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# Triangle packing in (sparse) tournaments: approximation and kernelization\*.

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## Abstract

Given a tournament  $\mathcal{T}$  and a positive integer  $k$ , the  $C_3$ -PACKING-T problem asks if there exists a least  $k$  (vertex-)disjoint directed 3-cycles in  $\mathcal{T}$ . This is the dual problem in tournaments of the classical minimal feedback vertex set problem. Surprisingly  $C_3$ -PACKING-T did not receive a lot of attention in the literature. We show that it does not admit a PTAS unless  $P=NP$ , and so is NP-complete, even if we restrict the considered instances to sparse tournaments, that is tournaments with a feedback arc set (FAS) being a matching. Focusing on sparse tournaments we provide a  $(1 + \frac{6}{c-1})$  approximation algorithm for sparse tournaments having a linear representation where all the backward arcs have “length” at least  $c$ . Concerning kernelization, we show that  $C_3$ -PACKING-T admits a kernel with  $\mathcal{O}(m)$  vertices, where  $m$  is the size of a given feedback arc set. In particular, we derive a  $\mathcal{O}(k)$  vertices kernel for  $C_3$ -PACKING-T when restricted to sparse instances. On the negative size, we show that  $C_3$ -PACKING-T does not admit a kernel of (total bit) size  $\mathcal{O}(k^{2-\epsilon})$  unless  $NP \subseteq \text{coNP}/\text{Poly}$ . The existence of a kernel in  $\mathcal{O}(k)$  vertices for  $C_3$ -PACKING-T remains an open question.

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## 1 Introduction and related work

### Tournament

A tournament  $\mathcal{T}$  on  $n$  vertices is an orientation of the edges of the complete undirected graph  $K_n$ . Thus, given a tournament  $\mathcal{T} = (V, A)$ , where  $V = \{v_i, i \in [n]\}$ , for each  $i, j \in [n]$ , either  $v_i v_j \in A$  or  $v_j v_i \in A$ . A tournament  $\mathcal{T}$  can alternatively be defined by an ordering  $\sigma(\mathcal{T}) = (v_1, \dots, v_n)$  of its vertices and a set of *backward arcs*  $\overleftarrow{A}_\sigma(\mathcal{T})$  (which will be denoted  $\overleftarrow{A}(\mathcal{T})$  as the considered ordering is not ambiguous), where each arc  $a \in \overleftarrow{A}(\mathcal{T})$  is of the form  $v_{i_1} v_{i_2}$  with  $i_2 < i_1$ . Indeed, given  $\sigma(\mathcal{T})$  and  $\overleftarrow{A}(\mathcal{T})$ , we can define  $V = \{v_i, i \in [n]\}$  and  $A = \overleftarrow{A}(\mathcal{T}) \cup \overrightarrow{A}(\mathcal{T})$  where  $\overrightarrow{A}(\mathcal{T}) = \{v_{i_1} v_{i_2} : (i_1 < i_2) \text{ and } v_{i_2} v_{i_1} \notin \overleftarrow{A}(\mathcal{T})\}$  is the set of forward arcs of  $\mathcal{T}$  in the given ordering  $\sigma(\mathcal{T})$ . In the following,  $(\sigma(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  is called a *linear representation* of the tournament  $\mathcal{T}$ . For a backward arc  $e = v_j v_i$  of  $\sigma(\mathcal{T})$  the *span value* of  $e$  is  $j - i - 1$ . Then  $\text{minspan}(\sigma(\mathcal{T}))$  (resp.  $\text{maxspan}(\sigma(\mathcal{T}))$ ) is simply the minimum (resp. maximum) of the span values of the backward arcs of  $\sigma(\mathcal{T})$ .

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\* This is the extended version of the corresponding ESA 2017 conference paper



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A set  $A' \subseteq A$  of arcs of  $\mathcal{T}$  is a *feedback arc set* (or *FAS*) of  $\mathcal{T}$  if every directed cycle of  $\mathcal{T}$  contains at least one arc of  $A'$ . It is clear that for any linear representation  $(\sigma(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  of  $\mathcal{T}$  the set  $\overleftarrow{A}(\mathcal{T})$  is a FAS of  $\mathcal{T}$ . A tournament is *sparse* if it admits a FAS which is a matching. We denote by  $C_3$ -PACKING-T the problem of packing the maximum number of vertex disjoint triangles in a given tournament, where a triangle is a directed 3-cycle. More formally, an input of  $C_3$ -PACKING-T is a tournament  $\mathcal{T}$ , an output is a set (called a *triangle packing*)  $S = \{t_i, i \in [|S|]\}$  where each  $t_i$  is a triangle and for any  $i \neq j$  we have  $V(t_i) \cap V(t_j) = \emptyset$ , and the objective is to maximize  $|S|$ . We denote by  $opt(\mathcal{T})$  the optimal value of  $\mathcal{T}$ . We denote by  $C_3$ -PERFECT-PACKING-T the decision problem associated to  $C_3$ -PACKING-T where an input  $\mathcal{T}$  is positive iff there is a triangle packing  $S$  such that  $V(S) = V(\mathcal{T})$  (which is called a *perfect triangle packing*).

### Related work

We refer the reader to Appendix where we recall the definitions of the problems mentioned bellow as well as the standard definitions about parameterized complexity and approximation. A first natural related problem is 3-SET-PACKING as we can reduce  $C_3$ -PACKING-T to 3-SET-PACKING by creating an hyperedge for each triangle.

**Classical complexity / approximation.** It is known that  $C_3$ -PACKING-T is polynomial if the tournament does not contain the three forbidden sub-tournaments described in [5]. From the point of view of approximability, the best approximation algorithm is the  $\frac{4}{3} + \epsilon$  approximation of [7] for 3-SET-PACKING, implying the same result for  $K_3$ -PACKING and  $C_3$ -PACKING-T. Concerning negative results, it is known [10] that even  $K_3$ -PACKING is MAX SNP-hard on graphs with maximum degree four. We can also mention the related "dual" problem FAST and FVST that received a lot of attention with for example the NP-hardness and PTAS for FAS in [6] and [13] respectively, and the  $\frac{7}{3}$  approximation and inapproximability results for FVST in [14].

**Kernelization.** We precise that the implicitly considered parameter here is the size of the solution. There is a  $\mathcal{O}(k^2)$  vertex kernel for  $K_3$ -PACKING in [15], and even a  $\mathcal{O}(k^2)$  vertex kernel for 3-SET-PACKING in [1], which is obtained by only removing vertices of the ground set. This remark is important as it directly implies a  $\mathcal{O}(k^2)$  vertex kernel for  $C_3$ -PACKING-T (see Section 4). Let us now turn to negative results. There is a whole line of research dedicated to finding lower bounds on the size of polynomial kernels. The main tool involved in these bounds is the weak composition introduced in [11] (whose definition is recalled in Appendix). Weak composition allowed several almost tight lower bounds, with for example the  $\mathcal{O}(k^{d-\epsilon})$  for  $d$ -SET-PACKING and  $\mathcal{O}(k^{d-4-\epsilon})$  for  $K_d$ -PACKING of [11]. These results were improved in [8] to  $\mathcal{O}(k^{d-\epsilon})$  for PERFECT  $d$ -SET-PACKING,  $\mathcal{O}(k^{d-1-\epsilon})$  for  $K_d$ -PACKING, and leading to  $\mathcal{O}(k^{2-\epsilon})$  for PERFECT  $K_3$ -PACKING. Notice that negative results for the "perfect" version of problems (where parameter  $k = \frac{n}{d}$ , where  $d$  is the number of vertices of the packed structure) are stronger than for the classical version where  $k$  is arbitrary. Kernel lower bound for these "perfect" versions is sometimes referred as *sparsification lower bounds*.

### Our contributions

Our objective is to study the approximability and kernelization of  $C_3$ -PACKING-T. On the approximation side, a natural question is a possible improvement of the  $\frac{4}{3} + \epsilon$  approximation implied by 3-SET-PACKING. We show that, unlike FAST,  $C_3$ -PACKING-T does not admit

a PTAS unless  $P=NP$ , even if the tournament is sparse. We point out that, surprisingly, we were not able to find any reference establishing a negative result for  $C_3$ -PACKING-T, even for the NP-hardness. As these results show that there is not much room for improving the approximation ratio, we focus on sparse tournaments and followed a different approach by looking for a condition that would allow ratio arbitrarily close to 1. In that spirit, we provide a  $(1 + \frac{6}{c-1})$  approximation algorithm for sparse tournaments having a linear representation with  $\text{minspan}$  at least  $c$ . Concerning kernelization, we complete the panorama of sparsification lower bounds of [12] by proving that  $C_3$ -PERFECT-PACKING-T does not admit a kernel of (total bit) size  $\mathcal{O}(n^{2-\epsilon})$  unless  $NP \subseteq \text{coNP}/\text{Poly}$ . This implies that  $C_3$ -PACKING-T does not admit a kernel of (total bit) size  $\mathcal{O}(k^{2-\epsilon})$  unless  $NP \subseteq \text{coNP}/\text{Poly}$ . We also prove that  $C_3$ -PACKING-T admits a kernel of  $\mathcal{O}(m)$  vertices, where  $m$  is the size of a given FAS of the instance, and that  $C_3$ -PACKING-T restricted to sparse instances has a kernel in  $\mathcal{O}(k)$  vertices (and so of total size bit  $\mathcal{O}(k \log(k))$ ). The existence of a kernel in  $\mathcal{O}(k)$  vertices for the general  $C_3$ -PACKING-T remains our main open question.

## 2 Specific notations and observations

Given a linear representation  $(\sigma(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  of a tournament  $\mathcal{T}$ , a triangle  $t$  in  $\mathcal{T}$  is a triple  $t = (v_{i_1}, v_{i_2}, v_{i_3})$  with  $i_l < i_{l+1}$  such that either  $v_{i_3}v_{i_1} \in \overleftarrow{A}(\mathcal{T})$ ,  $v_{i_3}v_{i_2} \notin \overleftarrow{A}(\mathcal{T})$  and  $v_{i_2}v_{i_1} \notin \overleftarrow{A}(\mathcal{T})$  (in this case we call  $t$  a *triangle with backward arc*  $v_{i_3}v_{i_1}$ ), or  $v_{i_3}v_{i_1} \notin \overleftarrow{A}(\mathcal{T})$ ,  $v_{i_3}v_{i_2} \in \overleftarrow{A}(\mathcal{T})$  and  $v_{i_2}v_{i_1} \in \overleftarrow{A}(\mathcal{T})$  (in this case we call  $t$  a *triangle with two backward arcs*  $v_{i_3}v_{i_2}$  and  $v_{i_2}v_{i_1}$ ).

Given two tournaments  $\mathcal{T}_1, \mathcal{T}_2$  defined by  $\sigma(\mathcal{T}_1)$  and  $\overleftarrow{A}(\mathcal{T}_1)$  we denote by  $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2$  the tournament called the concatenation of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , where  $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2)$  is the concatenation of the two sequences, and  $\overleftarrow{A}(\mathcal{T}) = \overleftarrow{A}(\mathcal{T}_1) \cup \overleftarrow{A}(\mathcal{T}_2)$ . Given a tournament  $\mathcal{T}$  and a subset of vertices  $X$ , we denote by  $\mathcal{T} \setminus X$  the tournament  $\mathcal{T}[V(\mathcal{T}) \setminus X]$  induced by vertices  $V(\mathcal{T}) \setminus X$ , and we call this operation *removing  $X$  from  $\mathcal{T}$* . Given an arc  $a = uv$  we define  $h(a) = v$  as the head of  $a$  and  $t(a) = u$  as the tail of  $a$ . Given a linear representation  $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  and an arc  $a \in \overleftarrow{A}(\mathcal{T})$ , we define  $s(a) = \{v : h(a) < v < t(a)\}$  as the *span* of  $a$ . Notice that the span value of  $a$  is then exactly  $|s(a)|$ .

Given a linear representation  $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  and a vertex  $v \in V(\mathcal{T})$ , we define the degree of  $v$  by  $d(v) = (a, b)$ , where  $a = |\{vu \in \overleftarrow{A}(\mathcal{T}) : u < v\}|$  is called the *left degree* of  $v$  and  $b = |\{uv \in \overleftarrow{A}(\mathcal{T}) : u > v\}|$  is called the *right degree* of  $v$ . We also define  $V_{(a,b)} = \{v \in V(\mathcal{T}) | d(v) = (a, b)\}$ . Given a set of pairwise distinct pairs  $D$ , we denote by  $C_3$ -PACKING-T <sup>$D$</sup>  the problem  $C_3$ -PACKING-T restricted to tournaments such that there exists a linear representation where  $d(v) \in D$  for all  $v$ . Notice that when  $D_M = \{(0, 1), (1, 0), (0, 0)\}$ , instances of  $C_3$ -PACKING-T <sup>$D_M$</sup>  are the sparse tournaments.

Finally let us point out that it is easy to decide in polynomial time if a tournament is sparse or not, and if so, to give a linear representation whose FAS is a matching. The corresponding algorithm is detailed in Appendix in Lemma 23. Thus, in the following, when considering a sparse tournament we will assume that a linear ordering of it where backward arcs form a matching is also given.

## 3 Approximation for sparse tournaments

### 3.1 APX-hardness for sparse tournaments

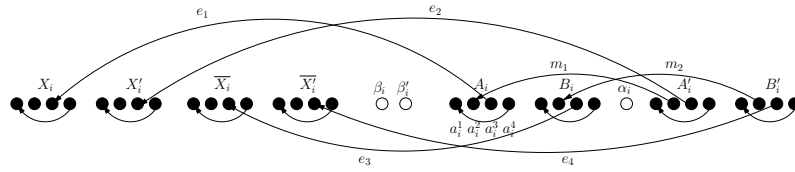
In this subsection we prove that  $C_3$ -PACKING-T <sup>$D_M$</sup>  is APX-hard by providing a  $L$ -reduction (see Definition 17 in appendix) from Max 2-SAT(3), which is known to be APX-hard [2, 3].

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Recall that in the MAX 2-SAT(3) problem where each clause contains exactly 2 variables and each variable appears in at most 3 clauses (and at most twice positively and once negatively).

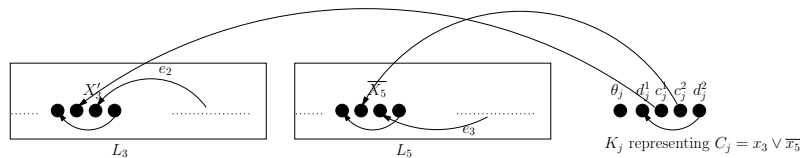
**Definition of the reduction** Let  $\mathcal{F}$  be an instance of MAX 2-SAT(3). In the following, we will denote by  $n$  the number of variables in  $\mathcal{F}$  and  $m$  the number of clauses. Let  $\{x_i, 1 \in [n]\}$  be the set of variables of  $\mathcal{F}$  and  $\{C_j, j \in [m]\}$  its set of clauses.

