Node Overlap Removal for 1D Graph Layout: Proof of Theorem 1

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Introduction

In this report, we give the complete proof of the theorem 1 of the paper [1]. This theorem states that the method described in the paper meets 4 requirements. The proof is given for each of these requirements.

Requirement 1 (Optimally using the segment length)

Let us define the nodes v_1 and $v_{|V|}$ such that $v_1 = \sigma^{-1}(1)$ and $v_{|V|} = \sigma^{-1}(|V|)$.

$$
\begin{aligned}
f(v_1) &= p'(v_1) - \frac{s(v_1)}{2} + \sum_{i=1}^{\sigma(v_1)} s(\sigma^{-1}(i)) && (1)\\
&= 0 - \frac{s(v_1)}{2} + s(v_1) && (2)\\
&= \frac{s(v_1)}{2} && (3)
\end{aligned}
$$

$$
\begin{aligned}
f(v_{|V|}) &= p'(v_{|V|}) - \frac{s(v_{|V|})}{2} + \sum_{i=1}^{\sigma(v_{|V|})} s(\sigma^{-1}(i)) && (4)\\
&= l - \sum_{v \in V} s(v) - \frac{s(v_{|V|})}{2} + \sum_{v \in V} s(v) && (5)\\
&= l - \frac{s(v_{|V|})}{2} && (6)
\end{aligned}
$$

We therefore obtain the required values for the first and last node positions.
□

Requirement 2 (No overlapping)

We first develop the definition of $f(v)$ as follows, given that $(u, v) \in V^2$ and $\sigma(u) < \sigma(v)$:

$$f(v) = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{7}$$

$$= p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) \tag{8}$$

We isolate from $\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))$ the nodes ordered until u through the σ ordering function:

$$\begin{aligned} f(v) = {} & p'(u) + p'(v) - p'(u) - \frac{s(v)}{2} \\ & + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) \end{aligned} \tag{9}$$

By adding $\frac{s(u)}{2} - \frac{s(u)}{2}$ to the right term of the equation, we identify the definition of $f(u)$ and rewrite $f(v)$ w.r.t. $f(u)$:

$$\begin{aligned} f(v) = {} & \left(p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \right) \\ & + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \end{aligned} \tag{10}$$

$$= f(u) + p'(v) - p'(u) - \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{11}$$

We extract from $\sum_{i=\sigma(u)+1}^{\sigma(v)} s(\sigma^{-1}(i))$ the size of v, thus obtaining an expression of $f(v)$ as a sum of positive terms, provided that $p'(v) - p'(u)$ is known to be positive.

$$f(v) = f(u) + p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{12}$$

which is equivalent to:

$$f(v) - f(u) = p'(v) - p'(u) + \frac{s(v)}{2} + \sum_{i=\sigma(u)+1}^{\sigma(v)-1} s(\sigma^{-1}(i)) + \frac{s(u)}{2} \tag{13}$$

For two nodes $(u, v) \in V^2$ such that $\sigma(u) < \sigma(v)$, $p'(v) - p'(u) \geq 0$. Moreover, the sum of node sizes $\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i))$ is positive. As a result:

$$f(v) - f(u) \geq \frac{s(u)}{2} + \frac{s(v)}{2} \tag{14}$$

and by extension:

$$\forall (u, v) \in V^2, |f(v) - f(u)| \geq s(u)/2 + s(v)/2. \tag{15}$$

□

Requirement 3 (Preserving initial ordering)

From the proof of Requirement 2, we know that for two nodes $(u, v) \in V^2$ such that $\sigma(u) < \sigma(v)$, $f(v) - f(u) \geq s(u)/2 + s(v)/2$. Since $s(u)/2 + s(v)/2$ is strictly positive, so is $f(v) - f(u)$. Hence, $\forall (u, v) \in V^2, \sigma(u) < \sigma(v) \Rightarrow f(u) < f(v)$. □

Requirement 4 (Preserving relative distances)

Lemma 1. *The distance between two consecutive nodes u and v in the final layout is equal to the difference $p'(v) - p'(u)$. More formally, given two nodes $(u, v) \in V^2$,*

$$\sigma(u) + 1 = \sigma(v) \Rightarrow f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) = p'(v) - p'(u)$$

Proof. Let us denote D as $f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2})$. We further develop D by applying the definition of $f(v)$:

$$D = p'(v) - \frac{s(v)}{2} + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) \tag{16}$$

$$= p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - (f(u) + \frac{s(u)}{2}) \tag{17}$$

Then by applying the definition of $f(u) = p'(u) - \frac{s(u)}{2} + \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)$:

$$D = p'(v) - s(v) + \sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - p'(u) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) \tag{18}$$

As $\sigma(v) = \sigma(u) + 1$, $\sum_{i=1}^{\sigma(v)} s(\sigma^{-1}(i)) - \sum_{i=1}^{\sigma(u)} s(\sigma^{-1}(i)) = s(v)$, the previous equation can be simplified to proove Lemma 1:

$$\begin{aligned} f(v) - \frac{s(v)}{2} - (f(u) + \frac{s(u)}{2}) &= p'(v) - s(v) - p'(u) + s(v) && (19) \\ &= p'(v) - p'(u) && (20) \end{aligned}$$

By applying Lemma 1, we need to prove that:

$$p(v) - p(u) \geq p(v') - p(u') \Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u')$$

We first develop $p'(v) - p'(u)$ by applying the definition of p:

$$p'(v) - p'(u) = \frac{p(v) - p_{min}}{p_{max} - p_{min}} \times C - \frac{p(u) - p_{min}}{p_{max} - p_{min}} \times C \tag{21}$$

where $C = (l - \sum_{x \in V} s(x))$, and $C \geq 0$. Then:

$$\begin{aligned} p'(v) - p'(u) &= C \left(\frac{p(v) - p(u)}{p_{max} - p_{min}} \right) && (22) \\ &= \frac{C}{p_{max} - p_{min}} \times (p(v) - p(u)) && (23) \end{aligned}$$

Since $C' = \frac{C}{p_{max} - p_{min}}$ is positive, we obtain that:

$$\begin{aligned} p(v) - p(u) \geq p(v') - p(u') &\Rightarrow C' (p(v) - p(u)) \geq C' (p(v') - p(u')) \\ p(v) - p(u) \geq p(v') - p(u') &\Rightarrow p'(v) - p'(u) \geq p'(v') - p'(u') \end{aligned}$$

□

References

1. S. Fadloun, P. Poncelet, J. Rabatel, M. Roche, and A. Sallaberry. Node Overlap Removal for 1D Graph Layout. Proceedings of the 21st International Conference Information Visualisation (IV'17), pp. 224-229, 2017.