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Node Overlap Removal for 1D Graph Layout: Proof of Theorem 1

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Introduction

In this report, we give the complete proof of the theorem 1 of the paper [1]. This theorem states that the method described in the paper meets 4 requirements. The proof is given for each of these requirements.

Requirement 1 (Optimally using the segment length)

Let us define the nodes $v_1$ and $v_{\mid V\mid}$ such that $v_1 = \sigma^{-1}(1)$ and $v_{\mid V\mid} = \sigma^{-1}(\mid V\mid)$.

\[
f(v_1) = p'(v_1) - \frac{s(v_1)}{2} + \sum_{i=1}^{\sigma(v_1)} s(\sigma^{-1}(i)) \tag{1}
\]
\[
= 0 - \frac{s(v_1)}{2} + s(v_1) \tag{2}
\]
\[
= \frac{s(v_1)}{2} \tag{3}
\]

\[
f(v_{\mid V\mid}) = p'(v_{\mid V\mid}) - \frac{s(v_{\mid V\mid})}{2} + \sum_{i=1}^{\sigma(v_{\mid V\mid})} s(\sigma^{-1}(i)) \tag{4}
\]
\[
= l - \sum_{v \in V} s(v) - \frac{s(v_{\mid V\mid})}{2} + \sum_{v \in V} s(v) \tag{5}
\]
\[
= l - \frac{s(v_{\mid V\mid})}{2} \tag{6}
\]

We therefore obtain the required values for the first and last node positions. \qed
Requirement 2 (No overlapping)

We first develop the definition of \( f(v) \) as follows, given that \((u,v) \in V^2\) and \(\sigma(u) < \sigma(v)\):

\[
f(v) = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{7}
\]

\[
= p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{8}
\]

We isolate from \(\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))\) the nodes ordered until \(u\) through the \(\sigma\) ordering function:

\[
f(v) = p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{9}
\]

By adding \(\frac{s(u)}{2} - \frac{s(u)}{2}\) to the right term of the equation, we identify the definition of \(f(u)\) and rewrite \(f(v)\) w.r.t. \(f(u)\):

\[
f(v) = \left( p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \right)
+ p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2}
\]

\[
= f(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{10}
\]

We extract from \(\sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i))\) the size of \(v\), thus obtaining an expression of \(f(v)\) as a sum of positive terms, provided that \(p'(v) - p'(u)\) is known to be positive.

\[
f(v) = f(u) + p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{12}
\]

which is equivalent to:

\[
f(v) - f(u) = p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{13}
\]
For two nodes \((u, v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(p'(v) - p'(u) \geq 0\). Moreover, the sum of node sizes \(\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))\) is positive. As a result:

\[
 f(v) - f(u) \geq \frac{s(u)}{2} + \frac{s(v)}{2}
\]  

(14)

and by extension:

\[
\forall (u, v) \in V^2, |f(v) - f(u)| \geq s(u)/2 + s(v)/2.
\]  

(15)

\[\square\]

**Requirement 3 (Preserving initial ordering)**

From the proof of Requirement 2, we know that for two nodes \((u, v) \in V^2\) such that \(\sigma(u) < \sigma(v)\), \(f(v) - f(u) \geq s(u)/2 + s(v)/2\). Since \(s(u)/2 + s(v)/2\) is strictly positive, so is \(f(v) - f(u)\). Hence, \(\forall (u, v) \in V^2, \sigma(u) < \sigma(v) \Rightarrow f(v) < f(u)\) \(\square\)

**Requirement 4 (Preserving relative distances)**

**Lemma 1.** The distance between two consecutive nodes \(u\) and \(v\) in the final layout is equal to the difference \(p'(v) - p'(u)\). More formally, given two nodes \((u, v) \in V^2\),

\[
\sigma(u) + 1 = \sigma(v) \Rightarrow f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - p'(u)
\]

Proof. Let us denote \(D\) as \(f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2})\). We further develop \(D\) by applying the definition of \(f(v)\):

\[
D = p'(v) - \frac{s(v)}{2} + \frac{\sigma(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - (f(u) + \frac{s(u)}{2})
\]  

(16)

\[
D = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - (f(u) + \frac{s(u)}{2})
\]  

(17)

Then by applying the definition of \(f(u) = p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i))\):

\[
D = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - p'(u) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i))
\]  

(18)
As $\sigma(v) = \sigma(u) + 1$, $\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) = s(v)$, the previous equation can be simplified to prove Lemma 1:

$$f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - s(v) - p'(u) + s(v)$$

$$= p'(v) - p'(u)$$

By applying Lemma 1, we need to prove that:

$$p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')$$

We first develop $p'(v) - p'(u)$ by applying the definition of $p$:

$$p'(v) - p'(u) = \frac{p(v) - p_{\min}}{p_{\max} - p_{\min}} \times C - \frac{p(u) - p_{\min}}{p_{\max} - p_{\min}} \times C$$

where $C = (l - \sum_{x \in V} s(x))$, and $C \geq 0$. Then:

$$p'(v) - p'(u) = C \left( \frac{p(v) - p(u)}{p_{\max} - p_{\min}} \times C \right)$$

$$= \frac{C}{p_{\max} - p_{\min}} \times (p(v) - p(u))$$

Since $C' = \frac{C}{p_{\max} - p_{\min}}$ is positive, we obtain that:

$$p(v) - p(u) \geq p(v') - p(u') \Rightarrow C' (p(v) - p(u)) \geq C' (p(v') - p(u'))$$

$$p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')$$

$\square$

References