We now define a reduction  $f$  which maps an instance  $\mathcal{F}$  of MAX 2-SAT(3) to an instance  $\mathcal{T}$  of  $C_3$ -PACKING- $T^{DM}$ . For each variable  $x_i$  with  $i \in [n]$ , we create a tournament  $L_i$  as follows and we call it *variable gadget*. We refer the reader to Figure 1 where an example of variable gadget is depicted. Let  $\sigma(L_i) = (X_i, X'_i, \overline{X}_i, \overline{X}'_i, \{\beta_i\}, \{\beta'_i\}, A_i, B_i, \{\alpha_i\}, A'_i, B'_i)$ . We define  $C = \{X_i, X'_i, \overline{X}_i, \overline{X}'_i, A_i, B_i, A'_i, B'_i\}$ . All sets of  $C$  have size 4. We denote  $X_i = (x_i^1, x_i^2, x_i^3, x_i^4)$ , and we extend the notation in a straightforward manner to the other others sets of  $C$ . Let us now define  $\overleftarrow{A}(L_i)$ . For each set of  $C$ , we add a backward arc whose head is the first element and the tail is the last element (for example for  $X_i$  we add the arc  $x_i^4 x_i^1$ ). Then, we add to  $\overleftarrow{A}(L_i)$  the set  $\{e_1, e_2, e_3, e_4\}$  where  $e_1 = x_i^3 a_i^3$ ,  $e_2 = x_i'^3 a_i'^3$ ,  $e_3 = \overline{x}_i^3 b_i^3$ ,  $e_4 = \overline{x}_i'^3 b_i'^3$  and the set  $\{m_1, m_2\}$  where  $m_1 = a_i'^2 a_i^2$ ,  $m_2 = b_i'^2 b_i^2$  called the two *medium arcs* of the variable gadget. This completes the description of tournament  $L_i$ . Let  $L = L_1 \dots L_n$  be the concatenation of the  $L_i$ .



■ **Figure 1** Example of a variable gadget  $L_i$ .

For each clause  $C_j$  with  $j \in [1, m]$ , we create a tournament  $K_j$  with ordering  $\sigma(K_j) = (\theta_j, d_j^1, c_j^1, c_j^2, d_j^2)$  and  $\overleftarrow{A}(K_j) = \{d_j^2 d_j^1\}$ . We also define  $K = K_1 \dots K_m$ . Let us now define  $\mathcal{T} = LK$ . We add to  $\overleftarrow{A}(\mathcal{T})$  the following backward arcs from  $V(K)$  to  $V(L)$ . If  $C_j = l_{i_1} \vee l_{i_2}$  is a clause in  $\mathcal{F}$  then we add the arcs  $c_j^1 v_{i_1}, c_j^2 v_{i_2}$  where  $v_{i_c}$  is the vertex in  $\{x_{i_c}^2, x_{i_c}'^2, \overline{x}_{i_c}^2\}$  corresponding to  $l_{i_c}$ : if  $l_{i_c}$  is a positive occurrence of variable  $i_c$  we chose  $v_{i_c} \in \{x_{i_c}^2, x_{i_c}'^2\}$ , otherwise we chose  $v_{i_c} = \overline{x}_{i_c}^2$ . Moreover, we chose vertices  $v_{i_c}$  in such a way that for any  $i \in [n]$ , for each  $v \in \{x_i^2, x_i'^2, \overline{x}_i^2\}$  there exists a unique arc  $a \in \overleftarrow{A}(\mathcal{T})$  such that  $h(a) = v$ . This is always possible as each variable has at most two positive occurrences and one negative occurrence. Thus,  $x_i^2$  represent the first positive occurrence of variable  $i$ , and  $x_i'^2$  the second one. We refer the reader to Figure 2 where an example of the connection between variable and clause gadget is depicted.



■ **Figure 2** Example showing how a clause gadget is attached to variable gadgets.

Notice that vertices of  $\overline{X'_i}$  are never linked to the clauses gadget. However, we need this set to keep the variable gadget symmetric so that setting  $x_i$  to true or false leads to the same number of triangles inside  $L_i$ . This completes the description of  $\mathcal{T}$ . Notice that the degree of any vertex is in  $\{(0, 1), (1, 0), (0, 0)\}$ , and thus  $\mathcal{T}$  is an instance of  $C_3$ -PACKING- $T^{DM}$ .

Let us now distinguish three different types of triangles in  $\mathcal{T}$ . A triangle  $t = (v_1, v_2, v_3)$  of  $\mathcal{T}$  is called an *outer* triangle iff  $\exists j \in [m]$  such that  $v_2 = \theta_j$  and  $v_3 = c_j^l$  (implying that  $v_1 \in V(L)$ ), *variable inner* iff  $\exists i \in [n]$  such that  $V(t) \subseteq V(L_i)$ , and *clause inner* iff  $\exists j \in [m]$  such that  $V(t) \subseteq V(K_j)$ . Notice that a triangle  $t = (v_1, v_2, v_3)$  of  $\mathcal{T}$  which is neither outer, variable or clause inner has necessarily  $v_3 = c_j^l$  for some  $j$ , and  $v_2 \neq \theta_j$  ( $v_2$  could be in  $V(L)$  or  $V(K)$ ). In the following definition, for any  $Y \in C$  (where  $C = \{X_i, X'_i, \overline{X_i}, \overline{X'_i}, A_i, B_i, A'_i, B'_i\}$ ) with  $Y = (y^1, y^2, y^3, y^4)$ , we define  $t_Y^2 = (y^1, y^2, y^4)$  and  $t_Y^3 = (y^1, y^3, y^4)$ . For example,  $t_{X'_i}^2 = (x_i^1, x_i^2, x_i^4)$ . For any  $i \in [n]$ , we define  $P_i$  and  $\overline{P}_i$ , two sets of vertex disjoint variable inner triangles of  $V(L_i)$ , by:

- $P_i = \{t_{X_i}^3, t_{X'_i}^3, t_{\overline{X_i}}^2, t_{\overline{X'_i}}^2, t_{A_i}^3, t_{B_i}^3, t_{A'_i}^3, t_{B'_i}^3, (h(e_3), \beta_i, t(e_3)), (h(e_4), \beta'_i, t(e_4)), (h(m_1), \alpha_i, t(m_1))\}$
- $\overline{P}_i = \{t_{X_i}^2, t_{X'_i}^2, t_{\overline{X_i}}^3, t_{\overline{X'_i}}^3, t_{A_i}^2, t_{B_i}^2, t_{A'_i}^2, t_{B'_i}^2, (h(e_1), \beta_i, t(e_1)), (h(e_2), \beta'_i, t(e_2)), (h(m_2), \alpha_i, t(m_2))\}$

Notice that  $P_i$  (resp.  $\overline{P}_i$ ) uses all vertices of  $L_i$  except  $\{x_i^2, x_i^2\}$  (resp.  $\{\overline{x_i^2}, \overline{x_i^2}\}$ ). For any  $j \in [m]$ , and  $x \in [2]$  we define the set of clause inner triangle of  $K_j$ , that is  $Q_j^x = \{d_j^1, c_j^x, d_j^2\}$ .

Informally, setting variable  $x_i$  to true corresponds to create the 11 triangles of  $P_i$  in  $L_i$  (as leaving vertices  $\{x_i^2, x_i^2\}$  available allows to create outer triangles corresponding to satisfied clauses), and setting it to false corresponds to create the 11 triangles of  $\overline{P}_i$ . Satisfying a clause  $j$  using its  $x^{th}$  literal (represented by a vertex  $v \in V(L)$ ) corresponds to create triangle in  $Q_j^{3-x}$  as it leaves  $c_j^x$  available to create the triangle  $(v, \theta_j, c_j^x)$ . Our final objective (in Lemma 4) is to prove that satisfying  $k$  clauses is equivalent to find  $11n + m + k$  vertex disjoint triangles.

**Restructuration lemmas** Given a solution  $S$ , let  $I_i^L = \{t \in S : V(t) \subseteq V(L_i)\}$ ,  $I_j^K = \{t \in S : V(t) \subseteq V(K_j)\}$ ,  $I^L = \cup_{i \in [n]} I_i^L$  be the set of variable inner triangles of  $S$ ,  $I^K = \cup_{j \in [m]} I_j^K$  be the set of clause inner triangles of  $S$ , and  $O = \{t \in S \mid t \text{ is an outer triangle}\}$  be the set of outer triangles of  $S$ . Notice that *a priori*  $I^L, I^K, O$  does not necessarily form a partition of  $S$ . However, we will show in the next lemmas how to restructure  $S$  such that  $I^L, I^K, O$  becomes a partition.

► **Lemma 1.** *For any  $S$  we can compute in polynomial time a solution  $S' = \{t'_l, l \in [k]\}$  such that  $|S'| \geq |S|$  and for all  $j \in [m]$  there exists  $x \in [2]$  such that  $I_j'^K = Q_j^x$  and*

- *either  $S'$  does not use any other vertex of  $K_j$  ( $V(S') \cap V(K_j) = V(Q_j^x)$ )*
- *either  $S'$  contains an outer triangle  $t'_l = (v, \theta_j, c_j^{3-x})$  with  $v \in V(L)$  (implying  $V(S') \cap V(K_j) = V(K_j)$ )*

**Proof.** Consider a solution  $S = \{t_l, l \in [k]\}$ . Let us suppose that  $S$  does not verify the desired property. We say that  $j \in [m]$  satisfies  $(\star)$  iff there exists  $x \in [2]$  such that  $I_j^K = Q_j^x$  and either  $S$  does not use any other vertex of  $K_j$ , or  $S$  contains an outer triangle  $t_l = (v, \theta_j, c_j^{3-x})$  with  $v \in V(L)$ .

Let us restructure  $S$  to increase the number of  $j$  satisfying  $(\star)$ , which will be sufficient to prove the lemma. Consider the largest  $j \in [m]$  which does not satisfy  $(\star)$ . Let  $c = |I_j^K|$ . Notice that the only possible triangle of  $I_j^K$  contains  $a = d_j^2 d_j^1$ , implying  $c \leq 1$ .

If  $c = 1$ , let  $t \in I_j^K$  and  $v_0 = \{c_j^1, c_j^2\} \setminus V(t)$ . If  $v_0 \notin V(S)$ , then let us prove that  $\theta_j \notin V(S)$ . Indeed, by contradiction if  $\theta_j \in V(S)$ , let  $t' \in S$  such that  $\theta_j \in V(t')$ . As  $d(\theta_j) = (0, 0)$  we necessarily have  $t' = (u, \theta_j, w)$  with  $w = c_j^{x'}$  with  $j' \geq j$ , which contradicts the maximality



of  $j$ . Otherwise ( $v_0 \in V(S)$ ), then denoting by  $t'$  the triangle of  $S$  which contains  $v_0$  we must have  $t' = (u, v, v_0)$ . Indeed, we cannot have (for some  $u', v'$ )  $t' = (v_0, u', v')$  as there is no backward arc  $a$  with  $h(a) = v_0$  and we cannot have either  $t' = (u', v_0, v')$  as this would imply  $v' = c_j^{x'}$  for  $j' > j$  and again contradict the definition of  $j$ . As, again, by maximality of  $j$  we get  $\theta_j \notin V(S)$  (and since  $u\theta_j$  and  $\theta_j v_0$  are forward arcs), we can replace  $t'$  by the triangle  $(u, \theta_j, v_0)$  which is disjoint to the other triangles of  $S$ .

If  $c = 0$ . Notice first that by maximality of  $j$ ,  $d_j^2 \notin V(S)$  as  $d_j^2$  could only be used in a triangle  $t = (v, d_j^2, c_j^{x'})$  with  $j' > j$ . Let  $Z = V(S) \cap \{c_j^1, c_j^2\}$ . If  $|Z| = 0$ , then by maximality of  $j$  we get  $d_j^1 \notin V(S)$  and  $\theta_j \notin V(S)$ , and thus we add to  $S$  triangle  $(d_j^1, c_j^1, d_j^2)$ . If  $|Z| = 1$ , let  $c_j^x \in Z$  and  $t \in S$  such that  $c_j^x \in V(t)$ . By maximality of  $j$  we necessarily have  $t = (u, v, c_j^x)$  for some  $u, v$ . If  $v \neq \theta_j$  then by maximality of  $j$  we have  $\theta_j \notin V(S)$ , and thus we swap  $v$  and  $\theta_j$  in  $t$  and now suppose that  $\theta_j \in V(t)$ . This implies that  $d_j^1 \notin V(S)$  (before the swap we could have had  $v = d_j^1$ , but now by maximality of  $j$  we know that  $d_j^1$  is unused), and we add  $(d_j^1, c_j^{3-x}, d_j^2)$  to  $S$ . It only remains now case where  $|Z| = 2$ . If there exists  $t \in S$  with  $Z \subseteq V(t)$ , then  $t = (u, c_j^1, c_j^2)$ . Using the same arguments as above we get that  $\{\theta_j, d_j^1\} \cap V(S) = \emptyset$ , and thus we swap  $c_j^1$  by  $\theta_j$  in  $t$  and add  $(d_j^1, c_j^1, d_j^2)$  to  $S$ . Otherwise, let  $t_x \in S$  such that  $c_j^x \in V(t_x)$  for  $x \in [2]$ . This implies that  $t_x = (u_x, v_x, c_j^x)$ . If  $\theta_j \notin V(t_1) \cup V(t_2)$  then  $\theta_j \notin V(S)$  and we swap  $v_1$  with  $\theta_j$ . Therefore, from now on we can suppose that  $\theta_j \in V(t_x)$  for  $x \in [2]$ . Then, if  $d_j^1 \notin V(t_{3-x})$  then  $d_j^1 \notin V(S)$  and thus we swap  $v_{3-x}$  with  $d_j^1$  and we now assume that  $d_j^1 \in V(t_{3-x})$ . Finally, we remove  $t_{3-x}$  from  $S$  and add instead  $(d_j^1, c_j^{3-x}, d_j^2)$ .  $\blacktriangleleft$

► **Corollary 2.** *For any  $S$  we can compute in polynomial time a solution  $S'$  such that  $|S'| \geq |S|$ , and  $S'$  only contains outer, variable inner, and clause inner triangles. Indeed, in the solution  $S'$  of Lemma 1, given any  $t \in S'$ , either  $V(t)$  intersects  $V(K_j)$  for some  $j$  and then  $t$  is an outer or a clause inner triangle, or  $V(t) \subseteq V(L_i)$  for  $i \in [n]$  as there is no backward arc  $uv$  with  $u \in V(L_{i_1})$  and  $v \in V(L_{i_2})$  with  $i_1 \neq i_2$ .*

► **Lemma 3.** *For any  $S$  we can compute in polynomial time a solution  $S'$  such that  $|S'| \geq |S|$ ,  $S'$  satisfies Lemma 1, and for every  $i \in [n]$ ,  $I_i'^L = P_i$  or  $I_i'^L = \overline{P}_i$ .*

**Proof.** Let  $S_0$  be an arbitrary solution, and  $S$  be the solution obtained from  $S_0$  after applying Lemma 1. By Corollary 2, we partition  $S$  into  $S = I^L \cup I^K \cup O$ . Let us say that  $i \in [n]$  satisfies  $(\star)$  if  $I_i^L = P_i$  or  $I_i^L = \overline{P}_i$ . Let us suppose that  $S$  does not verify the desired property, and show how to restructure  $S$  to increase the number of  $i$  satisfying  $(\star)$  while still satisfying Lemma 1, which will prove the lemma.

