs. We describe some recent techniques for proving this property that are related to tree-like decompositions. We also provide an unified presentation of the current state of the art on this topic.

Keywords: Erdős–Pósa property, min-max theorems, tree decompositions, tree partitions, girth, graph minors, topological minors, graph immersions.

1 Introduction

A considerable part of combinatorics has been developed around min-max theorems. Min-max theorems usually identify dualities between certain objects in graphs, hypergraphs, and other combinatorial structures. The target is to prove that the absence of the primal object implies the presence of the dual one and vice versa.

A classic example of such a duality is Menger's theorem: the primal concept is the existence of k internally disjoint paths between two vertex sets S and T of a graph G, while the dual concept is a collection of k vertices that intersect all (S,T)-paths. Another example is Kőnig's theorem where the primal notion is the existence of a matching with k edges in a bipartite graph and the dual one is the existence of a vertex cover of size k. It is also known that, in case of general graphs, this duality becomes an approximate one, i.e., a vertex cover of size $2k$. In both aforementioned examples, the duality relates the notions of *packing* and *covering* of a collection $\mathcal{C}$ of combinatorial objects of a graph. In Menger's theorem $\mathcal{C}$ consists of all (S,T)-paths of G while in Kőnig's theorem $\mathcal{C}$ is the set of all edges of G. That way, both aforementioned min-max theorems can be stated, for some class of graphs $\mathcal{G}$ (called *host* class) and some gap function $f : \mathbb{N} \to \mathbb{N}$, as follows:

> *For every graph G in $\mathcal{G}$, either G contains k-vertex disjoint objects in $\mathcal{C}$ or it contains $f(k)$ vertices intersecting all objects in $\mathcal{C}$ that appear in G.*

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Clearly, for the case of Menger's theorem the host class is the class of all graphs while in the case of Kőnig's theorem the host class is restricted to the class of bipartite graphs. In both cases the derived duality is an *exact* one in the sense that f is the identity function. However, this is not the case if we want to extend the duality of Kőnig's theorem in the case of all graphs, where we can consider $f : k \mapsto 2k$ (i.e., we have an *approximate* duality).

One of the most celebrated results about packing/covering dualities was obtained by Paul Erdős and Lajos Pósa in 1965 where the object to cover and pack was the set of all cycles of G [EP65]. In this case the host class contains all graphs, while $f : k \mapsto O(k \cdot \log k)$. Moreover, Erdős and Pósa proved that this gap is optimal in the sense that it cannot be improved to a function $f : k \mapsto o(k \cdot \log k)$. This result motivated a long line of research for min-max dualities that are exact or approximate. Since then, a multitude of results on the Erdős–Pósa property have appeared for several combinatorial objects, including extensions to digraphs [LY78, Sey96, RRST96, HM13, GT11], rooted graphs [KK15, PW12, Joo14, BJS14], labeled graphs [KW06], signed graphs [HNP06, ADG04], hypergraphs [Alo02, Bou13, BT15], matroids [GK09], and other combinatorial structures [GL69] (see [Ree97] for a survey on this topic). Also it is worth to stress that Erdős–Pósa dualities have been useful in more applied domains. For example, in bioinformatics, they were useful for upper-bounding the number of fixed-points of a boolean networks [Ara08, ADG04, ARS16].

The purpose of this paper is twofold. We first describe some recent techniques for proving Erdős–Pósa-type results, mainly based on techniques related to tree-like decompositions of graphs (Section 3) and the parameter of girth (Section 4). We focused our presentation to the description of general frameworks that, we believe, might be useful for further investigations. In Section 5, we present negative results on classes defined by containment relations. Lastly, in Section 6, we provide an extensive update of results on the Erdős–Pósa property, reflecting the current progress on this vibrant area of graph theory.

2 Definitions

Unless otherwise mentioned, graphs in this paper are finite, undirected, do not have loops and they may have multiple edges. We call a graph *nontrivial* if it contains at least one edge. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of a graph G, respectively, and we set $|G| = |V(G)|$ and $\|G\| = |E(G)|$ (counting multiplicities). For every set X of vertices of a graph G, the subgraph of G *induced* by X, that we write $G[X]$, is the graph $\left(X, E \cap \binom{X}{2}\right)$. For every set X of vertices (resp. edges), we define $G \setminus X$ as the graph $G[V(G) \setminus X]$ (resp. $(V(G), E(G) \setminus X)$). The *degree* of a vertex v of a graph G, that we write $\deg_G(v)$ is the number of vertices adjacent to v in G. We drop the subscript when there is no ambiguity.

For $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, and G a graph, let $A_{\mathsf{x}}(G) = V(G)$ if $\mathsf{x} = \mathsf{v}$ and $A_{\mathsf{x}}(G) = E(G)$ if $\mathsf{x} = \mathsf{e}$. In this sense we use symbols v and e in order to distinguish the vertex and the edge variants of the properties/parameters that we are dealing with. A graph is *subcubic* its maximum degree is bounded by 3. For every $t \in \mathbb{N}$, we denote by θ_t the graph with two vertices and t edges.

Local operations. The operation of *contracting an edge* $\{x, y\}$ in a graph G introduces a new vertex v_{xy} and makes it adjacent with all neighbors of x and y and then deletes x and y. The operation of *lifting a pair of edges* $\{x, y\}$, $\{y, z\}$ in a graph G increases by one the multiplicity of the edge $\{x, z\}$ (or introduces this edge if it does not exist) and then reduces by one the multiplicities of $\{x, y\}$ and $\{y, x\}$.

Partial orders on graphs. Given two graphs H and G, we say that H is an *induced subgraph* of G if H can be obtained from G after removing vertices. Additionally, H is a *subgraph* of G if it can be obtained by some induced subgraph of G after removing edges. We also say that H is a *minor* (resp. *topological minor*) of G if it can be obtained by some subgraph of G after contracting edges (after contracting edges with some endpoint of degree at most 2). Finally, we say that a graph H is an *immersion* of a graph G if it can be obtained from some subgraph of G after lifting pairs of edges that share some common endpoint.

Given a graph H, we denote by $\mathcal{M}(H)$, $\mathcal{T}(H)$, $\mathcal{I}(H)$ the class of all graphs that contain H as a minor, topological minor, or immersion respectively.

Packings and covers. Let $\mathcal{H}$ be a family of graphs and let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. An x-*$\mathcal{H}$-cover* of G is a set $C \subseteq A_{\mathsf{x}}(G)$ such that $G \setminus C$ does not contain any subgraph isomorphic to a member of $\mathcal{H}$. An x-*$\mathcal{H}$-packing* in G is a collection of x-disjoint subgraphs of G, each being isomorphic to some graph of $\mathcal{H}$.

We denote by $\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G)$ the maximum size of an x-$\mathcal{H}$-packing and by $\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G)$ the minimum size of an x-$\mathcal{H}$-cover in G. Clearly, by definition, it always hold that $\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G) \leq \mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G)$, for every graph G.

The Erdős–Pósa property. Let $\mathcal{G}$ and $\mathcal{H}$ be two graph classes, and let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. We refer to $\mathcal{G}$ as the *host* graph class and by $\mathcal{H}$ as the *guest* graph class. We say that $\mathcal{H}$ has the x-*Erdős–Pósa property* for $\mathcal{G}$ if there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that the following holds:

$$\forall G \in \mathcal{G},\ \mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G) \leq f(\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G)).$$

Any function f satisfying the above inequality is called a *gap* of the x-Erdős–Pósa property of $\mathcal{H}$ for $\mathcal{G}$. When a class of graphs has the x-Erdős–Pósa property for the class of finite graphs, we simply say that it has the x-Erdős–Pósa property.

Rooted trees. A *rooted tree* is a pair (T, s) where T is a tree and $s \in V(T)$ is a vertex referred to as the *root*. Given a vertex $x \in V(T)$, the *descendants* of x in (T, s), denoted by $\mathrm{desc}_{(T,s)}(x)$, is the set containing each vertex w such that the unique path from w to s in T contains x. If y is a descendant of x and is adjacent to x, then it is a *child* of x.

Tree partitions. A *tree partition* of a graph G is a pair $\mathcal{D} = (\mathcal{X}, T)$ where T is a tree and $\mathcal{X} = \{X_t\}_{t \in V(T)}$ is a partition of $V(G)$ such that either $|T| = 1$ or for every $\{x, y\} \in E(G)$, there exists an edge $\{t, t'\} \in E(T)$ where $\{x, y\} \subseteq X_t \cup X_{t'}$. Given an edge $f = \{t, t'\} \in E(T)$, we define E_f as the set of edges with one endpoint in X_t and the other in $X_{t'}$. The *width* of $\mathcal{D}$ is defined as

$$\max\{\max\{|X_t|\}_{t \in V(T)}, \max\{\|G[X_t]\|\}_{t \in V(T)}, \max\{|E_f|\}_{f \in E(T)}\}.$$

The *tree partition width* of G is the minimum width over all tree partitions of G and will be denoted by $\mathbf{tpw}(G)$. Tree partitions have been introduced in [See85] (see also [Hal91]) and tree partition width has been defined for simple graphs in [DO96]. The extension of this definition for multigraphs is due to [CRST15a].

A *rooted tree partition* of a graph G is a triple $\mathcal{D} = (\mathcal{X}, T, s)$ where (T, s) is a rooted tree and $(\mathcal{X}, T)$ is a tree partition of G.

Tree decompositions. A *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X}$ is a family $\{X_t\}_{t \in V(T)}$ of subsets of $V(G)$ (called *bags*) indexed by elements of $V(T)$, such that the following holds

(i) $\bigcup_{t \in V(T)} X_t = V(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T),\ v \in X_t\}$ is connected.

The *width* of a tree decomposition $(T, \mathcal{X})$ is defined as equal to $\max_{t \in V(T)} |X_t| - 1$. The *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions.

3 The Erdős–Pósa property from graph decompositions

Let $\mathcal{H}$ be a graph class, $\mathbf{p}$ be a graph parameter, and $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. We say that a function $f\colon \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ is a *ceiling* for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$ if for every graph G, $\mathbf{p}(G) \leq f(\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G))$. Intuitively, there is a ceiling for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$ if a large value of $\mathbf{p}$ on a graph forces a large x-packing of elements of $\mathcal{H}$.

Given a graph parameter $\mathbf{p}$ and an integer k, we denote

$$\mathcal{G}_{\mathbf{p} \leq k} = \{G,\ \mathbf{p}(G) \leq k\}.$$

Theorem 3.1. *Let $\mathcal{H}$ be a class of graphs, $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $\mathbf{p}$ be a graph parameter, let $f\colon \mathbb{N} \to \mathbb{N}$ be a function and let $h_r\colon \mathbb{N} \to \mathbb{N}$ be a function, for every $r \in \mathbb{N}$. Suppose that the following two conditions hold:*

A. *f is a ceiling for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$;*

B. *for every $r \in \mathbb{N}$, $\mathcal{H}$ has the x-Erdős–Pósa property for $\mathcal{G}_{\mathbf{p} \leq r}$ with gap h_r;*

then $\mathcal{H}$ has the x-Erdős–Pósa property with gap $k \mapsto h_{f(k)}(k)$.

Proof. Let G be a graph and let $k = \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G)$. We have $\mathbf{p}(G) \leq f(k)$, by definition of a ceiling. Therefore, $G \in \mathcal{G}_{\mathbf{p} \leq f(k)}$, and thus $\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G) \leq h_{f(k)}(k)$. □

Theorem 3.1 will be used as a master theorem for the results of this section.

3.1 Vertex version and tree decompositions

In a breakthrough paper [CC13a], Chekuri and Chuzhoy proved that every graph of large treewidth can be partitioned into several subgraphs of large treewidth, with a polynomial dependency between the treewidth of the original graph, the one of the subgraphs, and the number of subgraphs. In particular they proved the next result.

Theorem 3.2 ([CC13a, Theorem 1.1])**.** *Let G be a graph with $\mathbf{tw}(G) = k$, and let h, r be two integers with $hr^2 \leq k / \operatorname{polylog} k$. Then there is a partition $G_1, \dots, G_h$ of G into vertex-disjoint subgraphs such that $\mathbf{tw}(G_i) \geq r$ for every $i \in \{1, \dots, h\}$.*

Besides, the results in [CC13c] provide a polynomial bound for the grid exclusion theorem. The $(p \times q)$-*grid* (for $p, q \in \mathbb{N}$) is the graph with vertex set $\{1, \dots, p\} \times \{1, \dots, q\}$ and edge set $\{\{(i,j), (i',j')\},\ |i - i'| + |j - j'| = 1\}$.

Theorem 3.3 ([CC13c, Theorem 1.1]). *There is constant δ such that every graph of treewidth k contains as a minor a $(\Omega(k^\delta) \times \Omega(k^\delta))$-grid.*

As every planar graph G is a minor of the every $(p \times p)$-grid for $p = |G| + 2\|G\|$ ([RST94, 1.5]), these two results can be combined to give the following polynomial ceiling for planar graphs.

Corollary 3.4 (see also the proof of [CC13a, Theorem 5.4]). *There is a function $f_h(k) = h^{O(1)} \cdot k \cdot (\log k)^{O(1)}$ such that, for every planar graph H on h edges, f_h is a ceiling for the triple $(\mathbf{tw}, \mathcal{M}(H), \mathsf{v})$.*

Indeed, according to Theorem 3.2, every graph of *large* treewidth can be partitioned into *many* disjoint subgraphs each with treewidth *large enough* (i.e. polynomial, according to Theorem 3.3) to force a *large* grid as a minor, which in turn contains the desired planar graph.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *superadditive* if $f(x) + f(y) \leq f(x+y)$ for every pair x, y of positive reals. The following argument has been first used in [FST11] (see also [CC13a, RST16, CRST15a]).

