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Introduction

In this report, we give the complete proof of the theorem 1 of the paper [1]. This theorem states that the method described in the paper meets 4 requirements. The proof is given for each of these requirements.

Requirement 1 (Optimally using the segment length)

Let us define the nodes $v_1$ and $v_{|V|}$ such that $v_1 = \sigma^{-1}(1)$ and $v_{|V|} = \sigma^{-1}(|V|)$.

\begin{align*}
  f(v_1) &= p'(v_1) - \frac{s(v_1)}{2} + \sum_{i=1}^{\sigma(v_1)} s(\sigma^{-1}(i)) \\
  &= 0 - \frac{s(v_1)}{2} + s(v_1) \\
  &= \frac{s(v_1)}{2} \tag{1}
\end{align*}

\begin{align*}
  f(v_{|V|}) &= p'(v_{|V|}) - \frac{s(v_{|V|})}{2} + \sum_{i=1}^{\sigma(v_{|V|})} s(\sigma^{-1}(i)) \\
  &= l - \sum_{v \in V} s(v) - \frac{s(v_{|V|})}{2} + \sum_{v \in V} s(v) \\
  &= l - \frac{s(v_{|V|})}{2} \tag{2}
\end{align*}

We therefore obtain the required values for the first and last node positions. \qed
Requirement 2 (No overlapping)

We first develop the definition of \( f(v) \) as follows, given that \( (u,v) \in V^2 \) and \( \sigma(u) < \sigma(v) \):

\[
\begin{align*}
    f(v) &= p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \quad (7) \\
    &= p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \quad (8)
\end{align*}
\]

We isolate from \( \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \) the nodes ordered until \( u \) through the \( \sigma \) ordering function:

\[
\begin{align*}
    f(v) &= p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \quad (9)
\end{align*}
\]

By adding \( \frac{s(u)}{2} - \frac{s(u)}{2} \) to the right term of the equation, we identify the definition of \( f(u) \) and rewrite \( f(v) \) w.r.t. \( f(u) \):

\[
\begin{align*}
    f(v) &= \left( p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \right) \\
    &\quad + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \quad (10)
\end{align*}
\]

\[
\begin{align*}
    f(v) &= f(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \quad (11)
\end{align*}
\]

We extract from \( \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \) the size of \( v \), thus obtaining an expression of \( f(v) \) as a sum of positive terms, provided that \( p'(v) - p'(u) \) is known to be positive.

\[
\begin{align*}
    f(v) &= f(u) + p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \quad (12)
\end{align*}
\]

which is equivalent to:

\[
\begin{align*}
    f(v) - f(u) &= p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \quad (13)
\end{align*}
\]
For two nodes \((u, v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(p'(v) - p'(u) \geq 0\). Moreover, the sum of node sizes \(\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))\) is positive. As a result:

\[
f(v) - f(u) \geq \frac{s(u)}{2} + \frac{s(v)}{2} \tag{14}
\]

and by extension:

\[
\forall (u, v) \in V^2, |f(v) - f(u)| \geq s(u)/2 + s(v)/2. \tag{15}
\]

\[\square\]

**Requirement 3 (Preserving initial ordering)**

From the proof of Requirement 2, we know that for two nodes \((u, v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(f(v) - f(u) \geq s(u)/2 + s(v)/2\). Since \(s(u)/2 + s(v)/2\) is strictly positive, so is \(f(v) - f(u)\). Hence, \(\forall (u, v) \in V^2, \sigma(u) < \sigma(v) \Rightarrow f(u) < f(v)\). \[\square\]

**Requirement 4 (Preserving relative distances)**

**Lemma 1.** The distance between two consecutive nodes \(u\) and \(v\) in the final layout is equal to the difference \(p'(v) - p'(u)\). More formally, given two nodes \((u, v) \in V^2\),

\[
\sigma(u) + 1 = \sigma(v) \Rightarrow f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - p'(u)
\]

**Proof.** Let us denote \(D\) as \(f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2})\). We further develop \(D\) by applying the definition of \(f(v)\):

\[
D = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) \tag{16}
\]

\[
= p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - (f(u) + \frac{s(u)}{2}) \tag{17}
\]

Then by applying the definition of \(f(u) = p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i))\):

\[
D = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - p'(u) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \tag{18}
\]
As \( \sigma(v) = \sigma(u) + 1, \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) = \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) = s(v) \), the previous equation can be simplified to prove Lemma 1:

\[
f(v) - \frac{s(v)}{2} = (f(u) + \frac{s(u)}{2}) = p'(v) - s(v) - p'(u) + s(v) \quad (19)
\]

\[
= p'(v) - p'(u) \quad (20)
\]

By applying Lemma 1, we need to prove that:

\[
p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')
\]

We first develop \( p'(v) - p'(u) \) by applying the definition of \( p \):

\[
p'(v) - p'(u) = \frac{p(v) - p(u)}{p_{max} - p_{min}} \times C - \frac{p(u) - p_{min}}{p_{max} - p_{min}} \times C \quad (21)
\]

where \( C = (l - \sum_{x \in V} s(x)) \), and \( C \geq 0 \). Then:

\[
p'(v) - p'(u) = C \left( \frac{p(v) - p(u)}{p_{max} - p_{min}} \right)
= \frac{C}{p_{max} - p_{min}} \times (p(v) - p(u)) \quad (22)
\]

\[
= \frac{C}{p_{max} - p_{min}} \times (p(v) - p(u)) \quad (23)
\]

Since \( C' = \frac{C}{p_{max} - p_{min}} \) is positive, we obtain that:

\[
p(v) - p(u) \geq p(v') - p(u') \Rightarrow C' (p(v) - p(u)) \geq C' (p(v') - p(u'))
\]

\[
p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')
\]

\[\square\]

References