Let  $Lft_i = X_i \cup X'_i \cup \overline{X}_i \cup \overline{X}'_i$  and  $Rgt_i = A_i \cup B_i \cup \{\alpha_i\} \cup A'_i \cup B'_i$  be two subset of vertices of  $V(L_i)$ . Given any solution  $\tilde{S}$  satisfying Lemma 1, we define the following sets. Let  $\tilde{S}^{Lft_i} = \{t \in \tilde{I}_i^L : V(t) \subseteq Lft_i\}$ ,  $\tilde{S}^{Rgt_i} = \{t \in \tilde{I}_i^L : V(t) \subseteq Rgt_i\}$ , and  $\tilde{S}^{Lft_i Rgt_i} = \{t \in \tilde{I}_i^L : V(t) \cap Lft_i \neq \emptyset \text{ and } V(t) \cap Rgt_i \neq \emptyset\}$ . Observe that these three sets define a partition of  $\tilde{I}_i^L$ , and that triangles of  $\tilde{S}^{Lft_i}$  are even included in  $W$  with  $W \in \{X_i, X'_i, \overline{X}_i, \overline{X}'_i\}$ . Let  $\tilde{S}^{O_i} = \{t \in \tilde{O} : V(t) \cap V(L_i) \neq \emptyset\}$  be the set of outer triangles of  $\tilde{S}$  intersecting  $L_i$ . We also define  $g_i(\tilde{S}) = (|\tilde{S}^{Lft_i}|, |\tilde{S}^{Lft_i Rgt_i}|, |\tilde{S}^{Rgt_i}|, |\tilde{S}^{O_i}|)$  and  $h_i(\tilde{S}) = |\tilde{S}^{Lft_i}| + |\tilde{S}^{Lft_i Rgt_i}| + |\tilde{S}^{Rgt_i}| + |\tilde{S}^{O_i}| = |\tilde{I}_i^L \cup \tilde{S}^{O_i}|$ .

Our objective is to restructure  $S$  into a solution  $S'$  with  $S' = (S \setminus (I_i^L \cup S^{O_i})) \cup (I_i'^L \cup S'^{O_i})$ . We will define  $I_i'^L$  and  $S'^{O_i}$  verifying the following properties  $(\Delta)$ :

- $\Delta_1$  :  $I_i'^L = P_i$  or  $I_i'^L = \overline{P}_i$ ,
- $\Delta_2$  :  $S'^{O_i} \subseteq S^{O_i}$ ,
- $\Delta_3$  :  $|(I_i'^L \cup S'^{O_i})| \geq |(I_i^L \cup S^{O_i})|$  (which is equivalent to  $h_i(S') \geq h_i(S)$ ),
- $\Delta_4$  : triangles of  $I_i'^L \cup S'^{O_i}$  are vertex disjoint.

Notice that  $\Delta_2$  and  $\Delta_4$  imply that all triangles of  $S'$  are still vertex disjoint. Indeed, as  $S$  satisfies Lemma 1, the only triangles of  $S$  intersecting  $L_i$  are  $I_i^L \cup S^{O_i}$ , and thus replacing them with  $I_i^L \cup S'^{O_i}$  satisfying the above property implies that all triangles of  $S'$  are vertex disjoint. Moreover,  $S'$  will still satisfy Lemma 1 even with  $S'^{O_i} \subseteq S^{O_i}$  as removing outer triangles cannot violate property of Lemma 1. Finally  $\Delta_3$  implies that  $|S'| \geq |S|$ . Thus, defining  $I_i^L$  and  $S'^{O_i}$  satisfying  $(\Delta)$  will be sufficient to prove the lemma. Let us now state some useful properties.

$$p_1 : |S^{Lft_i}| \leq 4$$

$$p_2 : |S^{Lft_i Rgt_i}| \leq 4 \text{ as for any } t \in S^{Lft_i Rgt_i} \text{ there exists } l \in [4] \text{ such that } V(t) \supseteq V(e_l).$$

$p_3$  :  $|S^{Rgt_i}| \leq 5$  (as  $|V(S^{Rgt_i})| = 17$ ). Let  $Z = V(S^{Lft_i Rgt_i}) \cap Rgt_i$ . Let us also prove that if  $Z \supseteq \{a_i^3, b_i^3\}$ ,  $Z \supseteq \{a_i'^3, b_i'^3\}$ ,  $Z \supseteq \{a_i^3, b_i'^3\}$  or  $Z \supseteq \{a_i'^3, b_i^3\}$  then  $|S^{Rgt_i}| \leq 4$ . For any  $W \in \{A_i, B_i, A_i', B_i'\}$ , let  $s_W$  be the unique arc  $a$  of  $\mathcal{T}$  such that  $V(a) \subseteq W$  and let  $m_W$  be the unique medium arc  $a$  such that  $V(a) \cap W \neq \emptyset$ . Let us call the  $\{s_W\}$  the four small arcs of the tournament induced by  $Rgt_i$ . Let  $\overleftarrow{A}(S^{Rgt_i}) = \{a \in \overleftarrow{A}(L_i) : \exists t \in S^{Rgt_i} \text{ such that } V(a) \subseteq V(t)\}$  be the set of backward arcs used by  $S^{Rgt_i}$ . Observe that arcs of  $\overleftarrow{A}(S^{Rgt_i})$  are small or medium arcs. Let us bound  $|\overleftarrow{A}(S^{Rgt_i})| = |S^{Rgt_i}|$ . Notice that for any  $W \in \{A_i, B_i, A_i', B_i'\}$ ,  $W \cap Z \neq \emptyset$  implies that  $\overleftarrow{A}(S^{Rgt_i})$  cannot contain both  $s_W$  and  $m_W$ . If  $S^{Rgt_i}$  contains the 4 small arcs then by previous remark  $S^{Rgt_i}$  cannot contain any medium arc, and thus  $|S^{Rgt_i}| \leq 4$ . If  $S^{Rgt_i}$  contains 3 small arcs then it can only contain one medium arc, implying  $|S^{Rgt_i}| \leq 4$ . Obviously, if  $|S^{Rgt_i}|$  contains 2 or less small arcs then  $|S^{Rgt_i}| \leq 4$ .

$p_4$  : property  $p_3$  implies that if  $|S^{Lft_i Rgt_i}| \geq 3$ , or if  $|S^{Lft_i Rgt_i}| = 2$  and triangles of  $S^{Lft_i Rgt_i}$  contain  $\{e_1, e_3\}$ ,  $\{e_1, e_4\}$ ,  $\{e_2, e_3\}$  or  $\{e_2, e_4\}$ , then  $|S^{Rgt_i}| \leq 4$  (where triangles of  $S^{Lft_i Rgt_i}$  contains  $\{e_i, e_j\}$  means that there exist  $t_1, t_2$  in  $S^{Lft_i Rgt_i}$  such that  $V(t_1) \supseteq V(e_i)$  and  $V(t_2) \supseteq V(e_j)$ ).

$p_5$  :  $|S^{O_i}| \leq 3$ . Moreover, if  $|S^{Lft_i}| = 4$  then  $|S^{O_i}| \leq 4 - |S^{Lft_i Rgt_i}|$ , and if  $|S^{Lft_i}| = 3$  and  $|S^{Lft_i Rgt_i}| = 4$  then  $|S^{O_i}| \leq 1$ . The last two inequalities come from the fact that for any  $W \in \{X_i, X_i', \overline{X}_i, \overline{X}_i'\}$ , we cannot have both  $t_1 \in S^{O_i}$ ,  $t_2 \in S^{Lft_i Rgt_i}$  and  $t_3 \in S^{Lft_i}$  with  $V(t_i) \cap W \neq \emptyset$ .

Notice that if a solution  $S'$  satisfies  $I_i^L = P_i$  or  $I_i^L = \overline{P}_i$  then  $g_i(S') = (4, 2, 5, z)$  where  $z \in [2]$ , and  $h_i(S') = 11 + z$ . In the following we write  $(u_1^1, u_2^1, u_3^1, u_4^1) \leq (u_1^2, u_2^2, u_3^2, u_4^2)$  iff  $u_i^1 \leq u_i^2$  for any  $i \in [4]$ . Let us describe informally the following argument which will be used several times. Let  $z = |S^{O_i}|$ . If  $z \leq 1$  or if  $z = 2$  but the two corresponding outer triangles do not use one vertex in  $X_i \cup X_i'$  and one vertex in  $\overline{X}_i$ , then we will be able to "save" all these outer triangles (while creating the optimal number of variable inner triangles in  $L_i$ ), meaning that  $S'^{O_i} = S^{O_i}$ , as either  $P_i$  or  $\overline{P}_i$  will leave vertices of  $S^{O_i} \cap Lft_i$  available for outer triangles. Let us proceed by case analysis according to the value  $|S^{Lft_i Rgt_i}|$ . Remember that  $|S^{Lft_i Rgt_i}| \leq 4$  according to  $p_2$ .

Case 1:  $|S^{Lft_i Rgt_i}| \leq 1$ . According to  $p_1, p_3$  we get  $g_i(S) \leq (4, 1, 5, z)$  where  $z \in [3]$ . In this case,  $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x}_i^2\}$  and  $I_i^L = P_i$  verify  $(\Delta)$ . In particular, we have  $h_i(S') \geq h_i(S)$  as  $g_i(S') \geq (4, 2, 5, z - 1)$ .

Case 2:  $|S^{Lft_i Rgt_i}| = 2$ . Let  $g_i(S) = (x, 2, y, z)$ . If  $x \leq 3$ , then  $g_i(S) \leq (3, 2, 5, z)$  by  $p_3$  and we set  $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x}_i^2\}$  and  $I_i^L = P_i$ . This satisfies  $(\Delta)$  as in particular we have  $h_i(S') \geq h_i(S)$  as  $g_i(S') \geq (4, 2, 5, z - 1)$ . Let us now turn to case where  $x = 4$ . Let  $S^{Lft_i Rgt_i} = \{t_1, t_2\}$ . Let us first suppose that triangles of  $S^{Lft_i Rgt_i}$  contain  $\{e_i, e_j\}$  with  $\{e_i, e_j\} \in \{\{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}\}$ . By  $p_4$  we get  $y \leq 4$ , implying  $g_i(S) \leq (4, 2, 4, z)$ . In this case,  $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x}_i^2\}$  and  $I_i^L = P_i$  verify  $(\Delta)$ .



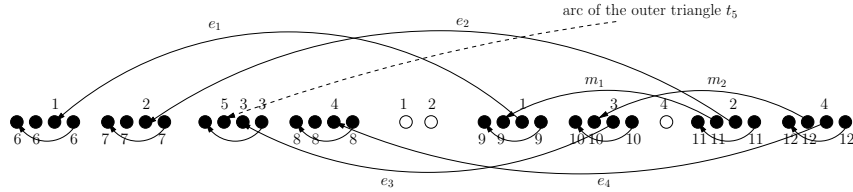
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In particular, we have  $h_i(S') \geq h_i(S)$  as  $g_i(S') = (4, 2, 5, z - 1)$ . Let us suppose now that  $t_1$  contains  $e_1$  and  $t_2$  contains  $e_2$  (case (2a)), or  $t_1$  contains  $e_3$  and  $t_2$  contains  $e_4$  (case (2b)). In both cases we have  $g_i(S) \leq (4, 2, 5, z)$  where  $z \in [2]$  by  $p_5$ . More precisely,  $p_5$  implies that  $\{W \in \{X_i, X'_i, \overline{X}_i, \overline{X}'_i\} : W \cap V(S^{O_i})\} \neq \emptyset$  is included in  $\{X, X'_i\}$  (case 2b) or in  $\overline{X}_i$  (case 2a). Thus, in case (2a) we define  $S'^{O_i} = S^{O_i}$  and  $I_i'^L = \overline{P}_i$ . In case (2b) we define  $S'^{O_i} = S^{O_i}$  and  $I_i'^L = P_i$ . In both cases these sets verify  $(\Delta)$  as in particular  $g_i(S') = (4, 2, 5, z)$ .

Case 3:  $|S^{Lft_iRgt_i}| = 3$ . In this case  $g_i(S) \leq (x, 3, 4, z)$  by  $p_4$ . If  $x \leq 3$ , the sets  $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$  and  $I_i'^L = P_i$  verify  $(\Delta)$ . In particular, we have  $h_i(S') \geq h_i(S)$  as  $g_i(S') \geq (4, 2, 5, z - 1)$ . If  $x = 4$  then  $z \leq 1$  by  $p_5$ . Thus, we define  $I_i'^L = P_i$  if  $V(S^{O_i}) \cap (X_i \cup X'_i) \neq \emptyset$ , and  $I_i'^L = \overline{P}_i$  otherwise, and  $S'^{O_i} = S^{O_i}$ . These sets satisfy  $(\Delta)$  as in particular  $g_i(S') = (4, 2, 5, z)$ .

Case 4:  $|S^{Lft_iRgt_i}| = 4$ . Let  $g_i(S) = (x, 4, y, z)$ . If  $x = 4$  then  $z \leq 0$  by  $p_5$  and  $y \leq 3$  as  $x + 4 + y \leq \frac{|V(L_i)|}{3}$ .