Lemma 3.5. *Let $\mathcal{H}$ be a family of connected graphs. If f is a superadditive ceiling for $(\mathbf{tw}, \mathcal{H}, \mathsf{v})$ then $\mathcal{H}$ has the v-Erdős–Pósa property with gap $k \mapsto 6 \cdot f(k) \log(k+1)$.*

Proof. Let us show the following for every integer k: for every graph G, if $\mathsf{v\text{-}pack}_{\mathcal{H}}(G) = k$ then $\mathsf{v\text{-}cover}_{\mathcal{H}} \leq 6f(k)\log(k+1)$. The proof is by induction on k. The base case $k = 0$ is trivial. Let $k > 0$, and let us assume that the above statement holds for every non-negative integer $k' < k$ (induction hypothesis).

Let G be a graph such that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G) = k$. A *separation* of G of order $p \in \mathbb{N}$ is a pair (A, B) of subsets of $V(G)$ such that G has no edge with the one endpoint in $A \setminus B$ and the other one in $B \setminus A$, and $|A \cap B| = p$. We will rely on the following claim.

Claim 3.6. There is a separation (A, B) of order at most $\mathbf{tw}(G) + 1$ of G such that

$$\begin{aligned} \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) &\leq 2k/3 \quad \text{and} \\ \mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A]) &\leq 2k/3. \end{aligned}$$

Proof. We consider a special type of tree decomposition called *nice tree decomposition.* A triple $(T, r, \{X_t\}_{t \in V(T)})$ is said to be a *nice* tree decomposition of a graph G if $(T, \{X_t\}_{t \in V(T)})$ is a tree-decomposition where the following holds:

1. every vertex of T has degree at most 3;

2. (T, r) is a rooted tree and the bag of the root r is empty ($X_r = \emptyset$);

3. every vertex t of T is

   - either a *base node*, i.e. a leaf of T whose bag is empty ($X_t = \emptyset$) and different from the root;
   - or an *introduce node*, i.e. a vertex with only one child t' such that $X_t = X_{t'} \cup \{u\}$ for some $u \in V(G)$;
   - or a *forget node*, i.e. a vertex with only one child t' such that $X_t = X_{t'} \setminus \{u\}$ for some $u \in X_{t'}$;

- or a *join node*, i.e. a vertex with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.

It is known that every graph G has a nice tree decomposition with width $\mathbf{tw}(G)$ [Klo94]. We therefore can assume that $(T, r, (X_t)_{t \in V(T)})$ is a nice tree decomposition of G of optimal width. For each $t \in V(T)$, we define

$$G_t = G\left[\bigcup_{s \in \mathrm{desc}_{(T,r)}(t)} X_s\right] \quad \text{and} \quad G_t^- = G_t \setminus X_t.$$

Let t be a vertex of T at minimal distance from a leaf subject to the requirement $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t^-) > 2k/3$. Such a vertex exists, as $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_r^-) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_r) = k$. Observe that t is either a forget node, or a join node. Indeed, for every base node u we have $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_u^-) = 0$. Moreover, every introduce node u with child v satisfies $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_u^-) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_v^-)$, since $G_u^- = G_v^-$.
First case: t is a forget node with child u. We set $A = V(G_u)$ and $B = V(G) \setminus V(G_u^-)$. Observe that (A, B) is a separation and that we have $A \cap B = X_u$, therefore the order of (A, B) is at most $\mathbf{tw}(G) + 1$. If $k = 1$, then $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_u^-) = 0$ (by definition of t), whereas the fact that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A]) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G)$ implies $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A]) = 0 \le 2k/3$. When $k \ge 2$, we have the following inequalities:

$$\begin{aligned}
\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) &= \mathsf{v\text{-}pack}_{\mathcal{H}}(G_u^-) \\
&\ge \mathsf{v\text{-}pack}_{\mathcal{H}}(G_t^-) - 1 \quad \text{(as t is a forget node)} \\
&> \frac{2k}{3} - 1 \\
&\ge \frac{k}{3}.
\end{aligned}$$

When $k = 2$, the last inequality follows from the fact that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B])$ is an integer. Notice that we always have

$$\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) + \mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A]) \le k.$$

Together with the above inequality, this implies that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A]) \le 2k/3$, wheras it follows from the definition of t that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) \le 2k/3$.
Second case: t is a join node with children u_1, u_2. We set $A = V(G_{u_i})$ and $B = V(G) \setminus V(G_{u_i}^-)$, where u_i is a child of t such that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_i}^-) \ge k/3$. Such child exists because $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t^-) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_1}^-) + \mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_2}^-)$ (as t is a join node) and $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t^-) > 2k/3$, by definition of t. Here again, (A, B) is a separation and its order is at most $\mathbf{tw}(G) + 1$ given that $A \cap B = X_{u_i}$. The inequality $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) \le 2k/3$ follows from the definition of t and the choice of i ensures that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) \ge k/3$, hence $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A]) \le 2k/3$, as above. □

Observe that $\mathbf{tw}(G) \le f(k)$, by definition of f. According to Claim 3.6, there is a separation (A, B) of order at most $\mathbf{tw}(G) + 1$ in G such that $k_A, k_B \le \lfloor 2k/3 \rfloor$, where $k_A = \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B])$ and $k_B = \mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A])$. Moreover, since (A, B) is a separation, there is no connected graph of $G \setminus (A \cap B)$ that have vertices in both $G[A \setminus B]$ and $G[B \setminus A]$. Therefore, given that every graph of $\mathcal{H}$ is connected, we can construct a v-$\mathcal{H}$-cover of $G \setminus (A \cap B)$ by taking the union of a v-$\mathcal{H}$-cover of $G[A \setminus B]$ and of one of $G[B \setminus A]$. In other words, we have

$$\begin{aligned}
\mathsf{v\text{-}cover}_{\mathcal{H}}(G) &\le \mathsf{v\text{-}cover}_{\mathcal{H}}(G[A \setminus B]) + \mathsf{v\text{-}cover}_{\mathcal{H}}(G[B \setminus A]) + |A \cap B| \\
&\le \mathsf{v\text{-}cover}_{\mathcal{H}}(G[A \setminus B]) + \mathsf{v\text{-}cover}_{\mathcal{H}}(G[B \setminus A]) + f(k) + 1 \\
&\le 6f(k_A)\log(k_A + 1) + 6f(k_B)\log(k_B + 1) + f(k) + 1.
\end{aligned}$$

The last inequality above is obtained by applying the induction hypothesis on both $G[A \setminus B]$ and $G[B \setminus A]$. Notice that in the case where $k = 1$, we get $k_A = k_B = 0$ and we have $\mathsf{v\text{-}cover}_{\mathcal{H}}(G) \leq f(k) \leq 6 \cdot f(k)\log(k+1)$. Therefore we now assume $k \geq 2$. We can then deduce that $\frac{2k}{3} + 1 \leq \frac{7}{9}(k+1)$.

We then have:

$$\begin{aligned}
\mathsf{v\text{-}cover}_{\mathcal{H}}(G) &\leq 6 \cdot (f(k_A) + f(k_B)) \log\left(\frac{2k}{3} + 1\right) + f(k) + 1 \\
&\leq 6 \cdot f(k) \log\left(\frac{7(k+1)}{9}\right) + f(k) + 1 \qquad \text{(superadditivity of } f\text{)} \\
&\leq 6 \cdot f(k)\log(k+1) - 6 \cdot \log(9/7) f(k) + 2f(k) \\
&\leq 6 \cdot f(k)\log(k+1).
\end{aligned}$$

The second inequality also requires that f is monotone, which is the case because it is superadditive and it never takes negative values. □

From the fact that the function of Corollary 3.4 is superadditive, we get the following consequence of Lemma 3.5.

Corollary 3.7 (see also [CC13a] and [CC13c]). *There is a function $f_h(k) = h^{O(1)} \cdot k \operatorname{polylog}(k)$ such that, for every connected planar graph H with h edges, the class $\mathcal{M}(H)$ has the Erdős–Pósa property with gap f_h.*

Notice that the above proof strongly relies on the fact that H is connected. The non-connected case requires some more ideas that are originating from [RS86] (also used for forests in [FJW13]). We expose them hereafter. We will need the two next lemmas.

Lemma 3.8 ([RS86]). *Let q, k be two positive integers, let T be a tree and let $\mathcal{A}_1, \dots, \mathcal{A}_q$ be families of subtrees of T. Assume that for every $i \in \{1, \dots, q\}$, there are kq elements of $\mathcal{A}_i$ that are pairwise vertex-disjoint. Then for every $i \in \{1, \dots, q\}$, there are k elements $T_1^i, \dots, T_k^i$ of $\mathcal{A}_i$ such that*

$$T_1^1, \dots T_k^1, T_1^2, \dots T_k^2, \dots, T_1^q, \dots T_k^q$$

are all pairwise vertex-disjoint.

The next lemma is the Erdős–Pósa property of subtrees of a tree. It can be obtained from the fact that subtrees of a tree have the Helly property.

Lemma 3.9 (see [GL69]). *Let T be a tree and let $\mathcal{A}$ be a collection of subtrees of T. For every positive integer k, either T has (at least) k vertex disjoint subtrees that belong to $\mathcal{A}$, or T has a subset X of less than k vertices such that no subtree of $T \setminus X$ belongs to $\mathcal{A}$.*

We are now ready to deal with disconnected patterns.

Lemma 3.10 ([RS86]). *Let w be a positive integer and let H be a graph on q connected components. $\mathcal{M}(H)$ has the v-Erdős–Pósa property on the class of graphs of treewidth at most w with gap $k \mapsto (w-1)(kq-1)$.*

Proof. Let k be a positive integer. We want to show that either $\mathsf{v\text{-}pack}_{\mathcal{M}(H)}(G) \geq k$ or $\mathsf{v\text{-}cover}_{\mathcal{M}(H)}(G) \leq (w-1)(kq-1)$. Let $H_1, \dots, H_q$ be the connected components of H. Let $(T, \mathcal{X})$ be a tree-decomposition of G of width w. For every subgraph F of G, we denote by

$T(F)$ the subgraph of T induced by the bags containing vertices of F. Notice that $T(F)$ is connected if F is connected.

For every $i \in \{1, \dots, q\}$, we let $\mathcal{H}_i$ be the class of subgraphs of G that are isomorphic to a graph in $\mathcal{M}(H_i)$ and we consider the class $\mathcal{T}_i = \{T(F), F \in \mathcal{H}_i\}$.

If for every $i \in \{1, \dots, q\}$, $\mathcal{T}_i$ contains kq vertex-disjoint trees, then according to Lemma 3.8 there is a collection $\{T_i^j\}_{i\in\{1,\dots,q\},\ j\in\{1,\dots,k\}}$ of pairwise vertex-disjoint trees, with $T_i^j \in \mathcal{T}_i$ for every $i \in \{1, \dots, q\}$ and every $j \in \{1, \dots, k\}$. Observe that for every two subgraphs F, F' of G, if $T(F)$ and $T(F')$ are vertex-disjoint, then so are F and F'. Therefore G has a collection $\{F_i^j\}_{i\in\{1,\dots,q\},\ j\in\{1,\dots,k\}}$ of pairwise vertex-disjoint subgraphs such that F_i^j is isomorphic to an element of $\mathcal{H}_i$, for every $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, k\}$. Consequently, for every $j \in \{1, \dots, k\}$, $\bigcup_{i=1}^q F_i^j$ is a subgraph of G containing a graph isomorphic to a member of $\mathcal{M}(H)$, and these subgraphs are vertex-disjoint for distinct values of j. This proves that in this case, $\mathsf{v\text{-}pack}_{\mathcal{M}(H)}(G) \geq k$.

We therefore now assume that the above condition does not hold, namely there is an index $i \in \{1, \dots, q\}$ such that $\mathcal{T}_i$ contains less than kq vertex-disjoint trees. Lemma 3.9 implies the existence of a subset X with $|X| \leq kq - 1$ such that $T \setminus X$ is free from subtrees isomorphic to a member of $\mathcal{T}_i$. Let Y denote the union of the bags indexed by vertices in X. Observe that $|Y| \leq (w-1)|X| \leq (w-1)(kq-1)$. The choice of Y ensures that $G \setminus Y$ has no subgraph isomorphic to a member of $\mathcal{H}_i$. Hence $\mathsf{v\text{-}cover}_{\mathcal{M}(H_i)} \leq (w-1)(kq-1)$. We deduce $\mathsf{v\text{-}cover}_{\mathcal{M}(H)} \leq (w-1)(kq-1)$. □

Corollary 3.11. *For every planar graph H with h edges and q connected components, the class $\mathcal{M}(H)$ has the Erdős–Pósa property with gap $k \mapsto q \cdot h^{O(1)} \cdot k^2 \cdot \mathrm{polylog}(k)$.*

3.2 Edge version and tree partitions

The technique presented in the previous section to deal with hosts of bounded treewidth cannot be straighforwardly translated to the setting of the edge-Erdős–Pósa property. Indeed, in general, knowing that two vertex sets are separated by a small number of vertices does not give any information on the minimum number of edges separating these sets. For this reason, we consider alternative of treewidth that guarantees that small edge-separators can be found. However, to the best of our knowledge, it is not known whether the edge-Erdős–Pósa property always holds when the host graphs have bounded treewidth.

