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Node Overlap Removal for 1D Graph Layout: Proof of Theorem 1

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Introduction

In this report, we give the complete proof of the theorem 1 of the paper [1]. This theorem states that the method described in the paper meets 4 requirements. The proof is given for each of these requirements.

Requirement 1 (Optimally using the segment length)

Let us define the nodes $v_1$ and $v_{|V|}$ such that $v_1 = \sigma^{-1}(1)$ and $v_{|V|} = \sigma^{-1}(|V|)$.

\begin{align*}
f(v_1) &= p'(v_1) - \frac{s(v_1)}{2} + \sum_{i=1}^{\sigma(v_1)} s(\sigma^{-1}(i)) \quad (1) \\
&= 0 - \frac{s(v_1)}{2} + s(v_1) \quad (2) \\
&= \frac{s(v_1)}{2} \quad (3)
\end{align*}

\begin{align*}
f(v_{|V|}) &= p'(v_{|V|}) - \frac{s(v_{|V|})}{2} + \sum_{i=1}^{\sigma(v_{|V|})} s(\sigma^{-1}(i)) \quad (4) \\
&= l - \sum_{v \in V} s(v) - \frac{s(v_{|V|})}{2} + \sum_{v \in V} s(v) \quad (5) \\
&= l - \frac{s(v_{|V|})}{2} \quad (6)
\end{align*}

We therefore obtain the required values for the first and last node positions. \qed
Requirement 2 (No overlapping)

We first develop the definition of \( f(v) \) as follows, given that \((u, v) \in V^2 \) and \( \sigma(u) < \sigma(v) \):

\[
f(v) = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{7}
\]

\[
= p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{8}
\]

We isolate from \( \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \) the nodes ordered until \( u \) through the \( \sigma \) ordering function:

\[
f(v) = p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))
\]

\[
+ \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{9}
\]

By adding \( \frac{s(u)}{2} - \frac{s(u)}{2} \) to the right term of the equation, we identify the definition of \( f(u) \) and rewrite \( f(v) \) w.r.t. \( f(u) \):

\[
f(v) = \left( p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \right)
\]

\[
+ p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2}
\]

\[
= f(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{10}
\]

We extract from \( \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \) the size of \( v \), thus obtaining an expression of \( f(v) \) as a sum of positive terms, provided that \( p'(v) - p'(u) \) is known to be positive.

\[
f(v) = f(u) + p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{11}
\]

which is equivalent to:

\[
f(v) - f(u) = p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{12}
\]
For two nodes \((u,v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(p'(v) - p'(u) \geq 0\). Moreover, the sum of node sizes \(\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))\) is positive. As a result:

\[
f(v) - f(u) \geq \frac{s(u)}{2} + \frac{s(v)}{2}
\] (14)

and by extension:

\[
\forall (u,v) \in V^2, |f(v) - f(u)| \geq s(u)/2 + s(v)/2. \tag{15}
\]

> Requirement 3 (Preserving initial ordering)

From the proof of Requirement 2, we know that for two nodes \((u,v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(f(v) - f(u) \geq s(u)/2 + s(v)/2\). Since \(s(u)/2 + s(v)/2\) is strictly positive, so is \(f(v) - f(u)\). Hence, \(\forall (u,v) \in V^2, \sigma(u) < \sigma(v) \Rightarrow f(u) < f(v)\). □

> Requirement 4 (Preserving relative distances)

**Lemma 1.** The distance between two consecutive nodes \(u\) and \(v\) in the final layout is equal to the difference \(p'(v) - p'(u)\). More formally, given two nodes \((u,v) \in V^2\),

\[
\sigma(u) + 1 = \sigma(v) \Rightarrow f(v) - f(u) = \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - p'(u)
\]

**Proof.** Let us denote \(D\) as \(f(v) - f(u) = \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2})\). We further develop \(D\) by applying the definition of \(f(v)\):

\[
D = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2})
\] (16)

\[
= p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - (f(u) + \frac{s(u)}{2})
\] (17)

Then by applying the definition of \(f(u) = p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i))\):

\[
D = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - p'(u) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i))
\] (18)
As $\sigma(v) = \sigma(u) + 1$, $\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) = s(v)$, the previous equation can be simplified to prove Lemma 1:

\[
f(v) - \frac{s(v)}{2} = (f(u) + \frac{s(u)}{2}) = p'(v) - s(v) - p'(u) + s(v)
= p'(v) - p'(u) \tag{19}
\]

By applying Lemma 1, we need to prove that:

\[
p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')
\]

We first develop $p'(v) - p'(u)$ by applying the definition of $p$:

\[
p'(v) - p'(u) = \frac{p(v) - p(u)}{p_{\text{max}} - p_{\text{min}}} \times C - \frac{p(u) - p_{\text{min}}}{p_{\text{max}} - p_{\text{min}}} \times C \tag{21}
\]

where $C = (l - \sum_{x \in V} s(x))$, and $C \geq 0$. Then:

\[
p'(v) - p'(u) = C \left( \frac{p(v) - p(u)}{p_{\text{max}} - p_{\text{min}}} \right) \tag{22}
= \frac{C}{p_{\text{max}} - p_{\text{min}}} \times (p(v) - p(u)) \tag{23}
\]

Since $C' = \frac{C}{p_{\text{max}} - p_{\text{min}}}$ is positive, we obtain that:

\[
p(v) - p(u) \geq p(v') - p(u') \Rightarrow C' (p(v) - p(u)) \geq C' (p(v') - p(u'))
p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')
\]

\[\blacksquare\]

References