Thus, we set  $S'^{O_i} = S^{O_i} = \emptyset$ ,  $I_i'^L = P_i$  (which is arbitrary in this case), and we have property  $(\Delta)$  as  $g_i(S') \geq (4, 2, 5, 0)$ . If  $x = 3$  (this case is depicted Figure 3) then  $y \leq 4$  by  $p_3$  and  $z \leq 1$  by  $p_5$ , implying  $g_i(S) = (3, 4, 4, z)$ . Thus, we define  $I_i'^L = P_i$  if  $V(S^{O_i}) \cap (X_i \cup X'_i) \neq \emptyset$ , and  $I_i'^L = \overline{P}_i$  otherwise, and  $S'^{O_i} = S^{O_i}$ . These sets satisfy  $(\Delta)$  as in particular  $g_i(S') = (4, 2, 5, z)$ . Finally, if  $x \leq 2$  then  $g_i(S) \leq (2, 4, 4, z)$  by  $p_3$ . In this case,  $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$  and  $I_i'^L = P_i$  verify  $(\Delta)$ . In particular, we have  $h_i(S') \geq h_i(S)$  as  $g_i(S') \geq (4, 2, 5, z - 1)$ .



■ **Figure 3** Example showing a "bad shaped" solution of case 4 with  $g_i(S) = (3, 4, 4, 1)$ . We have  $S^{Lft_iRgt_i} = \{t_1, t_2, t_3, t_4\}$ ,  $S^{O_i} = \{t_5\}$ ,  $S^{Lft_i} = \{t_6, t_7, t_8\}$  and  $S^{Rgt_i} = \{t_9, t_{10}, t_{11}, t_{12}\}$ . The three vertices of triangle  $t_i$  are annotated with label  $l$ .



**Proof of the L-reduction** We are now ready to prove the main lemma (recall that  $f$  is the reduction from MAX 2-SAT(3) to  $C_3$ -PACKING- $T^{DM}$  described in Section 3.1), and also the main theorem of the section.

► **Lemma 4.** *Let  $\mathcal{F}$  be an instance of MAX 2-SAT(3). For any  $k$ , there exists an assignment  $a$  of  $\mathcal{F}$  satisfying at least  $k$  clauses if and only if there exists a solution  $S$  of  $f(\mathcal{F})$  with  $|S| \geq 11n + m + k$ , where  $n$  and  $m$  are respectively is the number of variables and clauses in  $\mathcal{F}$ . Moreover, in the  $\Leftarrow$  direction, assignment  $a$  can be computed from  $S$  in polynomial time.*

**Proof.** For any  $i \in [n]$ , let  $A_i = P_i$  if  $x_i$  is set to true in  $a$ , and  $A_i = \overline{P}_i$  otherwise. We first add to  $S$  the set  $\cup_{i \in [n]} A_i$ . Then, let  $\{C_{j_l}, l \in [k]\}$  be  $k$  clauses satisfied by  $a$ . For any  $l \in [k]$ , let  $i_l$  be the index of a literal satisfying  $C_{j_l}$ , let  $x \in [2]$  such that  $c_{j_l}^x$  corresponds to this literal, and let  $Z_l = \{x_{i_l}^2, x'_{i_l}{}^2\}$  if literal  $i_l$  is positive, and  $Z_l = \{x_{i_l}^2\}$  otherwise. For any  $j \in [m]$ , if  $j = i_l$  for some  $l$  (meaning that  $j$  corresponds to a satisfied clause), we add to  $S$  the triangle in  $Q_j^{3-x}$ , and otherwise we arbitrarily add the triangle  $Q_j^1$ . Finally, for any

$l \in [k]$  we add to  $S$  triangle  $t_l = (y_l, \theta_{j_l}, c_{j_l}^{x_l})$  where  $y_l \in Z_l$  is such that  $y_l$  is not already used in another triangle. Notice that such an  $y_l$  always exists as triangles of  $A_i, i \in [n]$  do not intersect  $Z_l$  (by definition of the  $A_i$ ), and as there are at most two clauses that are true due to positive literal, and one clause that is true due to a negative literal. Thus,  $S$  has  $11n + m + k$  vertex disjoint triangles.

Conversely, let  $S$  a solution of  $f(\mathcal{F})$  with  $|S| \geq 11n + m + k$ . By Lemma 3 we can construct in polynomial time a solution  $S'$  from  $S$  such that  $|S'| \geq |S|$ ,  $S'$  only contains outer, variable or clause inner triangles, for each  $j \in [m]$  there exists  $x \in [2]$  such that  $I_j^{K} = Q_j^x$ , and for each  $i \in [n]$ ,  $I_i^L = P_i$  or  $I_i^L = \overline{P}_i$ . This implies that the  $k' \geq k$  remaining triangles must be outer triangles. Let  $\{t'_l, l \in [k']\}$  be these  $k'$  outer triangles with  $t'_l = (y_l, \theta_{j_l}, c_{j_l}^{x_l})$ . Let us define the following assignment  $a$ : for each  $i \in [n]$ , we set  $x_i$  to true if  $I_i^L = P_i$ , and false otherwise. This implies that  $a$  satisfies at least clauses  $\{C_{j_l}, l \in [k']\}$ . ◀

► **Theorem 5.**  $C_3$ -PACKING- $T^{DM}$  is APX-hard, and thus does not admit a PTAS unless  $P = NP$ .

**Proof.** Let us check that Lemma 4 implies a  $L$ -reduction (whose definition is recalled in Definition 17 of appendix). Let  $OPT_1$  (resp.  $OPT_2$ ) be the optimal value of  $\mathcal{F}$  (resp.  $f(\mathcal{F})$ ). Notice that Lemma 4 implies that  $OPT_2 = OPT_1 + 11n + m$ . It is known that  $OPT_1 \geq \frac{3}{4}m$  (where  $m$  is the number of clauses of  $\mathcal{F}$ ). As  $n \leq m$  (each variable has at least one positive and one negative occurrence), we get  $OPT_2 = OPT_1 + 11n + m \leq \alpha OPT_1$  for an appropriate constant  $\alpha$ , and thus point (a) of the definition is verified. Then, given a solution  $S'$  of  $f(\mathcal{F})$ , according to Lemma 4 we can construct in polynomial time an assignment  $a$  satisfying  $c(a)$  clauses with  $c(a) \geq S' - 11n - m$ . Thus, the inequality (b) of Definition 17 with  $\beta = 1$  becomes  $OPT_1 - c(a) \leq OPT_2 - S' = OPT_1 + 11n + m - S'$ , which is true. ◀

Reduction of Theorem 5 does not imply the NP-hardness of  $C_3$ -PERFECT-PACKING-T as there remain some unused vertices. However, it is straightforward to adapt the reduction by adding backward arcs whose head (resp. tail) are before (resp. after)  $\mathcal{T}$  to consume the remaining vertices. This leads to the following result.

► **Theorem 6.**  $C_3$ -PERFECT-PACKING- $T^{DM}$  is NP-hard.

**Proof.** Let  $(\mathcal{F}, k)$  be an instance of the decision problem of  $MAX - 2 - SAT(3)$  and let  $\mathcal{T} = f(\mathcal{F})$  be the tournament defined in Section 3.1. Recall that we have  $\mathcal{T} = LK$ . Let  $N = |V(\mathcal{T})| = 35n + 5m$ ,  $x^* = 33n + 3m + 3k$  and  $n' = N - x^*$ . We now define  $\mathcal{T}'$  by adding  $2n'$  new vertices in  $\mathcal{T}$  as follows:  $V(\mathcal{T}') = R_1 V(\mathcal{T}) R_2$  with  $R_i = \{r_i^l, l \in [n']\}$ . We add to  $\overleftarrow{A}(\mathcal{T}')$  the set of arcs  $R = \{(r_2^l r_1^l), l \in [n']\}$  which are called the dummy arcs. We say that a triangle  $t = (u, v, w)$  is dummy iff  $(wu) \in R$  and  $v \in V(\mathcal{T})$ . Let us prove that there are at least  $k$  clauses satisfiable in  $\mathcal{F}$  iff there exists a perfect packing in  $\mathcal{T}'$ .

⇒

Given an assignment satisfying  $k$  clause we define a solution  $S$  with  $V(S) \subseteq V(\mathcal{T})$  as in Lemma 4 (triangles of  $P_i$  or  $\overline{P}_i$  for each  $i \in [n]$ , a triangle  $Q_j^x$  for each  $j \in [m]$ , and an outer triangle  $t_l$  with  $l \in [k]$  for each satisfied clause. We have  $|S| = 11n + m + k$ . This implies that  $|V(\mathcal{T}) \setminus V(S)| = n'$ , and thus we use  $n'$  remaining vertices of  $V(\mathcal{T})$  by adding to  $S$   $n'$  dummy triangles.

⇐

Let  $S'$  be a perfect packing of  $\mathcal{T}'$ . Let  $S = \{t \in S' : V(t) \subseteq V(\mathcal{T})\}$ . Let  $X = V(\mathcal{T}) \setminus V(S)$ . As  $S'$  is a perfect packing of  $\mathcal{T}'$ , vertices of  $X$  must be used by  $|X|$  dummy triangles of  $S'$ , implying  $|X| \leq n'$  and  $|S| \geq 11n + m + k$ . As  $S$  is set of vertex disjoint triangles of  $\mathcal{T}$  of size at least  $11n + m + k$ , this implies by Lemma 4 that at least  $k$  clauses are satisfiable in  $\mathcal{F}$ .

To establish the kernel lower bound of Section 4, we also need the NP-hardness of  $C_3$ -PERFECT-PACKING-T where instances have a slightly simpler structure (to the price of losing the property that there exists a FAS which is a matching).

► **Theorem 7.**  $C_3$ -PERFECT-PACKING-T remains NP-hard even restricted to tournament  $\mathcal{T}$  admitting the following linear ordering.

- $\mathcal{T} = LK$  where  $L$  and  $K$  are two tournaments
- tournaments  $L$  and  $K$  are "fixed":
  - $K = K_1 \dots K_m$  for some  $m$ , where for each  $j \in [m]$  we have  $V(K_j) = (\theta_j, c_j)$
  - $L = R_1 L_1 \dots L_n R_2$ , where each  $L_i$  has is a copy of the variable gadget of Section 3.1,  $R_i = \{r_i^l, l \in [n']\}$  where  $n' = 2n - m$ , and in addition  $\overleftarrow{L}$  also contains  $R = \{(r_2^l r_1^l), l \in [n']\}$  which are called the dummy arcs.

**Proof.** We adapt the reduction of Section 3.1, reducing now from 3-SAT(3) instead of MAX 2-SAT(3). Given  $\mathcal{F}$  be an instance of 3-SAT(3) with  $n$  variables  $\{x_i\}$  and  $m$  clauses  $\{C_j\}$ . For each variable  $x_i$  with  $i \in [n]$ , we create a tournament  $L_i$  exactly as in Section 3.1 and we define  $L = L_1 \dots L_n$ . For each clause  $C_j$  with  $j \in [m]$ , we create a tournament  $K_j$  with  $V(K_j) = (\theta_j, c_j)$ , and we define  $K = K_1 \dots K_m$ . Let us now define  $\mathcal{T} = LK$ . Now, we add to  $\overleftarrow{A}(\mathcal{T})$  the following backward arcs from  $V(K)$  to  $V(L)$  (again, we follow the construction of Section 3.1 except that now each  $c_j$  has degree  $(3, 0)$ ). If  $C_j = l_{i_1} \vee l_{i_2} \vee l_{i_3}$  is a clause in  $\mathcal{F}$  then we add the arcs  $c_j v_{i_1}, c_j v_{i_2}, c_j v_{i_3}$  where  $v_{i_c}$  is the vertex in  $\{x_{i_c}^2, x_{i_c}'^2, \overline{x_{i_c}^2}\}$  corresponding to  $l_{i_c}$ : if  $l_{i_c}$  is a positive occurrence of variable  $i_c$  we chose  $v_{i_c} \in \{x_{i_c}^2, x_{i_c}'^2\}$ , otherwise we chose  $v_{i_c} = \overline{x_{i_c}^2}$ . Moreover, we chose vertices  $v_{i_c}$  in such a way that for any  $i \in [n]$ , for each  $v \in \{x_i^2, x_i'^2, \overline{x_i^2}\}$  there exists a unique arc  $a \in \overleftarrow{A}(\mathcal{T})$  such that  $h(a) = v$ . This is always possible as each variable has at most 2 positive occurrences and 1 negative one.

Finally, we add  $2n'$  new vertices in  $\mathcal{T}$  as follows:  $V(\mathcal{T}) = R_1 V(L) R_2 V(K)$ ,  $R_i = \{r_i^l, l \in [n']\}$  where  $n' = 2n - m$ . We add to  $\overleftarrow{A}(\mathcal{T})$  the set of arcs  $R = \{(r_2^l r_1^l), l \in [n']\}$  which are called the dummy arcs. Notice that  $\mathcal{T}$  satisfies the claimed structure (defining the left part as  $R_1 L R_2$  and not only  $L$ ). We define an outer and variable inner triangle as in Section 3 (there are no more clause inner triangle), and in addition we say that a triangle  $t = (u, v, w)$  is dummy iff  $(wu) \in R$  and  $v \in V(L)$ . Let us prove that there is an assignment satisfying the  $m$  clauses of  $\mathcal{F}$  iff  $\mathcal{T}$  has a perfect packing.

⇒

Given an assignment satisfying the  $m$  clauses we define a solution  $S$  containing only outer, variable inner and dummy triangles. The variable inner triangle are defined as in Lemma 4 (triangles of  $P_i$  or  $\overline{P}_i$  for each  $i \in [n]$ ). For each clause  $j \in [m]$  satisfied by a literal  $l_{i_x}$  we create an outer triangle  $(v_{i_x}, \theta_j, c_j)$ . It remains now  $2n - m = n'$  vertices of  $L$ , that we use by adding  $n'$  dummy triangles to  $S$ .

⇐

Let  $S$  be a perfect packing of  $\mathcal{T}'$ . Notice that restructuration lemmas of Section 3 do not directly remain true because of the dummy arcs. However, we can adapt in a straightforward manner arguments of these lemmas, using the fact that  $S$  is even a perfect packing. Given a solution  $S$ , we define as in Section 3 set  $I_i^L = \{t \in S : V(t) \subseteq V(L_i)\}$ ,  $I^L = \cup_{i \in [n]} I_i^L$ ,  $O = \{t \in S : t \text{ is an outer triangle}\}$ , and  $D = \{t \in S : t \text{ is a dummy triangle}\}$ . Again, we do not claim (at this point) that  $S$  does not contain other triangles. Given any perfect packing  $S$  of  $\mathcal{T}$ , we can prove the following properties.