One possible edge-analogue of treewidth is tree partition width. Recall that θ_t is the graph with two vertices and t edges, for every $t \in \mathbb{N}$. The following uses [DO96, Theorem 1.2].

Lemma 3.12. *For every $t \in \mathbb{N}$, there exists a ceiling for the triple $(\mathbf{tpw}, \mathcal{M}(\theta_t), \mathsf{e})$.*

Proof. According to [DO96, Theorem 1.2], there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every $p \in \mathbb{N}$, every simple graph G satisfying $\mathbf{tpw}(G) \geq f(p)$ contains as a subgraph either a p-wall, or a p-path, or a p-star, or a p-fan. We omit the definition of these graphs here, but we note that each of them contains a $\mathsf{e}\text{-}\mathcal{M}(\theta_t)$-packing of size k as soon as $kt < p/2$.

Let G be a graph such that $\mathbf{tpw}(G) \geq f(2kt) \cdot kt$. If G has a multiedge e of multiplicity $\geq kt$, then it clearly contains an $\mathsf{e}\text{-}\mathcal{M}(\theta_t)$-packing of size k. Therefore we now assume that all edges of G have multiplicity less than kt. Observe that, if we denote by $\underline{G}$ the underlying simple graph of G, we have $\mathbf{tpw}(\underline{G}) \geq \frac{\mathbf{tpw}(G)}{kt}$. Hence $\mathbf{tpw}(\underline{G}) \geq f(2kt)$ and, by definition of f and the remark above, $\underline{G}$ contains a $\mathsf{e}\text{-}\mathcal{M}(\theta_t)$-packing of size k. As $\underline{G}$ is a subgraph of G, the aforementioned packing also belong to G, which proves the lemma. □

Let $\mathcal{H}$ be a class of graphs. We define $\tilde{\mathcal{H}}$ as the set of all the *subgraph-minimal elements of* $\mathcal{H}$, i.e.,

$$\tilde{\mathcal{H}} = \{H,\ H \in \mathcal{H} \text{ and none of the subgraphs of } H \text{ belongs to } \mathcal{H}\}.$$

We define $\Delta(\mathcal{H})$ as the maximum number of edges incident to a vertex in a graph of $\mathcal{H}$ (counting multiple edges). We also set $\tilde{\Delta}(\mathcal{H}) = \Delta(\tilde{\mathcal{H}})$.

Lemma 3.13. *For every graph H of h edges, it holds that $\tilde{\Delta}(\mathcal{M}(H)) \leq h, \tilde{\Delta}(\mathcal{T}(H)) \leq h, \tilde{\Delta}(\mathcal{I}(H)) \leq 2h$.*

Lemma 3.14. *Let $\mathcal{H}$ be a class of connected non-trivial graphs where $\tilde{\Delta}(\mathcal{H}) \leq d$. Then for every $r \in \mathbb{N}$, $\mathcal{H}$ has the* e*-Erdős–Pósa property on $\mathcal{G}_{\mathbf{tpw} \leq r}$ with gap $g_r(k) = k \cdot r \cdot (dr + 1)$.*

Proof. Let $r \in \mathbb{N}$. We will show the following for every $k \in \mathbb{N}$: for every graph $G \in \mathcal{G}_{\mathbf{tpw} \leq r}$, if $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$ then $\mathsf{e\text{-}cover}_{\mathcal{H}}(G) \leq g_r(k)$.

We proceed by induction. The base case $k = 0$ is trivial. We thus assume that $k > 0$ and that the above statement holds for every positive integer $k' < k$ (induction hypothesis).

Let $G \in \mathcal{G}_{\mathbf{tpw} \leq r}$ be a graph such that $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$. We assume that G is connected, as otherwise we can treat each connected component separately.

Let $(\{X_t\}_{t \in V(T)}, T, s)$ be an optimal tree partition decomposition of G. We define $G_t = G\left[\bigcup_{u \in \mathrm{desc}_{(T,s)}(t)} X_u\right]$. For every edge $\{u, v\}$ of T we denote by $E_{\{u,v\}}$ the edges of G with the one endpoint in X_u and the other one in X_v. Let t be a vertex of T of minimum distance from a leaf, subject to $\mathsf{e\text{-}pack}_{\mathcal{H}}(G_t) > 0$.

Let M be a subgraph-minimal subgraph of G_t isomorphic to some member of $\mathcal{H}$ and let $t_1, \ldots, t_p$ be the children of t such that $V(G_{t_i}) \cap V(M) \neq \emptyset$ for every $i \in \{1, \ldots, p\}$. By minimality of M, it has no vertex with more than $\tilde{\Delta}(\mathcal{H}) \leq d$ incident edges. As $|X_t| \leq r$, we deduce that $p \leq rd$.

Let $C = E(X_t) \cup \bigcup_{i=1}^{p} E_{\{t,t_i\}}$. Notice that $|C| \leq r + dr^2$. Let us consider then graph $G' = G \setminus C$. Let M' be a subgraph of G' that is isomorphic to some member of $\mathcal{H}$. By minimality of t, $\mathsf{e\text{-}pack}_{\mathcal{H}}(G_{t_i}) = 0$, for every $i \in \{1, \ldots, p\}$. Therefore, if M' contained an edge $e \in E(G_{t_i})$ (for some $i \in \{1, \ldots, p\}$), it would also contain an edge of $E(G) \setminus E(G_{t_i})$. Since every graph of $\mathcal{H}$ is connected, M' would also need to contain some edge of $E_{\{t,u_i\}}$ in order to be connected to edges of $E(G) \setminus E(G_{t_i})$. However $E(G') \cap E_{\{t,u_i\}} = \emptyset$. We deduce that for every subgraph M' of G' that is isomorphic to some member of $\mathcal{H}$, we have $E(M') \cap E(M) = \emptyset$. It follows that every e-$\mathcal{H}$-packing in G' is edge-disjoint with M.

Hence $\mathsf{e\text{-}pack}_{\mathcal{H}}(G') < k$, as otherwise a packing of size k in G' would, together with M, yield a packing of size $k+1$ in G whereas $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$. By applying the induction hypothesis on G', there is a subset $D \subseteq E(G')$ such that $\mathsf{e\text{-}pack}_{\mathcal{H}}(G' \setminus D) = 0$ and moreover $|D| \leq g_r(k-1)$. It is easy to see that $C \cup D$ is an e-$\mathcal{H}$-cover of G. Furthermore $|C \cup D| \leq r(dr+1) + g_r(k-1) = g_r(k)$, as required. ☐

An application of Lemma 3.14 is the following result, which also relies on Theorem 3.1 and Lemma 3.12.

Corollary 3.15 (see also [CRST15a])**.** *For every $r \in \mathbb{N}_{\geq 1}$, $\mathcal{M}(\theta_r)$ has the* e*-Erdős–Pósa property.*

However, according to the results in [DO96], the class of graphs H such that there is a ceiling for $(\mathbf{tpw}, \mathcal{M}(H), \mathsf{e})$ is rather limited. An alternative counterpart to treewidth might be the tree-cut width. We do not provide the definition here, but we refer the reader to the article where this parameter has been introduced [Wol15] (see also [GPT+16] for an alternative definition). The next result appeared in [GKRT16a] and is strongly based on the results of [Wol15].

Theorem 3.16. *For every planar subcubic graph H with h edges, there exists a ceiling for the triple* $(\mathbf{tcw}, \mathcal{I}(H), \mathsf{e})$.

The next Lemma is the counterpart of Lemma 3.14, especially for the case of immersion models for graphs of bounded tree-cut width.

Lemma 3.17 ([GKRT16a])**.** *Let t be a positive integer and let H be a connected non-trivial planar subcubic graph of h edges. Then $\mathcal{I}(H)$ has the* e*-Erdős–Pósa property on $\mathcal{G}_{\mathbf{tcw}\leq t}$ with gap $k \mapsto t^2hk$.*

Using Theorem 3.1, Lemma 3.17, and Theorem 3.16 we can also derive the following.

Corollary 3.18 ([GKRT16a])**.** *Let H be a connected non-trivial planar subcubic graph of h edges. Then $\mathcal{I}(H)$ has the* e*-Erdős–Pósa property with gap $k \mapsto (hk)^{O(1)}$.*

4 The Erdős–Pósa property from girth

In this section, we give another proof of the Erdős–Pósa Theorem that highlights a technique for proving more general Erdős–Pósa-type results. The technique can be informally summarized as follow. We prove that either G contains a *small* cycle or that it can be reduced to a smaller graph with the same packing and cover number. We then apply induction on either the graph where a small cycle has been deleted (in the first case), or on the reduced graph (in the second case). This technique has been successfully applied in [FJW13, CRST15a], for instance.

The girth of a graph is the minimum length of a cycle in this graph. Let us first recall the following result.

Lemma 4.1 ([Tho83], see also [Die05, Theorem 7.4.2])**.** *There is a constant $c \in \mathbb{R}$, such that for every $q \in \mathbb{N}_{\geq 1}$, every graph of minimum degree at least 3 and girth at least $c \log q$ contains K_q as a minor.*

A direct consequence of this result is the following trichotomy.

Corollary 4.2. *For every graph G and every integer $q > 1$, one of the following holds:*

(i) G has a cycle on at most $c \log q$ vertices;

(ii) G has a vertex of degree at most 2;

(iii) G contains K_q as a minor,

where c is the constant of Lemma 4.1.

We now prove the lemma that implies the classic Erdős–Pósa Theorem both for the vertex and its edge version. Recall that $A_{\mathsf{x}}(G)$ denotes $V(G)$ or $E(G)$, depending if $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$.

Lemma 4.3. *For every $q \in \mathbb{N}^+$ and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, the class $\mathcal{M}(\theta_2)$ has the* x*-Erdős–Pósa property for the class of graphs excluding K_q as a minor with gap $O(k \cdot \log q)$.*

Proof. We will prove that for every non-negative integer k and every K_q-minor-free graph G, either G has k x-disjoint cycles, or G has a subset $X \subseteq A_{\mathsf{x}}(G)$ of size at most $ck \log q$ such that $G \setminus X$ is a forest, where c is the constant of Lemma 4.1. We proceed by induction on the pair (k, G), with the well-founded order defined by $(k', G') \leq (k, G) \iff (k' \leq k$ and $|A_{\mathsf{x}}(G')| \leq |A_{\mathsf{x}}(G)|)$, for all graphs G, G' and non-negative integers k, k'.

The base cases corresponding to $k = 0$ or $|A_{\mathsf{x}}(G)| = 0$ are trivial. Let us now assume that $k \geq 1$, $|A_{\mathsf{x}}(G)| \geq 1$, and that the lemma holds for every pair (k', G') such that $(k', G') \leq (k, G)$.

According to Corollary 4.2, either G has a cycle C on at most $c \log q$ vertices, or it has a vertex v of degree at most two, or it contains K_q as a minor. The last case is not possible, as we require G to be K_q-minor-free.

Whenever the first case applies, we set $G' = G \setminus A_{\mathsf{x}}(C)$ and we consider the pair $(k-1, G')$. If G' contains $k-1$ x-disjoint cycles, then G contains k x-disjoint cycles obtained by adding C to those of G' and we are done. Otherwise, the induction hypothesis implies the existence of a subset $X' \subseteq A_{\mathsf{x}}(G')$ with $|X'| \leq c(k-1) \log q$ such that $G' \setminus X'$ is a forest. Then by definition of C, $X = X' \cup A_{\mathsf{x}}(C)$ has size at most $c \log q$ and $G \setminus X$ is a forest, as required.

In the second case, we delete v if it is isolated and we contract an edge e incident with it otherwise. Notice that since we cannot apply the first case, this contraction does not decrease the maximum number of x-disjoint cycles in G. Also, we can assume without loss of generality that v (respectively e) is not part of a minimum x-cover of cycles in G, as any vertex adjacent to v (respectively edge incident with e) covers all the cycles covered by v (respectively e). Therefore the obtained graph G' satisfies $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_2)}(G') = \mathsf{x\text{-}pack}_{\mathcal{M}(\theta_2)}(G')$ and $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_2)}(G') = \mathsf{x\text{-}cover}_{\mathcal{M}(\theta_2)}(G)$. It is not hard to see that $|A_{\mathsf{x}}(G')| < |A_{\mathsf{x}}(G)|$. Therefore we can apply the induction hypothesis on G' and obtain the desired result on G', that immediately translates to G by the above remarks. □

By setting $q = 3k$ and observing that every graph containing K_{3k} as a minor also contains k vertex-disjoint cycles (hence also edge-disjoint), Lemma 4.3 yields the vertex and edge versions of the classic Erdős–Pósa Theorem as a corollary.

The technique presented in this section has been used to show the following results.

Theorem 4.4 ([FJW13])**.** *For every forest H, $\mathcal{M}(H)$ has the* v*-Erdős–Pósa property with gap $O(k)$.*

Theorem 4.5 ([CRST15a], see also [FJS13] for the vertex case)**.** *For every positive integer r and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $\mathcal{M}(\theta_r)$ has the* x*-Erdős–Pósa property with gap $O(k \log k)$.*

Actually, the ideas in [CRST15a] permit us to replace $\mathcal{M}(\theta_2)$ by $\mathcal{M}(\theta_r), r \geq 2$ in Lemma 4.3.