- $S$  must contain exactly  $m$  outer triangles ( $|O| = m$ ). Indeed, for any  $j$  from  $m$  to  $1$ , the only way to use  $\theta_j$  is to create an outer triangle  $(u_j, \theta_j, c_j)$ . This implies that triangles of  $O$  consume exactly  $m$  disjoint vertices in  $L$ .
- for any  $i \in [n]$ , we must have  $|I_i^L| = 11$ . Indeed, let  $x$  be the number of vertices of  $L$  used in  $S$  (as  $S$  is a perfect packing we know that  $x = |L| = 35n$ ). The only triangles of  $S$  that can use a vertex of  $L$  are the outer, the variable inner and the dummy triangles, implying  $x \leq (\sum_{i \in [n]} |I_i^L|) + m + n'$  as  $|D| \leq n'$ . As  $|V(L_i)| = 35$  we have  $|I_i^L| \leq 11$  and thus we must have  $|I_i^L| = 11$  for any  $i$ .

Let us now consider the tournament  $\mathcal{T}_0 = \mathcal{T}[V(\mathcal{T}) \setminus V(R)]$  without the dummy arcs, and  $S_0 = \{t \in S : V(t) \subseteq V(\mathcal{T}_0)\}$ . We adapt in a straightforward way the notion of variable inner and outer triangle in  $\mathcal{T}_0$ . Observe that the variable inner and outer triangles of  $S$  and  $S_0$  are the same, and thus are both denoted respectively  $I_i^L$  and  $S^{O_i}$ . In particular,  $S_0$  still contains  $m$  outer triangle of  $\mathcal{T}_0$ . Now we simply apply proof of Lemma 3 on  $S_0$ . More precisely, Lemma 3 restructures  $S_0$  into a solution  $S'_0$  with  $S'_0 = (S_0 \setminus (I_i^L \cup S^{O_i})) \cup (I_i'^L \cup S'^{O_i})$ , where  $I_i'^L$  and  $S'^{O_i}$  satisfy properties  $(\Delta)$ . In particular, as  $|I_i^L| = |I_i'^L| = 11$ ,  $\Delta_3$  implies that  $|S'^{O_i}| \geq |S^{O_i}|$ , and thus that  $|S'_0| \geq |S_0| = m$ . Thus,  $S'_0$  satisfies  $I_i^L = P_i$  or  $I_i^L = \overline{P_i}$  for any  $i$ , and has  $m$  outer triangles. We can now define as in Lemma 4 from  $S'_0$  an assignment satisfying the  $m$  clauses. ◀

### 3.2 $(1 + \frac{6}{c-1})$ -approximation when backward arcs have large minspan

Given a set of pairwise distinct pairs  $D$  and an integer  $c$ , we denote by  $C_3$ -PACKING- $T_{\geq c}^D$  the problem  $C_3$ -PACKING- $T^D$  restricted to tournaments such that there exists a linear representation of minspan at least  $c$  and where  $d(v) \in D$  for all  $v$ . In all this section we consider an instance  $\mathcal{T}$  of  $C_3$ -PACKING- $T_{\geq c}^D$  with a given linear ordering  $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$  of minspan at least  $c$  and whose degrees belong to  $D_M$ . The motivation for studying the approximability of this special case comes from the situation of MAX-SAT( $c$ ) where the approximability becomes easier as  $c$  grows, as the derandomized uniform assignment provides a  $\frac{2^c}{2^c-1}$  approximation algorithm. Somehow, one could claim that MAX-SAT( $c$ ) becomes easy to approximate for large  $c$  as there many ways to satisfy a given clause. As the same intuition applies for tournament admitting an ordering with large minspan (as there are  $c-1$  different ways to use a given backward in a triangle), our objective was to find a polynomial approximation algorithm whose ratio tends to 1 when  $c$  increases.

Let us now define algorithm  $\Phi$ . We define a bipartite graph  $G = (V_1, V_2, E)$  with  $V_1 = \{v_a^1 : a \in \overleftarrow{A}(\mathcal{T})\}$  and  $V_2 = \{v_l^2 : v_l \in V_{(0,0)}\}$ . Thus, to each backward arc we associate a vertex in  $V_1$  and to each vertex  $v_l$  with  $d(v_l) = (0, 0)$  we associate a vertex in  $V_2$ . Then,  $\{v_a^1, v_l^2\} \in E$  iff  $(h(a), v_l, t(a))$  is a triangle in  $\mathcal{T}$ .

In phase 1,  $\Phi$  computes a maximum matching  $M = \{e_l, l \in [|M|]\}$  in  $G$ . For every  $e_l = \{v_{i_j}^1, v_l^2\} \in M$  create a triangle  $t_l^1 = (v_j, v_l, v_i)$ . Let  $S^1 = \{t_l^1, l \in [|M|]\}$ . Notice that triangles of  $S^1$  are vertex disjoint. Let us now turn to phase 2. Let  $\mathcal{T}^2$  be the tournament  $\mathcal{T}$  where we removed all vertices  $V(S^1)$ . Let  $(V(\mathcal{T}^2), \overleftarrow{A}(\mathcal{T}^2))$  be the linear ordering of  $\mathcal{T}^2$  obtained by removing  $V(S^1)$  in  $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$ . We say that three distinct backward edges  $\{a_1, a_2, a_3\} \subseteq \overleftarrow{A}(\mathcal{T}^2)$  can be packed into triangles  $t_1$  and  $t_2$  iff  $V(\{t_1, t_2\}) = V(\{a_1, a_2, a_3\})$  and the  $t_i$  are vertex disjoint. For example, if  $h(a_1) < h(a_2) < t(a_1) < h(a_3) < t(a_2) < t(a_3)$ , then  $\{a_1, a_2, a_3\}$  can be packed into  $(h(a_1), h(a_2), t(a_1))$  and  $(h(a_3), t(a_2), t(a_3))$  (recall that when  $\overleftarrow{A}(\mathcal{T})$  form a matching,  $(u, v, w)$  is triangle iff  $wu \in \overleftarrow{A}(\mathcal{T})$  and  $u < v < w$ ), and if  $h(a_1) < h(a_2) < t(a_2) < h(a_3) < t(a_3) < t(a_1)$ , then  $\{a_1, a_2, a_3\}$  cannot be packed into two

**XX:12 Triangle packing in (sparse) tournaments: approximation and kernelization.**

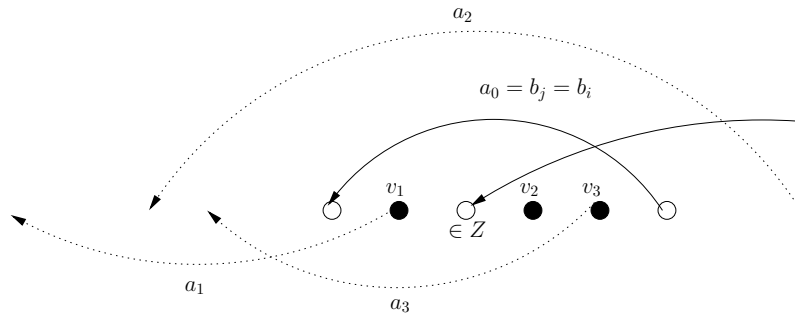
triangles. In phase 2, while it is possible,  $\Phi$  finds a triplet of arcs of  $Y \subseteq \overleftarrow{A}(\mathcal{T}^2)$  that can be packed into triangles, create the two corresponding triangles, and remove  $V(Y)$ . Let  $S^2$  be the triangle created in phase 2 and let  $S = S^1 \cup S^2$ .

► **Observation 8.** For any  $a \in \overleftarrow{A}(\mathcal{T})$ , either  $V(a) \subseteq V(S)$  or  $V(a) \cap V(S) = \emptyset$ . Equivalently, no backward arc has one endpoint in  $V(S)$  and the other outside  $V(S)$ .

According to Observation 8, we can partition  $\overleftarrow{A}(\mathcal{T}) = \overleftarrow{A}_0 \cup \overleftarrow{A}_1 \cup \overleftarrow{A}_2$ , where for  $i \in \{1, 2\}$ ,  $\overleftarrow{A}^i = \{a \in \overleftarrow{A}(\mathcal{T}) : V(a) \subseteq V(S^i)\}$  is the set of arcs used in phase  $i$ , and  $\overleftarrow{A}_0 =_{\text{def}} \{b_i, i \in [x]\}$  are the remaining unused arcs. Let  $\overleftarrow{A}_\Phi = \overleftarrow{A}_1 \cup \overleftarrow{A}_2$ ,  $m_i = |\overleftarrow{A}_i|$ ,  $m = m_0 + m_1 + m_2$  and  $m_\Phi = m_1 + m_2$  the number of arcs (entirely) consumed by  $\Phi$ . To prove the  $1 + f(\frac{6}{c-1})$  desired approximation ratio, we will first prove in Lemma 9 that  $\Phi$  uses at most all the arcs ( $m_A \geq (1 - \epsilon(c))m$ ), and in Theorem 10 that the number of triangles made with these arcs is "optimal". Notice that the latter condition is mandatory as if  $\Phi$  used its  $m_\Phi$  arcs to only create  $\frac{2}{3}(m_\Phi)$  triangles in phase 2 instead of creating  $m' \approx m_\Phi$  triangle with  $m'$  backward arcs and  $m'$  vertices of degree  $(0, 0)$ , we would have a  $\frac{3}{2}$  approximation ratio.

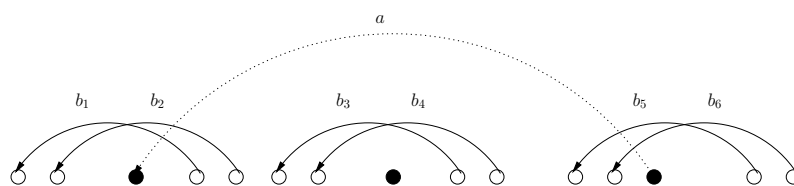
► **Lemma 9.** For any  $c \geq 2$ ,  $m_\Phi \geq (1 - \frac{6}{c+5})m$

**Proof.** In all this proof, the span  $s(a)$  is always considered in the initial input  $\mathcal{T}$ , and not in  $\mathcal{T}^2$ . For any  $i \in [x]$ , let us associate to each  $b_i \in \overleftarrow{A}_0$  a set  $B_i \subseteq \overleftarrow{A}_\Phi$  defined as follows (see Figure 4 for an example). Let  $b_j \in \overleftarrow{A}_0$  such that  $s(b_j) \subseteq s(b_i)$  and there does not exist a  $b_k \in \overleftarrow{A}_0$  such that  $s(b_k)$  included in  $s(b_j)$  (we may have  $b_j = b_i$ ). Let  $Z = V(\overleftarrow{A}_0) \cap s(b_j)$ . Notice that  $|Z| \leq 1$ , meaning that there is at most one endpoint of a  $b_l, l \neq j$  in  $s(b_j)$ , as otherwise we would be three arcs in  $\overleftarrow{A}_0$  that could be packed in two triangles. If there exists  $a \in \overleftarrow{A}_\Phi$  with  $s(a) \subseteq s(b_j)$  we define  $a_0 = a$ , and otherwise we define  $a_0 = b_j$ . Now, let  $v \in s(a_0) \setminus Z$ . Observe that  $V(\mathcal{T})$  is partitioned into  $V(\overleftarrow{A}_0) \cup V(\overleftarrow{A}_\Phi) \cup V_{(0,0)}$ . If  $v \in V_{(0,0)}$ , then there exists  $t_l^1 = (u, v, w)$  with  $wu \in \overleftarrow{A}_1$  (as otherwise the matching in phase 1 would not be maximal and we could add  $b_j$  and  $v$ ), and we add  $wu$  to  $B_i$ . Otherwise,  $v \in V(a)$  with  $a \in \overleftarrow{A}_\Phi$  (this arcs could have been used in phase 1 or phase 2), and we add  $a$  to  $B_i$ . Notice that as  $a_0$  does not properly contains another arc of  $\overleftarrow{A}_\Phi$ , all the added arcs are pairwise distinct, and thus  $|B_i| = |s(a_0) \setminus Z| \geq c - 1$ .



■ **Figure 4** On this example white vertices represent  $V(\mathcal{T}) \setminus V(S)$  (vertices not used by  $\Phi$ ), and black ones represent  $V(S)$ . In this case we have  $B_i = \{a_l, l \in [3]\}$ . Indeed, each  $v_l \in s(a_0) \setminus Z$ , for  $l \in [3]$ , brings  $a_l$  in  $B_i$ . In particular  $v_2 \in V_{(0,0)}$  and was used with  $a_2$  to create a triangle in phase 1.

Given  $a \in \overleftarrow{A}_\Phi$ , let  $B(a) = \{B_i, a \in B_i\}$ . Let us prove that  $|B(a)| \leq 6$  for any  $a \in \overleftarrow{A}_\Phi$ . For any  $v \in V(S)$ , let  $d_B(v) = |\{b_i : v \in s(b_i)\}|$ . Observe that  $d_B(v) \leq 2$ , as otherwise any



■ **Figure 5** Example where  $|B(a)| = 6$  for  $a \in \overleftarrow{A}_\Phi$ , where  $B(a) = \{b_l, l \in [6]\}$ .

triplet of arcs containing  $v$  in their span could be packed into two triangles (there are only 6 cases to check according to the  $3!$  possible ordering of the tail of these 3 arcs). For any  $a \in \overleftarrow{A}_1$ , let  $V'(a) = V(t^a)$  where  $t^a \in S$  is the triangle containing  $a$ , and for any  $a \in A_2$ , let  $V'(a) = V(a)$ . Observe that by definition of the  $B_i$ ,  $a \in B_i$  implies that  $b_i$  contributes to the degree  $d_B(v)$  for a  $v \in V'(a)$ . As in particular  $d_B(v)$  for any  $v \in V'(a)$ , this implies by pigeonhole principle that  $|B(a)| \leq 6$  (notice that this bound is tight as depicted Figure 5). Thus, if we consider the bipartite graph with vertex set  $(\overleftarrow{A}_0, \overleftarrow{A}_\Phi)$  and an edge between  $b_i \in \overleftarrow{A}_0$  and  $a \in \overleftarrow{A}_\Phi$  iff  $a \in B_i$ , the number of edges  $x$  of this graph satisfies  $|\overleftarrow{A}_0|(c-1) \leq x \leq 6|\overleftarrow{A}_\Phi|$ , implying the desired inequality as  $m_\Phi = m - m_0$ . ◀

▶ **Theorem 10.** For any  $c \geq 2$ ,  $\Phi$  is a polynomial  $(1 + \frac{6}{c-1})$  approximation algorithm for  $C_3$ -PACKING- $T_{\geq c}^{DM}$ .

**Proof.** Let  $OPT$  be an optimal solution. Let us define set  $OPT_i \subseteq OPT$  and  $\overleftarrow{A}_i^* \subseteq \overleftarrow{A}(\mathcal{T})$  as follows. Let  $t = (u, v, w) \in OPT$ . As the FAS of the instance is a matching, we know that  $wu \in \overleftarrow{A}(\mathcal{T})$  as we cannot have a triangle with two backward arcs. If  $d(v) = (0, 0)$  then we add  $t$  to  $OPT_1$  and  $wu$  to  $\overleftarrow{A}_1^*$ . Otherwise, let  $v'$  be the other endpoint of the unique arc  $a$  containing  $v$ . If  $v' \notin V(OPT)$ , then we add  $t$  to  $OPT_3$  and  $\{wu, a\}$  to  $\overleftarrow{A}_3^*$ . Otherwise, let  $t' \in OPT$  such that  $v' \in V(t')$ . As the FAS of the instance is a matching we know that  $v'$  is the middle point of  $t'$ , or more formally that  $t' = (u', v', w')$  with  $u'w' \in \overleftarrow{A}(\mathcal{T})$ . We add  $\{t, t'\}$  to  $OPT_2$  and  $\{wu, a, w'u'\}$  to  $\overleftarrow{A}_2^*$ . Notice that the  $OPT_i$  form a partition of  $OPT$ , and that the  $\overleftarrow{A}_i^*$  have pairwise empty intersection, implying  $|\overleftarrow{A}_1^*| + |\overleftarrow{A}_2^*| + |\overleftarrow{A}_3^*| \leq m$ . Notice also that as triangles of  $OPT_1$  correspond to a matching of size  $|OPT_1|$  in the bipartite graph defined in phase 1 of algorithm  $\Phi$ , we have  $|OPT_1| = |\overleftarrow{A}_1^*| \leq |\overleftarrow{A}_1|$ .

Putting pieces together we get (recall that  $S$  is the solution computed by  $\Phi$ ):  $|OPT| = |OPT_1| + |OPT_2| + |OPT_3| = |\overleftarrow{A}_1^*| + \frac{2}{3}|\overleftarrow{A}_2^*| + \frac{1}{2}|\overleftarrow{A}_3^*| \leq |\overleftarrow{A}_1^*| + \frac{2}{3}(|\overleftarrow{A}_2^*| + |\overleftarrow{A}_3^*|) \leq |\overleftarrow{A}_1^*| + \frac{2}{3}(m - |\overleftarrow{A}_1^*|) \leq \frac{1}{3}|\overleftarrow{A}_1| + \frac{2}{3}m$  and  $|S| = |S^1| + |S^2| = |\overleftarrow{A}_1| + \frac{2}{3}|\overleftarrow{A}_2| \geq |\overleftarrow{A}_1| + \frac{2}{3}((1 - \frac{6}{c+5})m - |\overleftarrow{A}_1|) = \frac{1}{3}|\overleftarrow{A}_1| + \frac{2}{3}(1 - \frac{6}{c+5})m$  which implies the desired ratio. ◀

## 4 Kernelization

In all this section we consider the decision problem  $C_3$ -PACKING- $T$  parameterized by the size of the solution. Thus, an input is a pair  $I = (\mathcal{T}, k)$  and we say that  $I$  is positive iff there exists a set of  $k$  vertex disjoint triangles in  $\mathcal{T}$ .

### 4.1 Positive results for sparse instances

Observe first that the kernel in  $\mathcal{O}(k^2)$  vertices for 3-SET PACKING of [1] directly implies a kernel in  $\mathcal{O}(k^2)$  vertices for  $C_3$ -PACKING- $T$ . Indeed, given an instance  $(\mathcal{T} = (V, A), k)$  of



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$C_3$ -PACKING-T, we create an instance  $(I' = (V, C), k)$  of 3-SET PACKING by creating an hyperedge  $c \in C$  for each triangle of  $\mathcal{T}$ . Then, as the kernel of [1] only removes vertices, it outputs an induced instance  $(\bar{I}' = I'[V'], k')$  of  $I$  with  $V' \subseteq V$ , and thus this induced instance can be interpreted as a subtournament, and the corresponding instance  $(\mathcal{T}[V'], k')$  is an equivalent tournament with  $\mathcal{O}(k^2)$  vertices.

As shown in the next theorem, as we could expect it is also possible to have kernel bounded by the number of backward arcs.

► **Theorem 11.**  $C_3$ -PACKING-T admits a polynomial kernel with  $\mathcal{O}(m)$  vertices, where  $m$  is the number of arcs in a given FAS of the input.

**Proof.** Let  $I = (\mathcal{T}, k)$  be an input of the decision problem associated to  $C_3$ -PACKING-T. Observe first that we can build in polynomial time a linear ordering  $\sigma(\mathcal{T})$  whose backward arcs  $\overleftarrow{A}(\mathcal{T})$  correspond to the given FAS. We will obtain the kernel by removing useless vertices of degree  $(0, 0)$ . Let us define a bipartite graph  $G = (V_1, V_2, E)$  with  $V_1 = \{v_a^1 : a \in \overleftarrow{A}(\mathcal{T})\}$  and  $V_2 = \{v_l^2 : v_l \in V_{(0,0)}\}$ . Thus, to each backward arc we associate a vertex in  $V_1$  and to each vertex  $v_l$  with  $d(v_l) = (0, 0)$  we associate a vertex in  $V_2$ . Then,  $\{v_a^1, v_l^2\} \in E$  iff  $(h(a), v_l, t(a))$  is a triangle in  $\mathcal{T}$ . By Hall's theorem, we can in polynomial time partition  $V_1$  and  $V_2$  into  $V_1 = A_1 \cup A_2$ ,  $V_2 = B_0 \cup B_1 \cup B_2$  such that  $N(A_2) \subseteq B_2$ ,  $|B_2| \leq |A_2|$ , and there is a perfect matching between vertices of  $A_1$  and  $B_1$  ( $B_0$  is simply defined by  $B_0 = V_2 \setminus (B_1 \cup B_2)$ ).

For any  $i, 0 \leq i \leq 2$ , let  $X_i = \{v_l \in V_{(0,0)} : v_l^2 \in B_i\}$  be the vertices of  $\mathcal{T}$  corresponding to  $B_i$ . Let  $V_{\neq(0,0)} = V(\mathcal{T}) \setminus V_{(0,0)}$ . Notice that  $|V_{\neq(0,0)}| \leq 2m$ . We define  $\mathcal{T}' = \mathcal{T}[V_{\neq(0,0)} \cup X_1 \cup X_2]$  the sub-tournament obtained from  $\mathcal{T}$  by removing vertices of  $X_0$ , and  $I' = (\mathcal{T}', k)$ . We point out that this definition of  $\mathcal{T}'$  is similar to the final step of the kernel of [1] as our partition of  $V_1$  and  $V_2$  (more precisely  $(A_1, B_0 \cup B_1)$ ) corresponds in fact to the crown decomposition of [1]. Observe that  $|V(\mathcal{T}')| \leq 2m + |A_1| + |A_2| \leq 3m$ , implying the desired bound of the number of vertices of the kernel.

It remains to prove that  $I$  and  $I'$  are equivalent. Let  $k \in \mathbb{N}$ , and let us prove that there exists a solution  $S$  of  $\mathcal{T}$  with  $|S| \geq k$  iff there exists a solution  $S'$  of  $\mathcal{T}'$  with  $|S'| \geq k$ . Observe that the  $\Leftarrow$  direction is obvious as  $\mathcal{T}'$  is a subtournament of  $\mathcal{T}$ . Let us now prove the  $\Rightarrow$  direction. Let  $S$  be a solution of  $\mathcal{T}$  with  $|S| \geq k$ . Let  $S = S_{(0,0)} \cup S_1$  with  $S_{(0,0)} = \{t \in S : t = (h(a), v, t(a)) \text{ with } v \in V_{(0,0)}, a \in \overleftarrow{A}(\mathcal{T})\}$  and  $S_1 = S \setminus S_{(0,0)}$ . Observe that  $V(S_1) \cap V_{(0,0)} = \emptyset$ , implying  $V(S_1) \subseteq V_{\neq(0,0)}$ . For any  $i \in [2]$ , let  $S_{(0,0)}^i = \{t \in S_{(0,0)} : t = (h(a), v, t(a)) \text{ with } v \in V_{(0,0)}, v_a^1 \in A_i\}$  be a partition of  $S_{(0,0)}$ . We define  $S' = S_1 \cup S_{(0,0)}^2 \cup S_{(0,0)}^1$ , where  $S_{(0,0)}^1$  is defined as follows. For any  $v_a^1 \in A_1$ , let  $v_{\mu(a)}^2 \in B_1$  be the vertex associated to  $v_a^1$  in the  $(A_1, B_1)$  matching. To any triangle  $t = (h(a), v, t(a)) \in S_{(0,0)}^1$  we associate a triangle  $f(t) = (h(a), v_{\mu(a)}, t(a)) \in S_{(0,0)}^1$ , where by definition  $v_{\mu(a)} \in X_1$ . For the sake of uniformity we also say that any  $t \in S_1 \cup S_{(0,0)}^2$  is associated to  $f(t) = t$ .

Let us now verify that triangles of  $S'$  are vertex disjoint by verifying that triangles of  $S_{(0,0)}^1$  do not intersect another triangle of  $S'$ . Let  $f(t) = (h(a), v_{\mu(a)}, t(a)) \in S_{(0,0)}^1$ . Observe that  $h(a)$  and  $t(a)$  cannot belong to any other triangle  $f(t')$  of  $S'$  as for any  $f(t'') \in S'$ ,  $V(f(t'')) \cap V_{\neq(0,0)} = V(t'') \cap V_{\neq(0,0)}$  (remember that we use the same notation  $V_{\neq(0,0)}$  to denote vertices of degree  $(0, 0)$  in  $\mathcal{T}$  and  $\mathcal{T}'$ ). Let us now consider  $v_{\mu(a)}$ . For any  $f(t') \in S_1$ , as  $V(f(t')) \cap V_{(0,0)} = \emptyset$  we have  $v_{\mu(a)} \notin V(f(t'))$ . For any  $f(t') = (h(a'), v_l, t(a')) \in S_{(0,0)}^2$ , we know by definition that  $v_{a'}^1 \in A_2$ , implying that  $v_l^2 \in B_2$  (and  $v_l \in X_2$ ) as  $N(A_2) \subseteq B_2$  and thus that  $v_l \neq v_{\mu(a)}$ . Finally, for any  $f(t') = (h(a'), v_l, t(a')) \in S_{(0,0)}^1$ , we know that  $v_l = v_{\mu(a')}$ , where  $a \neq a'$ , leading to  $v_l \neq v_{\mu(a)}$  as  $\mu$  is a matching. ◀

Using the previous result we can provide a  $\mathcal{O}(k)$  vertices kernel for  $C_3$ -PACKING-T restricted to sparse tournaments.

► **Theorem 12.**  *$C_3$ -PACKING-T restricted to sparse tournaments admits a polynomial kernel with  $\mathcal{O}(k)$  vertices, where  $k$  is the size of the solution.*

**Proof.** Let  $I = (\mathcal{T}, k)$  be an input of the decision problem associated to  $C_3$ -PACKING-T such that  $\mathcal{T}$  is a sparse tournament. We say that an arc  $a$  is a *consecutive backward arc* of  $\sigma(\mathcal{T})$  if it is a backward arc of  $\mathcal{T}$  and  $a = v_{i+1}v_i$  with  $v_i$  and  $v_{i+1}$  being consecutive in  $\sigma(\mathcal{T})$ . If  $\mathcal{T}$  admits a consecutive backward arc  $v_i v_{i+1}$  then we can exchange  $v_i$  and  $v_{i+1}$  in  $\mathcal{T}$ . The backward arcs of the new linear ordering is exactly  $\overleftarrow{A}(\mathcal{T}) \setminus v_{i+1}v_i$  and so is still a matching. Hence we can assume that  $\mathcal{T}$  does not contain any consecutive backward arc. Now if  $|\overleftarrow{A}(\mathcal{T})| < 5k$  then by Theorem 11 we have a kernel with  $\mathcal{O}(k)$  vertices. Otherwise, if  $|\overleftarrow{A}(\mathcal{T})| \geq 5k$  we will prove that  $T$  is a YES instance of  $C_3$ -PACKING-T. Indeed we can greedily produce a family of  $k$  vertex disjoint triangles in  $T$ . For that consider a backward arc  $v_j v_i$  of  $\mathcal{T}$ , with  $i < j$ . As  $v_j v_i$  is not consecutive there exists  $l$  with  $i < l < j$  and we select the triangle  $v_i v_j v_l$  and remove the vertices  $v_i$ ,  $v_l$  and  $v_j$  from  $\mathcal{T}$ . Denote by  $\mathcal{T}'$  the resulting tournament and let  $\sigma(\mathcal{T}')$  be the order induced by  $\sigma(\mathcal{T})$  on  $\mathcal{T}'$ . So we loose at most 2 backward arcs in  $\sigma(\mathcal{T}')$  ( $v_j v_i$  and a possible backward arc containing  $v_l$ ) and create at most 3 consecutive backward arcs by the removing of  $v_i$ ,  $v_l$  and  $v_j$ . Reducing these consecutive backward arcs as previously, we can assume that  $\sigma(\mathcal{T}')$  does not contain any consecutive backward arc and satisfies  $|\overleftarrow{A}(\mathcal{T}')| \geq |\overleftarrow{A}(\mathcal{T})| - 5 \geq 5(k - 1)$ . Finally repeating inductively this arguments, we obtain the desired family of  $k$  vertex-disjoint triangles in  $\mathcal{T}$ , and  $\mathcal{T}$  is a YES instance of  $C_3$ -PACKING-T. ◀

## 4.2 No (generalised) kernel in $\mathcal{O}(k^{2-\epsilon})$

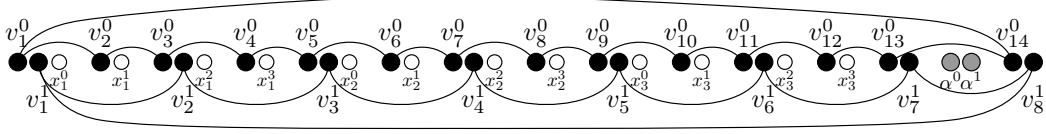
In this section we provide an OR-cross composition (see Definition 21 in Appendix) from  $C_3$ -PERFECT-PACKING-T restricted to instances of Theorem 7 to  $C_3$ -PERFECT-PACKING-T parameterized by the total number of vertices.