To extend the idea of Lemma 4.3 in order to prove that some graph class $\mathcal{H}$ has the x-Erdős–Pósa property with gap $f\colon \mathbb{N} \to \mathbb{N}$, one should show that for every positive integer k and every graph G with $\mathsf{x\text{-}pack}_{\mathcal{H}}(G) \leq k$,

- either there is a graph G' with $\mathsf{x\text{-}pack}_{\mathcal{H}}(G) = \mathsf{x\text{-}pack}_{\mathcal{H}}(G')$ and $\mathsf{x\text{-}cover}_{\mathcal{H}}(G) = \mathsf{x\text{-}cover}_{\mathcal{H}}(G')$ and such that $|G'| + \|G'\| < |G| + \|G\|$ (reduction case);
- or G has a subgraph isomorphic to a member of $\mathcal{H}$ on at most $f(k)/k$ vertices/edges (progress case).

In both proofs of Theorem 4.4 and Theorem 4.5, the reduction case is done using the graph theoretic notion of a protrusion introduced in [BFL+09a, BFL+09b] (or variants of it).

Roughly speaking, the idea is to identify large parts of the graph that have constant treewidth (or constant tree partition width, in case of Theorem 4.5) and a small interface towards the rest of the graph and then prove that they can be replaced by smaller ones without changing the packing or the cover number.

5 Results in terms of containment relations

For every partial order $\preceq$ on graphs, and for every graph H, let

$$\mathcal{G}_{\preceq}(H) = \{G \mid \ H \preceq G\}.$$

For every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, we define

$$\mathcal{EP}^{\mathsf{x}}_{\preceq} \quad = \quad \{H \mid \mathcal{G}_{\preceq}(H) \text{ has the } \mathsf{x}\text{-Erdős–Pósa property}\}$$

A general question on the Erdős–Pósa property is to characterize $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ for several containment relations. In this section we mainly provide some negative results about this problem. We start with the following easy observation.

Lemma 5.1. *If $\preceq$ is the subgraph or the induced subgraph relation, $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, and H is a non-trivial graph, then $\mathcal{G}_{\preceq}(H)$ has the x-Erdős–Pósa property, with gap $f : k \mapsto k \cdot |A_{\mathsf{x}}(H)|$. In other words, $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ is the set of all graphs.*

Proof. Let H and G be two graphs and let $k = \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{G}_{\preceq}(H)}(G)$. Let $M_1, \ldots, M_k$ be a v-$\mathcal{G}_{\preceq}(H)$-packing (respectively e-$\mathcal{G}_{\preceq}(H)$-packing) of size k with the minimal number of vertices (respectively edges). Observe that in this case, $|M_i| = |H|$ (respectively $\|M_i\| = \|H\|$) for every $i \in \{1, \ldots, k\}$. Let $X = \bigcup_{i=1}^{k} V(M_i)$ (respectively $X = \bigcup_{i=1}^{k} E(M_i)$). As the packing we consider is of size k, the graph $G \setminus X$ does not have any subgraph isomorphic to a member of $\mathcal{G}_{\preceq}(H)$. Hence X is an v-$\mathcal{G}_{\preceq}(H)$-cover (respectively e-$\mathcal{G}_{\preceq}(H)$-cover), and besides we have $|X| = k \cdot |H|$ (respectively $|X| = k \cdot \|H\|$). □

Notice that in case $\mathsf{x} = \mathsf{v}$, it is not necessary to demand that H is non-trivial in the statement of Lemma 5.1.

5.1 Some negative results

Let us now state several negative results on the Erdős–Pósa property of classes related to topological minors.

In the proofs below, we use the notion of *Euler genus* of a graph G. The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [MT01] for more details on graph embeddings. The *Euler genus* $\gamma(G)$ of a graph G is the minimum Euler genus of a surface where G can be embedded.

Lemma 5.2. *Let H be a non-planar graph. Then $\mathcal{T}(H)$ does not have the v-Erdős–Pósa property.*

Proof. Informally, we will construct, for every positive integer k, a graph G_k by "thickening" the vertices and edges of H. From the non-planarity of H and the way this graph is constructed, we will deduce that $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G_k) = 1$. On the other hand, the connectivity provided by the thickening of H will ensure that the removal of any $k-1$ vertices will leave at least one subdivision of H unaltered.

For every integers $k > 0$ and d, we denote by $\Gamma_{d,k}$ the graph obtained from a grid of width dk and height $d+k-1$ by adding k vertices $a_1, \dots, a_k$ (that we call *apices*) and connecting a_1 to the d first vertices on the first row of the grid (starting from the left), a_2 to the d next vertices, and so on. For every $i \in \{0, \dots, d-1\}$, the set of vertices at indices $\{ik+j,\ j \in \{0, \dots, k-1\}\}$ on the last row of $\Gamma_{d,k}$ is called the *i-th port* of $\Gamma_{d,k}$. We will refer to the vertex at index $ik+j$ of the last row as the j-th vertex of the i-th port. See Figure 1 for a drawing of $\Gamma_{4,3}$. On this drawing, the ports are $U_0, \dots, U_3$.

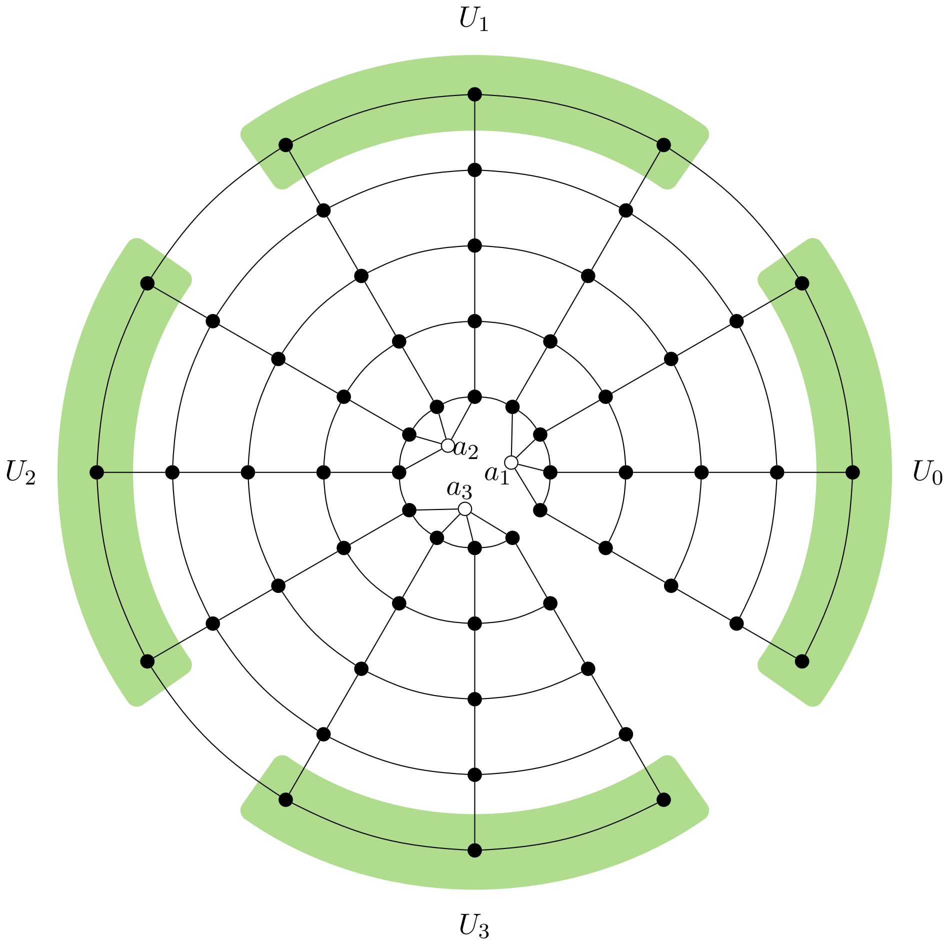

Figure 1: The gadget $\Gamma_{4,3}$ used in Lemma 5.2.

Let k be a positive integer. For every vertex v of H, we arbitrarily choose an ordering of its neighbors and we denote by $\sigma_v(u)$ the rank of u in this ordering (ranging from 0 to $\deg(v)-1$), for every neighbor u of v. We also let F_v be a copy of the graph $\Gamma_{\deg(v),k}$.

The graph G_k can be constructed from the disjoint union of the graphs of $\{F_v,\ v \in V(H)\}$ by adding, for every pair u, v of adjacent vertices, the edge connecting the i-th vertex of the

$\sigma_v(u)$-th port of F_v to the i-th vertex of the $\sigma_u(v)$-th port of F_u, for every $i \in \{0, \ldots, k-1\}$. Informally, we connect the vertices of the $\sigma_v(u)$-th port of F_v to the vertices of the $\sigma_u(v)$-th port of F_u using "parallel" edges. Figure 2 depicts the graph G_k when $G = K_5$ and $k = 3$. This graph contains a subdivision of K_5 but not two vertex-disjoint ones, and the removal of any two vertices leaves one subdivision of K_5 unaltered.

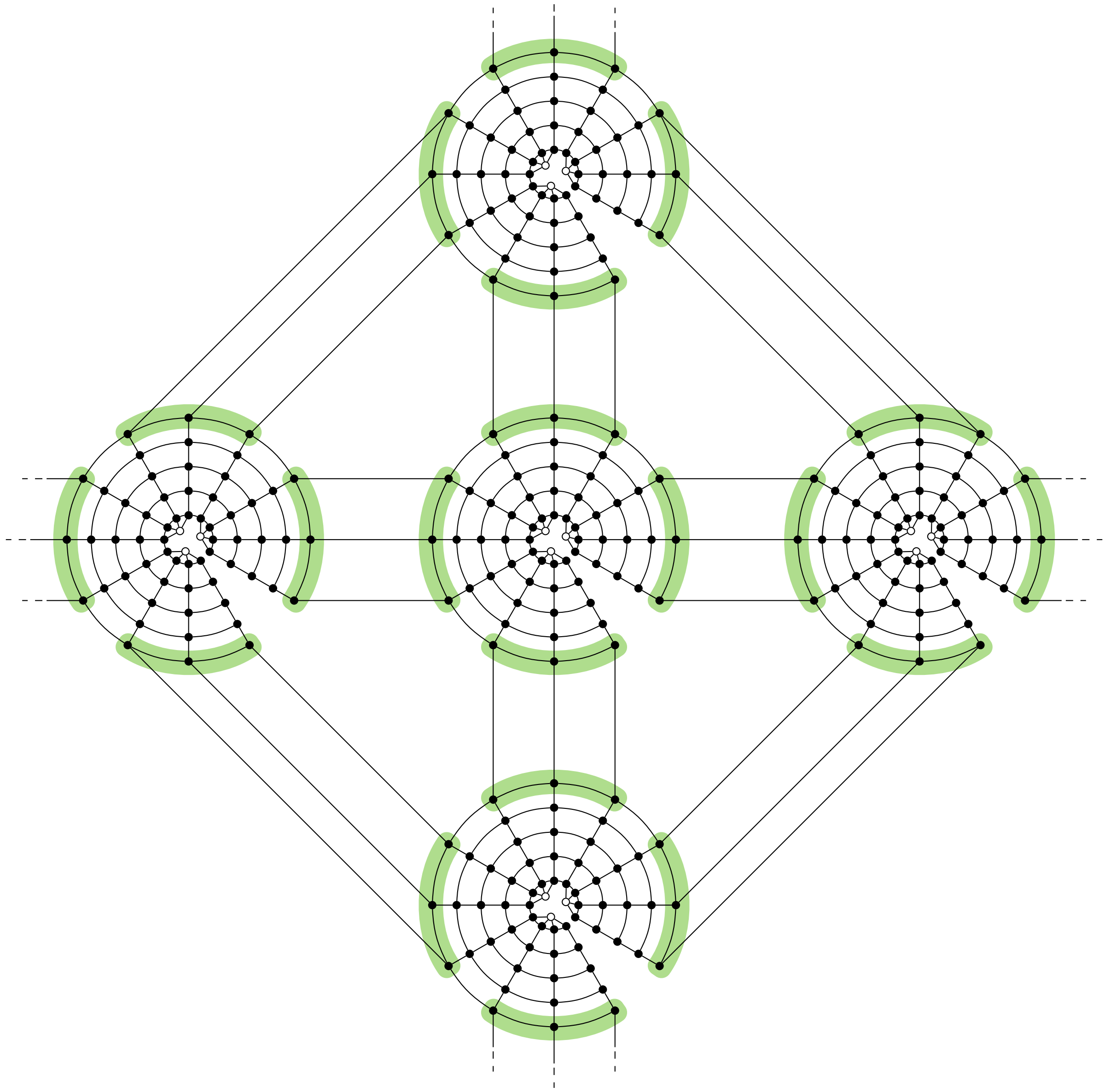

Figure 2: The "thickening" of K_5 for $k = 3$. Edges with dashed ends are connected to the aligned edges at the opposite side of the figure.

It can be easily checked that the Euler genus of G_k and H are equal. As H is not planar, the Euler genus of the disjoint union of two copies of H is larger than the one of H (see [BHK62]) and we get that $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G) < 2$. On the other hand, our construction ensures that $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G) \geq 1$.

Let us now show that for every subset $X \subseteq V(G_k)$ with $|X| < k$ we have $\mathsf{v}\text{-}\mathsf{pack}_{\mathcal{T}(H)}(G \setminus X) \geq 1$. This would complete the proof, since $\{G_k,\ k \in \mathbb{N}_{\geq 1}\}$ would be an infinite family of graphs that have no v-$\mathcal{T}(H)$-packings of size 2 but where a minimum v-$\mathcal{T}(H)$-cover can be arbitrarily large.

Let u and v be two adjacent vertices of H, and let $d = \deg(v)$. For every $i \in \{0, \dots, k-1\}$, let C_i denote the vertices that are

- either in the same column of F_u as the i-th vertex of the $\sigma_u(v)$-th port of F_u;
- or in the same column of F_v as the i-th vertex of the $\sigma_v(u)$-th port of F_v.

The family $\{C_i,\ i \in \{1, \dots, k\}\}$ contains k vertex disjoint elements, therefore at least one of them does not contain any vertex from X (as $|X| < k$). Therefore, for every edge $\{u, v\}$ of H there is an edge $f(\{u, v\})$ between a vertex x of the $\sigma_u(v)$-th port of F_u and a vertex y of the $\sigma_v(u)$-th port of F_v such that no vertex of the same column as x in F_u (respectively y in F_v) belong to X. Using the same argument we can show that for every vertex $v \in V(H)$ there is an apex a such that the columns of F_v adjacent to a are free of vertices of X. Also we know that at least d rows do not contain vertices from X, as the grid of F_v has height $d + k - 1$. Therefore F_v contains as a subgraph a grid S_v such that:

1. some apex a is adjacent to d vertices of the first row of S_v;
2. for every edge $\{u, v\} \in E(H)$, the edge $f(\{u, v\})$ of G_k shares one vertex of the last row of S_v;
3. no vertex of the last row of S_v belongs to two edges $f(\{u, v\})$ and $f(\{u', v\})$ for some distinct neighbors u, u' of v;
4. S_v has height and width at least d;
5. S_v does not contain any vertex of X.

We deduce that $F_v \setminus X$ contains d paths $P_0^v, \dots, P_{d-1}$ that have only the apex a as common vertex and such that P_i connects a to an endpoint of $f(\{v, u_i\})$, where u_i is the neighbor of v of rank i, for every $i \in \{0, \dots, d-1\}$. It is now easy to see that the graph

$$G_k \left[\bigcup_{v \in V(H)} \bigcup_{i=0}^{\deg_H(v)-1} V(P_i^v) \right]$$

contains a subdivision of H that does not contain any vertex of X. This concludes the proof. □

The proof of Lemma 5.2 can be adapted to the setting of the edge-Erdős–Pósa property under the additional requirement that the pattern is subcubic.

Lemma 5.3. *Let H be a subcubic non-planar graph. Then $\mathcal{T}(H)$ does not have the* e-*Erdős–Pósa property.*

Proof. Let k be a positive integer. We use the same construction of G_k as in the proof of Lemma 5.2 with the following modifications: each vertex v of degree $d \geq 4$ of G_k is replaced by a subcubic tree, the leaves of which are the neighbors of v. Let us call G'_k the graph we obtain. It is not hard to see that the genus of G'_k and G_k are equal. Moreover, as G'_k is subcubic, every e-$\mathcal{T}(H)$-packing is also an v-$\mathcal{T}(H)$-packing. We then obtain as previously that $\mathsf{e\text{-}pack}_{\mathcal{T}(H)}(G'_k) = 1$. The argument to show that $\mathsf{e\text{-}cover}_{\mathcal{T}(H)}(G'_k) \geq k$ is identical to that used in the proof of Lemma 5.2. □

In fact, Lemma 5.2 and Lemma 5.3 can be used to prove that more general classes do not have the Erdős–Pósa property, as follows. As we will see in Corollary 5.5 and Corollary 5.6, the conditions of Lemma 5.4 already encompass several well-studied classes.

Lemma 5.4. *Let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, let H be a non-planar graph and let $\mathcal{H}$ be a class of graphs such that:*

(i) $\mathcal{T}(H) \subseteq \mathcal{H}$; and

(ii) H is graph of minimum Euler genus in $\mathcal{H}$;

(iii) if $\mathsf{x} = \mathsf{e}$, then H is subcubic.

Then $\mathcal{H}$ does not have the x-Erdős–Pósa property.

Proof. Let k be a positive integer. We again consider the graphs G_k and G'_k constructed from H as in the proofs of Lemma 5.2 and Lemma 5.3. Let J_k be G_k if $\mathsf{x} = \mathsf{v}$ and $J_k = G'_k$ if $\mathsf{x} = \mathsf{e}$. Let us show that $\mathsf{v\text{-}pack}_{\mathcal{H}}(J_k) = 1$. For this, let us assume that there is an x-$\mathcal{H}$-packing $F_1, \dots, F_p$, for some $p \in \mathbb{N}_{\geq 2}$ in J_k. It is crucial to note that in both the cases $\mathsf{x} = \mathsf{v}$ and $\mathsf{x} = \mathsf{e}$, the subgraphs $F_1, \dots, F_p$ are vertex-disjoint. In fact, when $\mathsf{x} = \mathsf{v}$, this follows from the definition of a v-$\mathcal{H}$-packing, and if $\mathsf{x} = \mathsf{e}$ it is because G'_k is subcubic. Recall that $\gamma(G)$ denotes the Euler genus of G, i.e. the minimum Euler genus of a surface where G can be embedded, and that our construction ensures that $\gamma(J_k) = \gamma(H)$ (see the proofs of Lemma 5.2 and Lemma 5.3). Then we have:

$$\begin{aligned}
\gamma(J_k) &\geq \gamma(F_1 \cup \dots \cup F_p) && \\
&= \sum_{i=1}^{p} \gamma(F_i) && \text{(see [BHK62])} \\
&\geq p \cdot \gamma(H) && \text{(see below)} \\
\gamma(J_k) &> \gamma(H) && \text{(as } p \geq 2).
\end{aligned}$$

The inequality $\gamma(J_k) \geq p \cdot \gamma(H)$ come from the requirements (i)-(ii), which imply $\gamma(F_i) \geq \gamma(H)$ for every $i \in \{1, \dots, p\}$. The last inequality contradicts the fact that $\gamma(J_k) = \gamma(H)$. Therefore $\mathsf{v\text{-}pack}_{\mathcal{H}}(J_k) = 1$. On the other hand,

$$\mathsf{v\text{-}cover}_{\mathcal{H}}(J_k) \geq \mathsf{v\text{-}cover}_{\mathcal{T}(H)}(J_k) \geq k.$$

The last inequality can be found in the proof of Lemma 5.2 or Lemma 5.3 (depending if $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$). This concludes the proof. □

Corollary 5.5. *For every non-planar graph H, none of $\mathcal{I}(H)$ and $\mathcal{M}(H)$ have the v-Erdős–Pósa property.*

Corollary 5.6. *For every subcubic non-planar graph H, none of $\mathcal{I}(H)$ and $\mathcal{M}(H)$ have the e-Erdős–Pósa property.*

Corollary 5.6 can be strengthened by dropping the degree condition on H when considering minor models of H, as follows.

Lemma 5.7. *For every non-planar graph H, $\mathcal{M}(H)$ does not have the* e*-Erdős–Pósa property.*

Proof. Let k be a positive integer. Again we use the graph G'_k constructed as in Lemma 5.3. We modify it by replacing every apex a by a subcubic tree, the leaves of which are the neighbors of a. Let G''_k denote the graph that we obtain. Observe that G''_k is subcubic. Therefore, using the same argument as in the proof of Lemma 5.3 we can show that $\mathsf{e}\text{-}\mathsf{pack}_{\mathcal{M}(H)}(G) = 1$. In the sequel we use the terminology of the proof of Lemma 5.2. Let F''_v denote the graph obtained from F_v by replacing every vertex u of degree at least 4 by a subcubic tree, the leaves of which are the neighbors of u, for every $v \in V(H)$. The proof that $\mathsf{e}\text{-}\mathsf{cover}_{\mathcal{M}(H)}(G) \geq k$ goes as in the proof of Lemma 5.2, except that we obtain, for every $v \in V(H)$, that $F''_v \setminus X$ contains a tree, the leaves of which are endpoints of $f(\{v, u_i\})$ for $i \in \{0, \dots, d-1\}$ (instead of paths connecting an apex to endpoints of $f(\{v, u_i\})$). Fortunately this is enough to guarantee that $G''_k \setminus X$ contains H as a minor, and we are done. □

Let us now summarize results related to the most common containment relations.

Subgraphs and induced subgraphs: $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ is the class of all graphs, both for $\preceq$ being the subgraph and induced subgraph relation, for every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$ (Lemma 5.1).

Minors: $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{m}}}$ is the class of planar graphs [RS86]. Recently, an extension of this characterization, for strongly connected directed graphs, appeared in [AKKW16]. About the edge version, the authors of [RST16] proved that $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{m}}}$ includes the class $\{\theta_r\}_{r \in \mathbb{N}_{\geq 1}}$, and we show in Lemma 5.7 that $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{m}}}$ is a subclass of planar graphs (see also [CRST15a]).

Topological Minors: $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ has been characterized in [LPW14]. There are trees that do not belong to $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ [Tho88]. The class $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ does not contain any non-planar graph (Lemma 5.2) and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$ does not contain any non-planar subcubic graph (Lemma 5.3). Analogous characterizations for the case of strongly connected digraphs have recently appeared in [AKKW16].

Immersions: As proved in [GKRT16a], $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ contains all planar subcubic graphs and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{imm}}}$ contains all non-trivial, connected, planar subcubic graphs. Moreover, $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ does not contain any non-planar graph (Corollary 5.5) and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{imm}}}$ does not contain any subcubic non-planar graph (Corollary 5.6). On the other hand there is a 3-connected planar graph of maximum degree 4 that belongs to none of $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ and $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ [GKRT16a].

6 Summary of results

In the following sections we list positive and negative results on the Erdős–Pósa property, and open problems.

Let us define the notation used in all the tables of Subsection 6.1 and Subsection 6.2. The fourth column of the tables gives the type of the packings/covers the current line is about. The character v (respectively e) refers to vertex-disjoint (respectively edge-disjoint) packings and vertex (respectively edge) covers. We write v/e when the mentioned result holds for both the vertex and the edge version. The symbol $\mathsf{v}_{1/p}$ (resp. $\mathsf{e}_{1/p}$) for some $p \in \mathbb{N}$ indicates that the packing is allowed to use at most p times each vertex (resp. each edge) and that the cover contains vertices (resp. edges). Finally, w stands for vertex covers and packings where every vertex v of the host graph can be used at most $w(v)$ times by every packing, where w is a function mapping reals to the vertices of the host graph. The more specific definitions are given in the corresponding sections.

6.1 Positive results

We provide a series of tables presenting known results on the Erdős–Pósa property of some graph classes, sorted depending on the pattern. Results related to other structures (matroids, hypergraphs, geometry) and to fractional versions are not mentioned here.

A dash in the "gap" column means that the authors did not explicitly provided a gap function, even though one might be computable from the proof. The fourth column refers to the type of packing/cover, as defined above.

6.1.1 Acyclic patterns

Let G be a graph. For every $S, T \subseteq V(G)$, an (S,T)-*path* of G is a path with the one endpoint in S and the other one in T. An S-*path* is a path with both endpoints (which are distinct) in S. If $\mathcal{S}$ is a collection of subsets of $V(G)$, an $\mathcal{S}$-*path* is a path that has endpoints in two different elements of $\mathcal{S}$. A generalization of these settings have been introduced in [MW15], where the pairs of vertices that can be connected by a path are specified by an auxilliary graph. If $S \subseteq V(G)$ and H (demand graph) is a graph with vertex set S, a path of G is said to be H-*valid* if its endpoints are adjacent vertices of H.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Kőn31]	K_2	bipartite	v	k
[LY78] [Lov76]	directed cuts	any digraph	e	k
[Men27]	(S,T)-paths	any	v/e	k
[Grü38]	directed (S,T)-paths	any digraph	v/e	k
[Gal64]	S-paths	any	v	$2k$
[Mad78b]	$\mathcal{S}$-paths	any	v	see [Sch01]
[Mad78a]	$\mathcal{S}$-paths	any	e	see [SS04]
[CGG+06]	non-zero directed S-paths	edge-group-labeled digraphs	v	$2k-2$
[MW15]	H-valid paths, H with no matching of size t	any	v	$2^{2^{O(k+t)}}$
[FJW13], Th. 4.4	$\mathcal{M}(H)$, H forest	any	v	$O_{\mathcal{H}}(k)$

6.1.2 Triangles

A graph is *flat* if every edge belongs to at most two triangles.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Tuz90]	triangles	planar graphs	e	$2k$
		G with $\|\|G\|\| \geq 7\|G\|^2/16$	e	$2k$
		tripartite graphs	e	7k/3
[Kri95]	triangles	$\mathcal{T}(K_{3,3})$-free graphs	e	$2k$
[HK88]	triangles	tripartite graphs	e	$1.956k$
[Hax99]	triangles	any	e	$(3 - \frac{3}{23})k$
[ALBT11]	triangles	odd-wheel-free graphs	e	$2k$
		4-colorable graphs		
[HKT11]	triangles	K_4-free planar graphs	e	$3k/2$
		K_4-free flat graphs		

6.1.3 Cycles

The statement of the results in [DZ02, DXZ03] requires additional definition. An odd ring is a graph obtained from an odd cycle by replacing every edge $\{u, v\}$ by either a triangle containing $\{u, v\}$, or by two triangles on vertices $\{u, a, b\}$ and $\{v, c, d\}$ together with the edges $\{b, c\}$ and $\{a, d\}$. We denote by $\mathcal{G}_1$ the class of graphs with no induced subdivision of the following: $K_{2,3}$, a wheel, or an odd ring. We denote by $\mathcal{G}_2$ the class of graphs with no induced subdivision of the following: $K_{3,3}$, a wheel, or an odd ring.