**Definition of the instance selector** The next lemma build a special tournament, called an *instance selector* that will be useful to design the cross composition.

► **Lemma 13.** *For any  $\gamma = 2^{\gamma'}$  and  $\omega$  we can construct in polynomial time (in  $\gamma$  and  $\omega$ ) a tournament  $P_{\omega, \gamma}$  such that*

- *there exists  $\gamma$  subsets of  $\omega$  vertices  $X^i = \{x_j^i : j \in [\omega]\}$ , that we call the distinguished set of vertices, such that*
  - *the  $X^i$  have pairwise empty intersection*
  - *for any  $i \in [\gamma]$ , there exists a packing  $S$  of triangles of  $P_{\omega, \gamma}$  such that  $V(P_{\omega, \gamma}) \setminus V(S) = X^i$  (using this packing of  $P_{\omega, \gamma}$  corresponds to select instance  $i$ )*
  - *for any packing  $S$  of triangles of  $P_{\omega, \gamma}$  with  $|V(S)| = |V(P_{\omega, \gamma})| - \omega$  there exists  $i \in [\gamma]$  such that  $V(P_{\omega, \gamma}) \setminus V(S) \subseteq X^i$*
- $|V(P_{\omega, \gamma})| = \mathcal{O}(\omega\gamma)$ .

**Proof.** Let us first describe vertices of  $P_{\omega, \gamma}$ . For any  $i \in [\gamma - 1]_0$  (where  $[x]_0$  denotes  $\{0, \dots, x\}$ ) let  $X^i = \{x_j^i : j \in [\omega]\}$ , and let  $X = \cup_{i \in [\gamma - 1]_0} X^i$ . For any  $l \in [\gamma' - 1]_0$ , let  $V^l = \{v_k^l, k \in [|V^l|]\}$  be the vertices of level  $l$  with  $|V^l| = \omega\gamma/2^l + 2$ , and  $V = \cup_{l \in [\gamma' - 1]_0} V^l$ . Finally, we add a set  $\alpha = \{\alpha^l : l \in [\gamma' - 1]_0\}$  containing one dummy vertex for each level and finally set  $V(P_{\omega, \gamma}) = X \cup V \cup \alpha$ . Observe that  $|V(P_{\omega, \gamma})| = \omega\gamma + \sum_{l \in [\gamma' - 1]_0} (|V^l| + 1) =$



■ **Figure 6** An example of the instance selector, where  $\omega = 3$  and  $\gamma = 4$ . All depicted arcs are backward arcs.

$\mathcal{O}(\omega\gamma)$ . Let us now describe  $\sigma(P_{\omega,\gamma})$  and  $\overleftarrow{A}(P_{\omega,\gamma})$  recursively. Let  $P_{\omega,\gamma}^0$  be the tournament such that  $\sigma(P_{\omega,\gamma}^0) = (v_1^0, x_1^0, v_2^0, x_1^1, \dots, v_\gamma^0, x_1^{\gamma-1}) (v_{\gamma+1}^0, x_2^0, \dots, v_{2\gamma}^0, x_2^{\gamma-1}) \dots (v_{(\omega-1)\gamma+1}^0, x_\omega^0, \dots, v_{\omega\gamma}^0, x_\omega^{\gamma-1}) (v_{\omega\gamma+1}^0, \alpha^1, v_{\omega\gamma+2}^0)$  and  $\overleftarrow{A}(P_{\omega,\gamma}^0) = Z_P^0$  where  $Z_P^0 = A_P^0 \cup A_P^{\prime 0}$  with  $A_P^0 = \{v_{k+1}^0 v_k^0 : k \in [|V^0| - 2]\}$  and  $A_P^{\prime 0} = \{v_{|V^0|}^0 v_{|V^0|-1}^0, v_{|V^0|}^0 v_1^0\}$ .

Then, given a tournament  $P_{\omega,\gamma}^l$  with  $0 \leq l < \gamma' - 1$ , we construct the tournament  $P_{\omega,\gamma}^{l+1}$  such that the vertices of  $P_{\omega,\gamma}^{l+1}$  are those of  $P_{\omega,\gamma}^l$  to which are added the set  $V^{l+1}$ . For  $j \in [|V^{l+1}| - 2]$ , we add the vertex  $v_j^{l+1}$  of  $V^{l+1}$  just after the vertex  $v_{2j-1}^l$  in the order of  $P_{\omega,\gamma}^l$ , and we for  $i \in \{0, 1\}$  we add vertex  $v_{|V^{l+1}|-i}^{l+1}$  just after  $v_{|V^l|-i}^l$ . Similarly, we add the vertex  $\alpha^{l+1}$  just after the vertex  $\alpha^l$ . The backward arcs of  $P_{\omega,\gamma}^{l+1}$  are defined by:  $\overleftarrow{A}(P_{\omega,\gamma}^{l+1}) = \overleftarrow{A}(P_{\omega,\gamma}^l) \cup Z_P^{l+1}$  where  $Z_P^{l+1} = A_P^{l+1} \cup A_P^{\prime l+1}$  are called the *arcs of level l*, with  $A_P^{l+1} = \{v_{k+1}^{l+1} v_k^{l+1} : k \in [|V^{l+1}| - 2]\}$  and  $A_P^{\prime l+1} = \{v_{|V^{l+1}|}^{l+1} v_{|V^{l+1}|-1}^{l+1}, v_{|V^{l+1}|}^{l+1} v_1^{l+1}\}$ . We can now define our gadget tournament  $P_{\omega,\gamma}$  as the tournament corresponding to  $P_{\omega,\gamma}^{\gamma'-1}$ . We refer the reader to Figure 6 where an example of the gadget is depicted, where  $\omega = 3$  and  $\gamma = 4$ .

In all the following given  $i \in [\gamma - 1]_0$  we call the last  $x$  bits (resp. the  $x^{th}$  bit)  $i$  its  $x$  right most (resp. the  $x^{th}$ , starting from the right) bits in the binary representation of  $i$ . Let us now state the following observations.

- $\Delta_1$  The vertices of  $X$  have degree  $(0, 0)$  in  $P_{\omega,\gamma}$ .
- $\Delta_2$  For any  $l \in [\gamma' - 1]_0$ , the extremities of the arcs of level  $l$  are exactly  $V^l$  ( $V(Z_P^l) = V^l$ ) and the arcs of  $Z_P^l$  induce an even circuit on  $V^l$ .
- $\Delta_3$  For any  $a \in A_P^l$ , the span of  $a$  contains  $2^l$  consecutive vertices of  $X$ , more precisely  $s(a) \cap X = \{x_j^i, \dots, x_j^{i+2^l-1}\}$  for  $j \in [m]$  and  $i$  such that the  $l - 1$  last bits of  $i$  are equal to 0.
- $\Delta_4$  There is a unique partition  $Z_P^l = Z_P^{l,0} \cup Z_P^{l,1}$  such that  $|Z_P^{l,0}| = |Z_P^{l,1}| = \mu^l$ , the size of a maximum matching of backward arcs in  $P_{\omega,\gamma}[V^l]$ , such that each  $Z_P^{l,x}$  is a matching (for any  $a, a' \in Z_P^{l,x}, V(a) \cap V(a') = \emptyset$ ), and such that  $\cup_{a \in Z_P^{l,x} \setminus A_P^l} s(a) \cap X$  is the set of all vertices  $x_j^i$  of  $X$  whose  $l^{th}$  bit of  $i$  is  $x$ .

Now let us first prove that for any  $i \in [\gamma - 1]_0$ , there exists an packing  $S$  of  $P_{\omega,\gamma}$  such that  $V(P_{\omega,\gamma}) \setminus V(S) = X^i$ . Let  $(x_{\gamma'-1} \dots x_0)$  be the binary representation of  $i$ . Let us define recursively  $S = \cup_{l \in [\gamma'-1]_0} S_l$  in the following way. We maintain the invariant that for any  $l$ , the remaining vertices of  $X$  after defining  $\cup_{z \in [l]_0} S_z$  are all the vertices of  $X$  having their  $l$  last bits equal to  $(x_{l-1}, \dots, x_0)$ . We define  $S_l$  as the  $\mu^l - 1$  triangles  $\{(h(a), x_a, t(a), a \in Z_P^{l,1-x_l}) \setminus A_P^l\}$  such that  $x_a$  is the unique remaining vertex of  $X$  in  $s(a)$  (by  $\Delta_3$  and our invariant of the  $S_{\leq l}$ , there remains exactly one vertex in  $s(a)$ , and by  $\Delta_4$  these  $\mu^l - 1$  triangles consume all remaining vertices of  $X$  whose  $l^{th}$  bit is  $1 - x_l$ ), and a last triangle using an arc in  $A_P^l$  with  $t = (v_{|V^0|}^l, \alpha^l, v_{|V^0|-1}^l)$  if  $x_l = 1$  and  $t = (v_0^l, \alpha^l, v_{|V^0|}^l)$  otherwise. Thus, by our invariant, the remaining vertices of  $X$  after defining  $S$  are exactly  $X^i$ . As  $S$  also consumes  $\alpha$  and  $V$  we have  $V(P_{\omega,\gamma}) \setminus V(S) = X^i$ . Notice that this definition of  $S$  shows that  $|V(P_{\omega,\gamma})| - m = |V(S)| = 3 \sum_{l \in [\gamma'-1]_0} \mu^l$

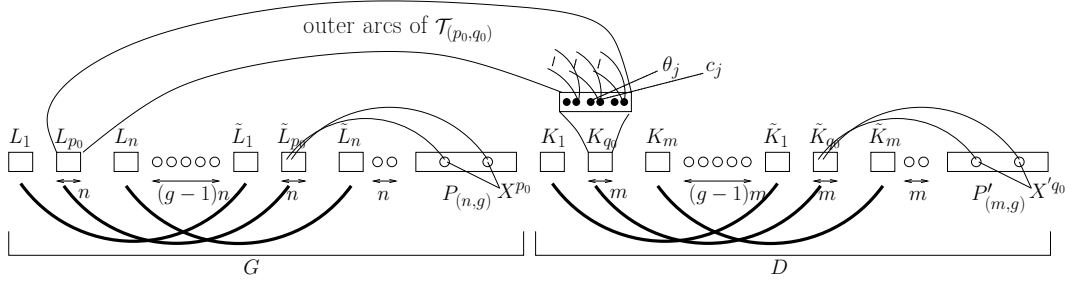
Let us now prove that for any packing  $S$  of  $P_{\omega,\gamma}$  with  $|V(S)| = |V(P_{\omega,\gamma})| - m =$

$3 \sum_{l \in [\gamma' - 1]_0} \mu^l$ , there exists  $i \in [\gamma]$  such that  $V(P_{\omega, \gamma}) \setminus V(S) \subseteq X^i$ . Let  $t_1, \dots, t_\mu$  be the triangles of  $S$ . For any  $t_k$  of  $S$ , we associate one backward arc  $a_k$  of  $t_k$  (if there are two backward arcs, we pick one arbitrarily). Let  $Z = \{a_k : k \in [|S|]\}$  and for every  $l \in [\gamma' - 1]_0$  let  $Z^l = \{a_k \in A : V(a_k) \subset V^l\}$  the set of the backward arcs which are between two vertices of level  $l$ . Notice that the  $Z^l$ 's form a partition of  $Z$ . For any  $l \in [\gamma' - 1]_0$ , we have  $|Z^l| \leq \mu^l$  as two arcs of  $Z^l$  correspond to two different triangles of  $S$ , implying that  $Z^l$  is a matching. Furthermore, as  $|S| = |Z| = \sum_{l \in [\gamma' - 1]_0} |Z^l| = \mu = \sum_{l \in [\gamma']} \mu^l$ , we get the equality  $|Z^l| = \mu^l$  for any  $l \in [\gamma' - 1]_0$ . This implies that for each  $Z^l$  there exists  $x$  such that  $Z^l = Z_P^{l,x}$ , implying also that  $V(Z^l) = V^l$ , and  $V(Z) = V$ .

Let  $A^l = Z^l \setminus A_P^l$ ,  $S^l = \{t_k \in S : a_k \in A^l\}$ . We can now prove by induction that all the remaining vertices  $R_l = X \setminus V(\cup_{x \in [l]_0} S^l)$  have the same  $l$  last bits. Notice that since all vertices of  $V$  are already used, and as triangles of  $S^l$  cannot use a dummy vertex in  $\alpha$ , all triangles of  $S^l$  must be of the form  $(h(a_k), x, t(a_k))$  with  $x \in X$ . As  $A^l = Z_P^{l,x} \setminus A_P^l$ , by  $\Delta_4$  we know that  $\cup_{a \in A^l} s(a) \cap X$  contains all the remaining vertices of  $X$ , and thus of  $R_{l-1}$ , whose  $l^{th}$  bit is  $x$ . Moreover, by  $\Delta_3$  we know that for any  $a \in A^l$  we have  $|R_{l-1} \cap s(a)| \leq 1$  because as  $a \in A_P^l$  we know exactly the structure of  $s(a) \cap X$ , and if there remain two vertices in  $s(a) \cap X$  then their last  $l - 1$  last bits would be different. Thus, as triangles of  $S^l$  remove on vertex in the span of each  $a \in A^l$ , they remove all vertices of  $R_{l-1}$  whose  $l^{th}$  bit is  $x$ , implying the desired result.  $\blacktriangleleft$

**Definition of the reduction** We suppose given a family of  $t$  instances  $F = \{\mathcal{I}_l, l \in [t]\}$  of  $C_3$ -PERFECT-PACKING-T restricted to instances of Theorem 7 where  $\mathcal{I}_l$  asks if there is a perfect packing in  $\mathcal{T}_l = L_l K_l$ . We chose our equivalence relation in Definition 21 such that there exist  $n$  and  $m$  such that for any  $l \in [t]$  we have  $|V(L_l)| = n$  and  $|V(K_l)| = m$ . We can also copy some of the  $t$  instances such that  $t$  is a square number and  $g = \sqrt{t}$  is a power of two. We reorganize our instances into  $F = \{\mathcal{I}_{(p,q)} : 1 \leq p, q \leq g\}$  where  $\mathcal{I}_{(p,q)}$  asks if there is a perfect packing in  $\mathcal{T}_{(p,q)} = L_p K_q$ . Remember that according to Theorem 7, all the  $L_p$  are equals, and all the  $K_q$  are equals. We point out that the idea of using a problem on "bipartite" instances to allow encoding  $t$  instances on a "meta" bipartite graph  $G = (A, B)$  (with  $A = \{A_i, i \in \sqrt{t}\}$ ,  $B = \{B_i, i \in \sqrt{t}\}$ ) such that each instance  $p, q$  is encoded in the graph induced by  $G[A_i \cup B_i]$  comes from [8]. We refer the reader to Figure 7 which represents the different parts of the tournament. We define a tournament  $G = LM_G \tilde{L} \tilde{M}_G P_{(n,g)}$ , where  $L = L_1 \dots L_g$ ,  $\tilde{M}_G$  is a set of  $n$  vertices of degree  $(0, 0)$ ,  $M_G$  is a set of  $(g - 1)n$  vertices of degree  $(0, 0)$ ,  $\tilde{L} = \tilde{L}_1 \dots \tilde{L}_g$  where each  $\tilde{L}_p$  is a set of  $n$  vertices, and  $P_{(n,g)}$  is a copy of the instance selector of Lemma 13. Then, for every  $p \in [g]$  we add to  $G$  all the possible  $n^2$  backward arcs going from  $\tilde{L}_p$  to  $L_p$ . Finally, for every distinguished set  $X^p$  of  $P_{(n,g)}$  (see in Lemma 13), we add all the possible  $n^2$  backward arcs from  $X^p$  to  $\tilde{L}_p$ .