The results on directed cycles also need few more definitions. A digraph is *strongly planar* if it has a planar drawing such that for every vertex v, the edges with head v form an interval in the cyclic ordering of edges incident with v (definition from [GT11]). An *odd double circuit* is a digraph obtained from an undirected circuit of odd length more than 2 by replacing each edge by a pair of directed edges, one in each direction. F_7 is the digraph obtained from the directed cycle on vertices $v_1, \ldots, v_7, v_1$, by adding the edges creating the directed cycle $v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_1$. We denote by $\mathcal{F}$ the class of digraphs with no butterfly minor isomorphic to an odd double circuit, or F_7 (for the definition of butterfly minors of digraphs see [GT11, JRST01, AKKW16]).

Results related to cycles with length constraints, with prescribed vertices, or to extensions of cycles are presented in the forthcoming tables.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[EP65]	cycles	any	v	$O(k \log k)$
[Sim67]	cycles	any	v	$(4 + o(1))\, k \log k$
[Vos68]	cycles	any	v	$(2 + o(1))\, k \log k$
[Die05]	cycles	any	e	$(2 + o(1)) k \log k$
[DZ02]	cycles	$\mathcal{G}_1$, weighted	w	k
[DXZ03]	cycles	$\mathcal{G}_2$	v	k
[KLL02]	cycles	planar graphs	v	$5k$
		outerplanar graphs	v	$2k$
[MYZ13]	cycles	planar graphs	v	$3k$
			e	$4k - 1$
[RRST96]	directed cycles	any digraph	v	–
[RS96]	directed cycles	planar digraphs	v	$O(k \log(k) \log \log k)$
[GT11]	directed cycles	strongly planar digraphs	v	k
		$\mathcal{F}$		
[LY78]	directed cycles	planar digraphs	e	k
[Sey96]	directed cycles	eulerian digraphs with a linkless embedding in 3-space	e	k
[HJW16]	cycles non homologuous to zero	embedded graphs	$\mathsf{v}_{1/2}$	–

6.1.4 Cycles with length constraints

The class of cycles (resp. directed cycles) of length at least t is referred to as $\mathcal{C}_{\geq t}$ (resp. $\vec{\mathcal{C}}_{\geq t}$). For every positive integer k with, we say that a graph is *k-near bipartite* if every set X of vertices contains a stable set of size at least $|X|/2 - k$.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Ree99]	odd cycles	planar graphs	v	superexponential
[FHRV05]	odd cycles	planar graphs	v	$10k$
[Tho01]	odd cycles	$2^{3^{9k}}$-connected graphs	v	$2k-2$
[RR01]	odd cycles	$576k$-connected graphs	v	$2k-2$
[KR09]	odd cycles	$24k$-connected graphs	v	$2k-2$
[Ree99]	odd cycles	k-near bipartite graphs	v	–
[KN07]	odd cycles	embeddable in an orientable surface of Euler genus t	v/e	–
[BR00]	odd cycles	any	e	–
[KV04] [FHRV05]	odd cycles	planar graphs	e	$2k$
[KK16]	odd cycles	4-edge-connected graphs	e	$2^{2^{O(k \log k)}}$
[Ree99]	odd cycles	any	$\mathsf{v}_{1/2}$	–
[NL]	even cycles	any	v	–
[Tho88]	cycles of length 0 mod t	any	v	–
[KW06]	non-zero cycles	$(15k/2)$-connected group-labeled graphs	v	$2k-2$
[Wol11]	non-zero cycles	group-labeled graphs, c.f. [Wol11]	v	$c^{k^{c'}}$ for some c, c'
	cycles of non-zero length mod $2t+1$	any		
[HJW16]	doubly non-zero cycles, c.f. [HJW16]	doubly group-labeled graphs	$\mathsf{v}_{1/2}$	–
	odd cycles non homologuous to zero	embedded graphs		
[NL]	$\mathcal{C}_{\geq t}$	any	v	–
[BBR07]	$\mathcal{C}_{\geq t}$	any	v	$(13 + o_t(1))tk^2$
[FH14]	$\mathcal{C}_{\geq t}$	any	v	$(6t + 4 + o_t(1))k \log k$
[MNŠW16]	$\mathcal{C}_{\geq t}$	any	v	$6kt + (10 + o(1))k \log k$
[HM13]	$\vec{\mathcal{C}}_{\geq 3}$	any digraph	v	–

6.1.5 Extensions of cycles

A *dumb-bell* is a graph obtained by connecting two cycles by a (non-trivial) path.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Sim67]	dumb-bells	any	v	$(4000 + o(1))k \log k$
[FLM+13]	$\mathcal{M}(\theta_t)$	any	v	$O(t^2 k^2)$
[FJS13]	$\mathcal{M}(\theta_t)$	any	v	$O_t(k \log k)$
[RST16]	$\mathcal{M}(\theta_t)$	any	e	$O(k^2 t^2 \operatorname{polylog} kt)$ $O(k^4 t^2 \operatorname{polylog} kt)$
[CRST15a], Cor. 3.15	$\mathcal{M}(\theta_t)$	any	v/e	$O_t(k \log k)$

6.1.6 Minor models

For every digraph D, we denote by $\vec{\mathcal{M}}(D)$ (respectively $\vec{\mathcal{T}}(G)$, $\vec{\mathcal{I}}(G)$) the class of all digraphs that contain D as a directed minor (respectively directed topological minor, directed immersion). Refer to [CS11, CFS12, FS13] for a definition of these notions.

We also denote by $\vec{\mathcal{M}}_{\bowtie}(D)$ (respectively $\vec{\mathcal{T}}_{\bowtie}(G)$) the class of all digraphs that contain D as a butterfly-minor (respectively as a butterfly topological minor). $\vec{\mathcal{P}}$ (respectively $\vec{\mathcal{W}}$) is the class of all graphs that are butterfly minors of a cylindrical directed grid (respectively butterfly topological minors of a cylindrical directed wall). See for instance [AKKW16] for a definition of the cyclindrical directed grid and wall and [JRST01, AKKW16] for a definition of butterfly (topological) minors.

For every $s \in \mathbb{N}$, a digraph is said to be *s-semicomplete* if for every vertex v there are at most s vertices that are not connected to v by an arc (in either direction). A *semicomplete* digraph is a 0-semicomplete digraph.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[RS86], Lemma 3.10	$\mathcal{M}(H)$, H planar	any	v	–
		$\{G,\ \mathbf{tw}(G) \leq t\}$	v	$(t-1)(k\,\mathbf{cc}(H) - 1)$
[DKW12]	$\mathcal{M}(K_t)$	$O(kt)$-connected graphs	v	–
[FST11]	$\mathcal{M}(H)$, H planar connected	K_t-minor free	v	$O_{H,t}(k)$
[RT13a]	$\mathcal{M}(H)$, $\mathbf{pw}(H) \leq 2$ and H connected	any	v	$2^{O(\lvert H\rvert^2)} \cdot k^2 \log k$
[CC13a]+ [CC13c], Cor. 3.7	$\mathcal{M}(H)$, H planar connected	any	v	$O(\lvert H\rvert^{O(1)} \cdot k \operatorname{polylog} k)$
[CRST15a], Cor. 3.15	$\mathcal{M}(H)$, H connected	$\{G,\ \mathbf{tpw}(G) \leq t\}$	v/e	$O_{H,t}(k)$
	$\mathcal{M}(\theta_{t,t'})$	simple graphs	e	–
[AKKW16]	$\vec{\mathcal{M}}_{\bowtie}(H)$, $H \in \vec{\mathcal{P}}$ and H strongly-connected	any digraph	v	–

6.1.7 Subdivisions

For every $t \in \mathbb{N}$, $\mathcal{T}_{(0 \bmod t)}(H)$ denotes the class of subdivisions of H where every edge is subdivided $0 \bmod t$ times. $\mathcal{L}$ is a graph class defined in the (unpublished) manuscript [LPW14]. See the previous section for the definition of $\vec{\mathcal{T}}(G)$ and $\vec{\mathcal{W}}$.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Tho88]	$\mathcal{T}_{(0 \bmod t)}(H)$, H planar subcubic	any	v	–
[LPW14]	$\mathcal{T}(H), H \in \mathcal{L}$	any	v	–
[AKKW16]	$\vec{\mathcal{T}}(H)$, $H \in \vec{\mathcal{W}}$ and H strongly-connected	any digraph	v	–

6.1.8 Immersion expansions

A graph H is a *half-integral immersion* of a graph G is H is an immersion of the graph obtained by G after duplicating the multiplicity of all its edges. We denote by $\mathcal{I}_{1/2}(H)$ the class of all graphs containing H as a half-integral immersion. See above the definition of $\vec{\mathcal{I}}(G)$.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Liu15]	$\mathcal{I}(H)$	4-edge-connected	e	–
[GKRT16a], Lemma 3.17 Cor. 3.18	$\mathcal{I}(H)$, H planar subcubic connected non-trivial	any	e	$(\lVert H \rVert \cdot k)^{O(1)}$
	$\mathcal{I}(H)$, H connected non-trivial	$\{G, \mathbf{tpw}(G) \leq t\}$ $\{G, \mathbf{tcw}(G) \leq t\}$	e	$\lVert H \rVert \cdot t^2 \cdot k$
[Liu15]	$\mathcal{I}_{1/2}(H)$	any	$\mathsf{e}_{1/2}$	–

6.1.9 Patterns with prescribed vertices

Let us first present the two settings of Erdős–Pósa problems with prescribed vertices that we want to deal with here. The first type is when the guest class consists of fixed subgraphs of the host graph. For instance, one can consider a family $\mathcal{F}$ of (non necessarily disjoint) subtrees of a tree T, and compare the maximum number of disjoint elements in $\mathcal{F}$ with the minimum number of vertices/edges of T meeting all elements of $\mathcal{F}$. We will refer to these guest classes by words indicating that we are dealing with substructures (like "subtrees"). We stress that in this setting, the host class is allowed to contain one subgraph F of the host graph, but not one other subgraph F' even if F and F' are isomorphic. For every positive integer t, a t-path is a disjoint union of t paths, and a t-subpath of a t-path G is a subgraph that has a connected intersection with every connected component of G. The concept of t-forests and t-subforests is defined similarly.

In order to introduce the second type of problem, we need the following definition. Let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. If $\mathcal{H}$ is a class of graphs, G is a graph and $S \subseteq A_{\mathsf{x}}(G)$, then a S-$\mathcal{H}$-subgraph of G is a subgraph of G isomorphic to some member of $\mathcal{H}$ and that contain one edge/vertex of S. We are now interested in comparing, for every graph G and every $S \subseteq A_{\mathsf{x}}(G)$, the maximum number of S-$\mathcal{H}$-subgraph of G with the minimum number of elements of $A_{\mathsf{x}}(G)$ that meet all S-$\mathcal{H}$-subgraphs of G. We refer to these problems by prefixing the guest class with an "S" (like in "S-cycles"). The authors of [HJW16] consider (S_1, S_2)-cycles for $S_1, S_2 \subseteq V(G)$: such cycles must meet both of S_1 and S_2. A generalization of this type of problem has been introduced in [KM15]: instead of one set S, one considers three subsets S_1, S_2, S_3 of $V(G)$ and a (S_1, S_2, S_3)-subgraph is required to intersect at least two sets of S_1, S_2 and S_3. Note that some results on patterns with prescribed vertices have been stated in the table on acyclic patterns. Recall that $\mathbf{cc}(G)$ denotes the number of connected components of the graph G.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[HS58]	subpaths	paths	v	k
[GL69]	t-subpaths	t-paths	v	$O(k^{t!})$
	subgraphs H with $\mathbf{cc}(H) \leq t$	paths	v	–
	t-subforests	t-forests	v	–
[GL69]	subtrees of a tree	trees	v	k
[Kai97]	t-subpaths	t-paths	v	$(t^2 - t + 1)k$
[Alo98]	t-subpaths	t-paths	v	$2t^2k$
[Alo02]	subgraphs H with $\mathbf{cc}(H) \leq t$	trees	v	$2t^2k$
	subgraphs H with $\mathbf{cc}(H) \leq t$	$\{G,\ \mathbf{tw}(G) \leq w\}$	v	$2(w+1)t^2k$
[KKM11]	S-cycles	any	v	$O(k^2 \log k)$
[PW12]	S-cycles	any	v/e	$O(k \log k)$
[BJS14]	S-cycles $\cap\, \mathcal{C}_{\geq t}$	any	v	$O(tk \log k)$
[Joo14]	odd S-cycles	$50k$-connected graphs	v	$O(k)$
[KK13]	odd S-cycles	any	$\mathsf{v}_{1/2}$	–
[KK12]	directed S-cycles	all digraphs	$\mathsf{v}_{1/5}$	–
[KKKK13]	odd directed S-cycles	any digraph	$\mathsf{v}_{1/2}$	–
[HJW16]	(S_1, S_2)-cycles	any	v	–
[KM15]	(S_1, S_2, S_3)-$\mathcal{M}(H)$, H planar	any	v	–

6.1.10 Classes with bounded parameters

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Up.-bound on the gap
[Tho88]	any family of connected graphs	$\{G,\ \mathbf{tw}(G) \leq t\}$	v	$k(t+1)$
[FJW13]	$\{H,\ \mathbf{pw}(H) \geq t\}$	any	v	$O_t(k)$
[CRST15a]	any finite family of connected graphs	$\{G,\ \mathbf{tpw}(G) \leq t\}$	v/e	$O_t(k)$

6.2 Negative results

The next table presents lower bounds on the gap for several graph classes, as well as graph classes that do not have the Erdős–Pósa property. It indicates to which extend the results of the table of Subsection 6.1 are best possible. The notation used here are the same as in the previous section, where they are defined.

6.2.1 Cycles and paths

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Tuz90]	triangles	all graphs	e	$\geq 2k$
[EP65]	cycles	all graphs	v	$\Omega(k \log k)$
[Sim67]	cycles	all graphs	v	$> \left(\frac{1}{2} + o(1)\right) k \log k$
[Vos68]	cycles	all graphs	v	$\geq \left(\frac{1}{8} + o(1)\right) k \log k$
[KLL02]	cycles	planar graphs	v	$\geq 2k$
[MYZ13]	cycles	planar graphs	e	$\geq 4k - c$, $c \in \mathbb{N}$
[DNL87]	odd cycles	all graphs	v	arbitrary
[Ree99]	odd cycles	all graphs	e	arbitrary
[Tho01]	odd cycles	planar graphs	v	$\geq 2k - 2$
[KV04]	odd cycles	planar graphs	e	$\geq 2k$
[DNL87]	cycles of length $2^p q \bmod 2^r s$ with $p, q, r, s \in \mathbb{N}_{\geq 1}$, q, s odd and $p < r$	all graphs	v	arbitrary
[DNL87]	directed cycles of length $2^p q \bmod 2^r s$ as above	all digraphs	e	arbitrary
[DNL87]	directed odd cycles	all digraphs	e	arbitrary
[PW12]	S-cycles	all graphs	v	$\Omega(k \log k)$
[KK13]	odd S-cycles	all graphs	v	arbitrary
[KK12]	directed S-cycles	all diraphs	v/e	arbitrary
[KKKK13]	odd directed S-cycles	all digraphs	v	arbitrary
[FH14]	$\mathcal{C}_{\geq t}$	all graphs	v	$\Omega(k \log k)$, t fixed
				$\Omega(t)$, k fixed
[MNŠW16]	$\mathcal{C}_{\geq t}$	all graphs	v	$\geq (k-1)t$
				$\geq \frac{(k-1)\log k}{8}$
[Sim67]	dumb-bells	all graphs	v	$> (1 + o(1))k \log k$
[MW15]	H-valid paths, H with no matching of size t	all graphs	v	unavoidable dependency in t

6.2.2 Patterns related to containment relations

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
from [EP65]	$\mathcal{M}(H)$, H has a cycle	all graphs	v	$\Omega(k \log k)$
[RS86]	$\mathcal{M}(H)$, H non-planar	all graphs	v	arbitrary
Lemma 5.7	$\mathcal{M}(H)$, H non-planar	all graphs	e	arbitrary
Lemma 5.2	$\mathcal{T}(H)$, H non-planar	all graphs	v	arbitrary
[Tho88]	$\mathcal{T}_{(p \bmod t)}(H)$, H planar subcubic, $p \in \{1, \ldots, t-1\}$	all graphs	v	arbitrary
[Tho88]	$\mathcal{T}(H)$, for infinitely many trees H with $\Delta(H) = 4$	planar graphs	e	arbitrary
Lemma 5.3	$\mathcal{T}(H)$, H non-planar subcubic	all graphs	e	arbitrary
copying [Tho88]	$\mathcal{I}(H)$, for infinitely many trees H with $\Delta(H) = 4$	planar graphs	e	arbitrary
Cor. 5.5	$\mathcal{I}(H)$, H non-planar	all graphs	v	arbitrary
Cor. 5.6	$\mathcal{I}(H)$, H non-planar subcubic	all graphs	e	arbitrary
[GKRT16a]	$\mathcal{I}(H)$, for some 3-connected H with $\Delta(H) = 4$	planar graphs	e	arbitrary
[Liu15]	$\mathcal{I}(H)$, for every H	3-edge-connected graphs	e	arbitrary
[AKKW16]	$\vec{\mathcal{M}}_{\bowtie}(H)$, $H \notin \vec{\mathcal{P}}$ an H strongly-connected	all digraphs	v	arbitrary
	$\vec{\mathcal{T}}_{\bowtie}(H)$, $H \notin \vec{\mathcal{W}}$ and H strongly-connected			

6.3 Some questions and conjectures

Clearly, the most general question on the Erdős–Pósa property is to characterize the class $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ (defined in Section 5) for various instantiations of x and $\preceq$ and optimize the corresponding gap. In what follows we sample some related conjectures and questions that have appeared in the bibliography.

Question 6.1 ([Tho88]). Is it true that for every class $\mathcal{H}$ of graphs, either $\mathcal{H}$ has the v-Erdős–Pósa property or there is no integer q such for every graph G with $\textsf{v-pack}_{\mathcal{H}}(G) \leq 1$ it holds that $\textsf{v-cover}_{\mathcal{H}}(G) \leq q$. In particular, it is true when $\mathcal{H}$ consists of connected graphs and is closed under topological minors?

Conjecture 6.2 (Tuza's conjecture [Tuz90]). For every graph G it holds that

$$\textsf{e-cover}_{\{K_3\}}(G) \leq 2 \cdot \textsf{e-pack}_{\{K_3\}}(G).$$

Conjecture 6.3 ([BBR07]). Let $l \geq 6$ be an integer. Let G be a graph containing no $\textsf{v-}\mathcal{C}_{\geq l}$-packing of size 2. Then there exists a $\textsf{v-}\mathcal{C}_{\geq l}$-cover of G of size at most l.

Conjecture 6.4 (Jones' conjecture [KLL02]). Let $\mathcal{C}$ denote the class of all cycles. For every planar graph G, it holds that

$$\textsf{v-cover}_{\mathcal{C}}(G) \leq 2 \cdot \textsf{v-pack}_{\mathcal{C}}(G).$$

A *hole* is an induced cycle of lenght at least 4.

Question 6.5 ([JP16]). Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every graph G and every $k \in \mathbb{N}$, the following holds:

- G has k vertex-disjoint holes; or
- there is a set $X \subseteq V(G)$ such that $G \setminus X$ has no hole?

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References

[ADG04] Julio Aracena, Jacques Demongeot, and Eric Goles. Positive and negative circuits in discrete neural networks. *Neural Networks, IEEE Transactions on*, 15(1):77–83, Jan 2004.

[AKKW16] Saeed Akhoondian Amiri, Ken-ichi Kawarabayashi, Stephan Kreutzer, and Paul Wollan. The Erdős–Pósa property for directed graphs. *ArXiv preprint arXiv:0904.0727*, 2016.

[ALBT11] S. Aparna Lakshmanan, Csilla Bujtás, and Zsolt Tuza. Small edge sets meeting all triangles of a graph. *Graphs and Combinatorics*, 28(3):381–392, 2011.

[Alo98] Noga Alon. Piercing d -intervals. *Discrete & Computational Geometry*, 19(3):333–334, 1998.

[Alo02] Noga Alon. Covering a hypergraph of subgraphs. *Discrete Mathematics*, 257(2–3):249 – 254, 2002.

[Ara08] Julio Aracena. Maximum number of fixed points in regulatory boolean networks. *Bulletin of Mathematical Biology*, 70(5):1398–1409, 2008.

[ARS16] Julio Aracena, Adrien Richard, and Lilian Salinas. Number of fixed points and disjoint cycles in monotone Boolean networks. *ArXiv preprint arXiv:1602.03109*, February 2016.

[BBR07] Etienne Birmelé, J. Adrian Bondy, and Bruce A. Reed. The Erdős-Pósa property for long circuits. *Combinatorica*, 27(2):135–145, 2007.

[BFL+09a] Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (Meta) Kernelization. *ArXiv preprint 0904.0727, full version of [BFL+09b]*, 2009.

[BFL+09b] Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (Meta) Kernelization. In *Proceedings of FOCS 2009*, pages 629–638. IEEE Computer Society, 2009.

[BHK62] Joseph Battle, Frank Harary, and Yukihiro Kodama. Additivity of the genus of a graph. *Bulletin of the American Mathematical Society*, 68(6):565–568, 11 1962.

[BJS14] Henning Bruhn, Felix Joos, and Oliver Schaudt. Long cycles through prescribed vertices have the Erdős-Pósa property. *ArXiv preprint arXiv:1412.2894*, December 2014.

[Bou13] Nicolas Bousquet. *Hitting sets: VC-dimension and Multicut*. PhD thesis, Université Montpellier 2, 2013.

[BR00] Claude Berge and Bruce Reed. Optimal packings of edge-disjoint odd cycles. *Discrete Mathematics*, 211(1–3):197 – 202, 2000.

[BT15] Nicolas Bousquet and Stéphan Thomassé. Vc-dimension and Erdős–Pósa property. *Discrete Mathematics*, 338(12):2302 – 2317, 2015.

[CC13a] Chandra Chekuri and Julia Chuzhoy. Large-treewidth graph decompositions and applications. *ArXiv preprint arXiv:1304.1577, full version of [CC13b]*, 2013.

[CC13b] Chandra Chekuri and Julia Chuzhoy. Large-treewidth graph decompositions and applications. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 291–300. ACM, 2013.

[CC13c] Chandra Chekuri and Julia Chuzhoy. Polynomial Bounds for the Grid-Minor Theorem. *ArXiv preprint arXiv:1305.6577, full version of [CC14]*, May 2013.

[CC14] Chandra Chekuri and Julia Chuzhoy. Polynomial bounds for the grid-minor theorem. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 60–69. ACM, 2014.

[CFS12] Maria Chudnovsky, Alexandra Fradkin, and Paul Seymour. Tournament immersion and cutwidth. *Journal of Combinatorial Theory, Series B*, 102(1):93 – 101, 2012.

[CGG+06] Maria Chudnovsky, Jim Geelen, Bert Gerards, Luis Goddyn, Michael Lohman, and Paul Seymour. Packing non-zero a-paths in group-labelled graphs. *Combinatorica*, 26(5):521–532, 2006.

[CRST15a] Dimitris Chatzidimitriou, Jean-Florent Raymond, Ignasi Sau, and Dimitrios M. Thilikos. An $O(\log OPT)$-approximation for covering and packing minor models of θ_r. *ArXiv preprint arXiv:1510.03945, full version of [CRST15b]*, October 2015.

[CRST15b] Dimitris Chatzidimitriou, Jean-Florent Raymond, Ignasi Sau, and Dimitrios M. Thilikos. An $O(\log \mathrm{opt})$-approximation for covering/packing minor models of θ_r. In Laura Sanità and Martin Skutella, editors, *Approximation and Online Algorithms: 13th International Workshop, WAOA 2015, Patras, Greece, September 17-18, 2015. Revised Selected Papers*, pages 122–132, Cham, 2015. Springer International Publishing.

[CS11] Maria Chudnovsky and Paul Seymour. A well-quasi-order for tournaments. *Journal of Combinatorial Theory, Series B*, 101(1):47–53, 2011.

[Die05] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, third edition, 2005.

[DKW12] Reinhard Diestel, Ken-ichi Kawarabayashi, and Paul Wollan. The Erdős–Pósa property for clique minors in highly connected graphs. *Journal of Combinatorial Theory, Series B*, 102(2):454–469, 2012.

[DNL87] IJ Dejter and Víctor Neumann-Lara. Unboundedness for generalized odd cyclic transversality. In *Combinatorics (Eger, 1987), Colloq. Math. Soc. János Bolyai*, volume 52, pages 195–203, 1987.

[DO96] Guoli Ding and Bogdan Oporowski. On tree-partitions of graphs. *Discrete Mathematics*, 149(1–3):45 – 58, 1996.

[DXZ03] Guoli Ding, Zhenzhen Xu, and Wenan Zang. Packing cycles in graphs, II. *Journal of Combinatorial Theory, Series B*, 87(2):244 – 253, 2003.

[DZ02] Guoli Ding and Wenan Zang. Packing cycles in graphs. *Journal of Combinatorial Theory, Series B*, 86(2):381 – 407, 2002.

[EP65] Paul Erdős and Louis Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.

[FH14] Samuel Fiorini and Audrey Herinckx. A tighter Erdős-Pósa function for long cycles. *Journal of Graph Theory*, 77(2):111–116, 2014.

[FHRV05] Samuel Fiorini, Nadia Hardy, Bruce Reed, and Adrian Vetta. Approximate min-max relations for odd cycles in planar graphs. In *Integer Programming and Combinatorial Optimization*, pages 35–50. Springer, 2005.

[FJS13] Samuel Fiorini, Gwenaël Joret, and Ignasi Sau. Optimal Erdős–Pósa property for pumpkins. Manuscript, 2013.

[FJW13] Samuel Fiorini, Gwenaël Joret, and David R. Wood. Excluded forest minors and the Erdős-Pósa property. *Combinatorics, Probability & Computing*, 22(5):700–721, 2013.

[FLM+13] Fedor V Fomin, Daniel Lokshtanov, Neeldhara Misra, Geevarghese Philip, and Saket Saurabh. Quadratic upper bounds on the erdős–pósa property for a generalization of packing and covering cycles. *Journal of Graph Theory*, 74(4):417–424, 2013.

[FS13] Alexandra Fradkin and Paul Seymour. Tournament pathwidth and topological containment. *Journal of Combinatorial Theory, Series B*, 103(3):374 – 384, 2013.

[FST11] Fedor V. Fomin, Saket Saurabh, and Dimitrios M. Thilikos. Strengthening Erdős–Pósa property for minor-closed graph classes. *Journal of Graph Theory*, 66(3):235–240, 2011.

[Gal64] Tibor Gallai. Maximum-minimum sätze und verallgemeinerte faktoren von graphen. *Acta Mathematica Hungarica*, 12, 03 1964.

[GK09] Jim Geelen and Kasper Kabell. The Erdős–Pósa property for matroid circuits. *Journal of Combinatorial Theory, Series B*, 99(2):407–419, March 2009.

[GKRT16a] Archontia Giannopoulou, O-joung Kwon, Jean-Florent Raymond, and Dimitrios M. Thilikos. Packing and covering immersion models of planar subcubic graphs. *arXiv preprint arXiv:1602.04042, full version of [GKRT16b]*, 2016.

[GKRT16b] Archontia Giannopoulou, O-joung Kwon, Jean-Florent Raymond, and Dimitrios M. Thilikos. Packing and covering immersion models of planar subcubic graphs. In *Graph-Theoretic Concepts in Computer Science: 42nd International Workshop, WG 2016, Istanbul, Turkey, June 22-24, 2016, Revised Selected Papers*, volume 9941, page 74. Springer, 2016.

[GL69] András Gyárfás and Jenö Lehel. A Helly-type problem in trees. In *Combinatorial Theory and its applications*, pages 571–584. Colloquia Mathematica Societatis János Bolyai, 1969.

[GPT+16] Archontia C. Giannopoulou, Michał Pilipczuk, Dimitrios M. Thilikos, Jean-Florent Raymond, and Marcin Wrochna. Linear kernels for edge deletion problems to immersion-closed graph classes. *ArXiv preprint arxiv:1609.07780*, September 2016.

[Grü38] Tibor Grünwald. Ein neuer beweis eines mengerschen satzes. *Journal of the London Mathematical Society*, 1(3):188–192, 1938.

[GT11] Bertrand Guenin and Robin Thomas. Packing directed circuits exactly. *Combinatorica*, 31(4):397–421, 2011.

[Hal91] Rudolf Halin. Tree-partitions of infinite graphs. *Discrete Mathematics*, 97(1–3):203 – 217, 1991.

[Hax99] Penny Haxell. Packing and covering triangles in graphs. *Discrete Mathematics*, 195(1–3):251 – 254, 1999.

[HJW16] Tony Huynh, Felix Joos, and Paul Wollan. A unified Erdős–Pósa theorem for constrained cycles. *ArXiv preprints arXiv:1605.07082*, May 2016.

[HK88] Penny Haxell and Y. Kohayakawa. Packing and covering triangles in tripartite graphs. *Graphs and Combinatorics*, 14(1):1–10, 1988.

[HKT11] Penny Haxell, Alexandr Kostochka, and Stéphan Thomassé. Packing and covering triangles in k_4-free planar graphs. *Graphs and Combinatorics*, 28(5):653–662, 2011.

[HM13] Frédéric Havet and Ana Karolinna Maia. On disjoint directed cycles with prescribed minimum lengths. Research Report RR-8286, INRIA, April 2013.

[HNP06] Winfried Hochstättler, Robert Nickel, and Britta Peis. Two disjoint negative cycles in a signed graph. *Electronic Notes in Discrete Mathematics*, 25:107–111, 2006.

[HS58] András Hajnal and János Surányi. Über die auflösung von graphen in vollständige teilgraphen. *Annales Universitatis Scientarium Budapestinensis de Rolando Eötvös Nominatae, Sectio Mathematica*, 1:113–121, 1958.

[Joo14] Felix Joos. Parity linkage and the Erdős-Pósa property of odd cycles through prescribed vertices in highly connected graphs. *ArXiv preprint arXiv:1411.6554*, November 2014.

[JP16] Bart MP Jansen and Marcin Pilipczuk. Approximation and kernelization for chordal vertex deletion. *arXiv preprint arXiv:1605.03001, to appear in the Proceedings of the 28th ACM-SIAM Symposium on Discrete Algorithms, SODA 2017*, 2016.

[JRST01] Thor Johnson, Neil Robertson, Paul D. Seymour, and Robin Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138 – 154, 2001.

[Kai97] Tomás Kaiser. Transversals of d-intervals. *Discrete & Computational Geometry*, 18(2):195–203, 1997.

[KK12] Naonori Kakimura and Ken-ichi Kawarabayashi. Packing directed circuits through prescribed vertices bounded fractionally. *SIAM Journal on Discrete Mathematics*, 26(3):1121–1133, 2012.

[KK13] Naonori Kakimura and Ken-ichi Kawarabayashi. Half-integral packing of odd cycles through prescribed vertices. *Combinatorica*, 33(5):549–572, 2013.

[KK15] Naonori Kakimura and Ken-ichi Kawarabayashi. Fixed-parameter tractability for subset feedback set problems with parity constraints. *Theoretical Computer Science*, 576:61–76, 2015.

[KK16] Ken-ichi Kawarabayashi and Yusuke Kobayashi. Edge-disjoint odd cycles in 4-edge-connected graphs. *Journal of Combinatorial Theory, Series B*, 119:12–27, 2016.

[KKKK13] Ken-ichi Kawarabayashi, Daniel Král, Marek Krčál, and Stephan Kreutzer. Packing directed cycles through a specified vertex set. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 365–377. Society for Industrial and Applied Mathematics, 2013.

[KKM11] Naonori Kakimura, Ken-ichi Kawarabayashi, and Dániel Marx. Packing cycles through prescribed vertices. *Journal of Combinatorial Theory, Series B*, 101(5):378 – 381, 2011.

[KLL02] Ton Kloks, C. M. Lee, and Jiping Liu. New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. In *28th International Workshop on Graph Theoretic Concepts in Computer Science (WG 2002)*, volume 2573 of *LNCS*, pages 282–295. Springer, Berlin, 2002.

[Klo94] Tom Kloks. *Treewidth. Computations and Approximations*, volume 842. Springer, LNCS, 1994.

[KM15] O-jong Kwon and Dániel Marx. Erdős–Pósa property of planar-H-minor models with prescribed vertex sets. Manuscript, 2015.

[KN07] Ken-ichi Kawarabayashi and Atsuhiro Nakamoto. The erdős–pósa property for vertex- and edge-disjoint odd cycles in graphs on orientable surfaces. *Discrete Mathematics*, 307(6):764 – 768, 2007.

[Kőn31] Denés Kőnig. Gráfok és mátrixok, 1931.

[KR09] Ken-ichi Kawarabayashi and Bruce Reed. Highly parity linked graphs. *Combinatorica*, 29(2):215–225, 2009.

[Kri95] Michael Krivelevich. On a conjecture of tuza about packing and covering of triangles. *Discrete Mathematics*, 142(1–3):281 – 286, 1995.

[KV04] Daniel Král' and Heinz-Jürgen Voss. Edge-disjoint odd cycles in planar graphs. *Journal of Combinatorial Theory, Series B*, 90(1):107 – 120, 2004.

[KW06] Ken-ichi Kawarabayashi and Paul Wollan. Non-zero disjoint cycles in highly connected group labelled graphs. *Journal of Combinatorial Theory, Series B*, 96(2):296–301, 2006.

[Liu15] Chun-Hung Liu. Packing and Covering Immersions in 4-Edge-Connected Graphs. *ArXiv preprint arXiv:1505.00867*, May 2015.

[Lov76] László Lovász. On two minimax theorems in graph. *Journal of Combinatorial Theory, Series B*, 21(2):96 – 103, 1976.

[LPW14] Chun-Hung Liu, Luke Postle, and Paul Wollan. Erdős-Pósa property for the topological minors. Manuscript, 2014.

[LY78] Claudio L. Lucchesi and D. H. Younger. A minimax theorem for directed graphs. *Journal of the London Mathematical Society (2)*, 17(3):369–374, 1978.

[Mad78a] Wolfgang Mader. Über die maximalzahl kantendisjunkter a-wege. *Archiv der Mathematik*, 30(1):325–336, 1978.

[Mad78b] Wolfgang Mader. Über die maximalzahl kreuzungsfreier h-wege. *Archiv der Mathematik*, 31(1):387–402, 1978.

[Men27] Karl Menger. Zur allgemeinen kurventheorie. *Fundamenta Mathematicae*, 1(10):96–115, 1927.

[MNŠW16] Franck Mousset, Andreas Noever, Nemanja Škorić, and Felix Weissenberger. A tight Erdős-Pósa function for long cycles. *ArXiv preprint arXiv:1603.07588*, March 2016.

[MT01] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins University Press, Baltimore, MD, 2001.

[MW15] Dániel Marx and Paul Wollan. An exact characterization of tractable demand patterns for maximum disjoint path problems. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 642–661. SIAM, 2015.

[MYZ13] Jie Ma, Xingxing Yu, and Wenan Zang. Approximate min-max relations on plane graphs. *Journal of Combinatorial Optimization*, 26(1):127–134, 2013.

[NL] Víctor Neumann-Lara. Cyclic transversality for even and long cycles. Unpublished (see [DNL87, Tho88]).

[PW12] Matteo Pontecorvi and Paul Wollan. Disjoint cycles intersecting a set of vertices. *Journal of Combinatorial Theory, Series B*, 102(5):1134 – 1141, 2012.

[Ree97] Bruce A. Reed. *Tree width and tangles: A new connectivity measure and some applications*, pages 87–162. Cambridge University Press, 1997.

[Ree99] Bruce A. Reed. Mangoes and blueberries. *Combinatorica*, 19(2):267–296, 1999.

[RR01] Dieter Rautenbach and Bruce Reed. The Erdős–Pósa property for odd cycles in highly connected graphs. *Combinatorica*, 21(2):267–278, 2001.

[RRST96] Bruce A. Reed, Neil Robertson, Paul D. Seymour, and Robin Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

[RS86] Neil Robertson and Paul D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986.

[RS96] Bruce A. Reed and Bruce F. Shepherd. The Gallai-Younger conjecture for planar graphs. *Combinatorica*, 16(4):555–566, 1996.

[RST94] Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 62(2):323–348, 1994.

[RST16] Jean-Florent. Raymond, Ignasi. Sau, and Dimitrios M. Thilikos. An edge variant of the Erdős-Pósa property. *Discrete Mathematics*, 339(8):2027 – 2035, 2016.

[RT13a] Jean-Florent Raymond and Dimitrios M. Thilikos. Polynomial Gap Extensions of the Erdős-Pósa Theorem. *ArXiv preprint arXiv:1305.7376, full version of [RT13b]*, May 2013.

[RT13b] Jean-Florent Raymond and Dimitrios M. Thilikos. Polynomial gap extensions of the Erdős-Pósa theorem. In Jaroslav Nešetřil and Marco Pellegrini, editors, *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 13–18. Scuola Normale Superiore, 2013.

[Sch01] Alexander Schrijver. A short proof of mader's s-paths theorem. *Journal of Combinatorial Theory, Series B*, 82(2):319–321, 2001.

[See85] Detlef Seese. Tree-partite graphs and the complexity of algorithms. In *Proceedings of Fundamentals of Computation Theory*, volume 199 of *Lecture Notes in Computer Science*, pages 412–421. Springer Berlin Heidelberg, 1985.

[Sey96] Paul D. Seymour. Packing circuits in Eulerian digraphs. *Combinatorica*, 16(2):223–231, 1996.

[Sim67] Miklós Simonovits. A new proof and generalizations of a theorem of erdős and pósa on graphs without $k + 1$ independent circuits. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(1):191–206, 1967.

[SS04] András Sebő and László Szegő. The path-packing structure of graphs. In Daniel Bienstock and George Nemhauser, editors, *Integer Programming and Combinatorial Optimization: 10th International IPCO Conference, New York, NY, USA, June 7-11, 2004. Proceedings*, pages 256–270, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

[Tho83] Carsten Thomassen. Girth in graphs. *Journal of Combinatorial Theory, Series B*, 35(2):129 – 141, 1983.

[Tho88] Carsten Thomassen. On the presence of disjoint subgraphs of a specified type. *Journal of Graph Theory*, 12(1):101–111, 1988.

[Tho01] Carsten Thomassen. The Erdős-Pósa property for odd cycles in graphs of large connectivity. *Combinatorica*, 21(2):321–333, 2001.

[Tuz90] Zsolt Tuza. A conjecture on triangles of graphs. *Graphs and Combinatorics*, 6(4):373–380, 1990.

[Vos68] Heinz-Jürgen Voss. Some properties of graphs containing k independent circuits. In *Proceedings of Colloquium Tihany*, pages 321–334, 1968.

[Wol11] Paul Wollan. Packing cycles with modularity constraints. *Combinatorica*, 31(1):95–126, 2011.

[Wol15] Paul Wollan. The structure of graphs not admitting a fixed immersion. *Journal of Combinatorial Theory, Series B*, 110:47–66, 2015.