Now, in a symmetric way we define a tournament  $D = KM_D \tilde{K} \tilde{M}_D P'_{(m,g)}$ , where  $K = K_1 \dots K_g$ ,  $\tilde{M}_D$  is a set of  $m$  vertices of degree  $(0, 0)$ ,  $M_D$  is a set of  $(g - 1)m$  vertices of degree  $(0, 0)$ ,  $\tilde{K} = \tilde{K}_1 \dots \tilde{K}_g$  where each  $\tilde{K}_q$  is a set of  $m$  vertices, and  $P'_{(m,g)}$  is a copy of the instance selector of Lemma 13. Then, for every  $q \in [g]$  we add to  $G$  all the  $m^2$  possible backward arcs going from  $\tilde{K}_p$  to  $K_p$ . Finally, for every distinguished set  $X'^q$  of  $P'_{(m,g)}$  we add all the possible  $m^2$  backward arcs from  $X'^q$  to  $\tilde{K}_q$ . Finally, we define  $\mathcal{T} = GD$ . Let us add some backward arcs from  $D$  to  $G$ . For any  $p$  and  $q$  with  $1 \leq p, q \leq g$ , we add backward arcs from  $K_q$  to  $L_p$  such that  $\mathcal{T}[K_q L_p]$  corresponds to  $\mathcal{T}_{(p,q)}$ . Notice that this is possible as for any fixed  $p$ , all the  $\mathcal{T}_{(p,q)}$ ,  $q \in [g]$  have the same left part  $L_p$ , and the same goes for any fixed right part.



■ **Figure 7** A example of the weak composition. All depicted arcs are backward arcs. Bold arcs represents the  $n^2$  (or  $m^2$ ) possible arcs between the two groups.

**Restructuration lemmas** Given a set of triangles  $S$  we define  $S_{\subseteq P'} = \{t \in S \mid V(t) \subseteq P'_{(m,g)}\}$ ,  $S_{\subseteq P} = \{t \in S : V(t) \subseteq P_{(n,g)}\}$ ,  $S_{\tilde{M}_D} = \{t \in S : V(t) \text{ intersects } \tilde{K}, \tilde{M}_D \text{ and } P'_{(m,g)}\}$ ,  $S_{M_D} = \{t \in S : V(t) \text{ intersects } K, M_D \text{ and } \tilde{K}\}$ ,  $S_{\tilde{M}_G} = \{t \in S : V(t) \text{ intersects } \tilde{L}, \tilde{M}_G \text{ and } P_{(n,g)}\}$ ,  $S_{M_G} = \{t \in S : V(t) \text{ intersects } L, M_G \text{ and } \tilde{L}\}$ ,  $S_D = \{t \in S : V(t) \subseteq V(D)\}$ ,  $S_G = \{t \in S : V(t) \subseteq V(G)\}$ , and  $S_{GD} = \{t \in S : V(t) \text{ intersects } V(G) \text{ and } V(D)\}$ . Notice that  $S_G, S_{GD}, S_D$  is a partition of  $S$ .

► **Claim 13.1.** If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then  $|S_{\tilde{M}_D}| = m$  and  $|S_{M_D}| = (g-1)m$ . This implies that  $V(S_{\tilde{M}_D} \cup S_{M_D}) \cap V(\tilde{K}) = V(\tilde{K})$ , meaning that the vertices of  $\tilde{K}$  are entirely used by  $S_{\tilde{M}_D} \cup S_{M_D}$ .

**Proof.** We have  $|S_{\tilde{M}_D}| \leq m$  since  $|\tilde{M}_D| = m$ . We obtain the equality since the vertices of  $\tilde{M}_D$  only lie in the span of backward arcs from  $P'_{(m,g)}$  to  $\tilde{K}$ , and they are not the head or the tail of a backward arc in  $\mathcal{T}$ . Thus, the only way to use vertices of  $\tilde{M}_D$  is to create triangles in  $S_{\tilde{M}_D}$ , implying  $|S_{\tilde{M}_D}| \geq m$ . Using the same kind of arguments we also get  $|S_{M_D}| = (g-1)m$ . As  $|V(\tilde{K})| = gm$  we get the last part of the claim. ◀

► **Claim 13.2.** If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then there exists  $q_0 \in [g]$  such that  $\tilde{K}_S = \tilde{K}_{q_0}$ , where  $\tilde{K}_S = \tilde{K} \cap V(S_{\tilde{M}_D})$ .

**Proof.** Let  $S_{P'}$  be the triangles of  $S$  with at least one vertex in  $P'_{(m,g)}$ . As according to Claim 13.1 vertices of  $\tilde{K}$  are entirely used by  $S_{\tilde{M}_D} \cup S_{M_D}$ , the only way to consume vertices of  $P'_{(m,g)}$  is by creating local triangles in  $P'_{(m,g)}$  or triangles in  $S_{\tilde{M}_D}$ . In particular, we cannot have a triangle  $(u, v, w)$  with  $\{u, v\} \subseteq \tilde{K}$  and  $w \in P'_{(m,g)}$ , or with  $u \in \tilde{K}$  and  $\{v, w\} \subseteq P'_{(m,g)}$ . More formally, we get the partition  $S_{P'} = S_{\subseteq P'} \cup S_{\tilde{M}_D}$ . As  $S$  is a perfect packing and uses in particular all vertices of  $P'_{(m,g)}$  we get  $|V(S_{P'})| = |V(P'_{(m,g)})|$ , implying  $|V(S_{\subseteq P'})| = |V(P'_{(m,g)})| - m$  by Claim 13.1. By Lemma 13, this implies that there exists  $q_0 \in [g]$  such that  $X' \subseteq X'^{q_0}$  where  $X' = V(P'_{(m,g)}) \setminus V(S_{\subseteq P'})$ . As  $X'$  are the only remaining vertices that can be used by triangles of  $S_{\tilde{M}_D}$ , we get that the  $m$  triangles of  $S_{\tilde{M}_D}$  are of the form  $(u, v, w)$  with  $u \in \tilde{K}_{q_0}$ ,  $v \in \tilde{M}_D$ , and  $w \in X'$ . ◀

► **Claim 13.3.** If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then there exists  $q_0 \in [g]$  such that  $V(S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) = V(D) \setminus K_{q_0}$ .

**Proof.** By Claim 13.1 we know that  $|S_{M_D}| = (g-1)m$ . As by Claim 13.2 there exists  $q_0 \in [g]$  such that  $\tilde{K}_S = \tilde{K}_{q_0}$ , we get that the  $(g-1)m$  triangles of  $S_{M_D}$  are of the form  $(u, v, w)$  with  $u \in K \setminus K_{q_0}$ ,  $v \in M_D$ , and  $w \in \tilde{K} \setminus \tilde{K}_{q_0}$ . ◀

► **Lemma 14.** *If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then  $V(S_{GD}) \cap V(G) \subseteq V(L)$ . Informally, triangles of  $S_{GD}$  do not use any vertex of  $M_G, \tilde{L}, \tilde{M}_T$  and  $P_{n,g}$ .*

**Proof.** By Claim 13.3, there exists  $q_0 \in [g]$  such that  $V(S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) = V(D) \setminus K_{q_0}$ . By Theorem 7 we know that  $K_{q_0} = K_{(q_0,1)} \cdots K_{(q_0,m')}$  for some  $m'$  (we even have  $m' = \frac{m}{2}$ ), where for each  $j \in [m']$  we have  $V(K_{(q_0,j)}) = (\theta_j, c_j)$ . Moreover, for any  $p \in [g]$ , the last property of Theorem 7 ensures that for any  $a \in \overleftarrow{A}(\mathcal{T}_{(p,q_0)})$ ,  $V(a) \cap V(K_{q_0}) \neq \emptyset$  implies  $a = vc_j$  for  $v \in L_p$ . So no arc of  $\overleftarrow{A}(\mathcal{T}_{(p,q_0)})$ , and thus no arc of  $\mathcal{T}$  is entirely included in  $K_{q_0}$ . This implies that  $S$  cannot cover the vertices of  $K_{q_0}$  using triangles  $t$  with  $V(t) \subseteq V(K_{q_0})$ , and thus that all these vertices must be used by triangles of  $S_{GD}$ , implying that  $V(S_{GD}) \cap V(D) = K_{q_0}$ . The last property of Theorem 7 also implies that all the  $\theta_j$  have a left degree equal to 0 in  $\mathcal{T}$ , or equivalently that there is no arc  $a$  of  $\mathcal{T}$  such that  $t(a) = \theta_j$  and  $h(a) < \theta_j$ . Thus, by induction for any  $j$  from  $m'$  to 1, we can prove that the only way for triangles of  $S_{GD}$  to use  $\theta_j$  is to create a triangle  $t_j = (v, \theta_j, c_j)$  with necessarily  $v \in V(L)$ . ◀

Lemma 14 will allow us to prove Claims 14.1, 14.2 and 14.3 using the same arguments as in the right part (D) of the tournament as all vertices of  $M_G, \tilde{L}, \tilde{M}_T$  and  $P_{n,g}$  must be used by triangles in  $S_G$ .

► **Claim 14.1.** *If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then  $|S_{\tilde{M}_G}| = n$  and  $|S_{M_G}| = (g-1)n$ . This implies that  $V(S_{\tilde{M}_G} \cup S_{M_G}) \cap V(\tilde{L}) = V(\tilde{L})$ , meaning that vertices of  $\tilde{L}$  are entirely used by  $S_{\tilde{M}_G} \cup S_{M_G}$ .*

**Proof.** We have  $|S_{\tilde{M}_G}| \leq n$  since  $|\tilde{M}_G| = n$ . Lemma 14 implies that all vertices of  $\tilde{M}_G$  must be used by triangles of  $S_G$ , and thus using arcs whose both endpoints lie in  $V(G)$ . As vertices of  $\tilde{M}_G$  are not the head or the tail of a backward arc in  $\mathcal{T}$ , we get that the only way for  $S_G$  to use vertices of  $\tilde{M}_G$  is to create triangles in  $S_{\tilde{M}_G}$ , implying  $|S_{\tilde{M}_G}| \geq n$ . Using the same kind of arguments (and as all vertices of  $M_G$  must also be used by triangles of  $S_G$ ) we also get  $|S_{M_G}| = (g-1)n$ . As  $|V(\tilde{L})| = gn$  we get the last part of the claim. ◀

► **Claim 14.2.** *If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then there exists  $p_0 \in [g]$  such that  $\tilde{L}_S = \tilde{L}_{p_0}$ , where  $\tilde{L}_S = \tilde{L} \cap V(S_{\tilde{M}_G})$ .*

**Proof.** Lemma 14 implies that all vertices of  $\tilde{M}_G$  and  $P_{(n,g)}$  must be used by triangles in  $S_G$ . Let  $S_P$  be the triangles of  $S_G$  with at least one vertex in  $P_{n,g}$ . As according to Claim 14.1 vertices of  $\tilde{L}$  are entirely used by  $S_{\tilde{M}_G} \cup S_{M_G}$ , the only way for  $S_G$  to consume vertices of  $P_{n,g}$  is by creating local triangles in  $P_{n,g}$  or triangles in  $S_{\tilde{M}_G}$ . In particular, we cannot have a triangle  $(u, v, w)$  with  $\{u, v\} \subseteq \tilde{L}$  and  $w \in P_{n,g}$ , or with  $u \in \tilde{L}$  and  $\{v, w\} \subseteq P_{n,g}$ . More formally, we get the partition  $S_P = S_{\subseteq P} \cup S_{\tilde{M}_G}$ . As  $S_G$  uses in particular all vertices of  $P_{n,g}$  we get  $|V(S_P)| = |V(P_{n,g})|$ , implying  $|V(S_{\subseteq P})| = |V(P_{n,g})| - n$  by Claim 14.1. By Lemma 13, this implies that there exists  $p_0 \in [g]$  such that  $X \subseteq X^{p_0}$  where  $X = V(P_{n,g}) \setminus V(S_{\subseteq P})$ . As  $X$  are the only remaining vertices that can be used by triangles of  $S_{\tilde{M}_G}$ , we get that the  $n$  triangles of  $S_{\tilde{M}_G}$  are of the form  $(u, v, w)$  with  $u \in \tilde{L}_{p_0}$ ,  $v \in \tilde{M}_G$ , and  $w \in X$ . ◀

► **Claim 14.3.** *If there exists a perfect packing  $S$  of  $\mathcal{T}$ , then there exists  $p_0 \in [g]$  such that  $V(S_P \cup S_{\tilde{M}_G} \cup S_{M_G}) = V(G) \setminus L_{p_0}$ .*

**Proof.** By Claim 13.1 we know that  $|S_{M_G}| = (g-1)n$ . As by Claim 14.2 there exists  $p_0 \in [g]$  such that  $\tilde{L}_S = \tilde{L}_{p_0}$ , we get that the  $(g-1)n$  triangles of  $S_{M_G}$  are of the form  $(u, v, w)$  with  $u \in L \setminus L_{p_0}$ ,  $v \in M_G$ , and  $w \in \tilde{L} \setminus \tilde{L}_{p_0}$ . ◀



We are now ready to state our final claim is now straightforward as according Claim 13.3 and 14.3 we can define  $S_{(p_0, q_0)} = S \setminus ((S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) \cup (S_P \cup S_{\tilde{M}_G} \cup S_{M_G}))$ .

► **Claim 14.4.** If there exists a perfect packing  $S$  of  $\mathcal{T}$ , there exists  $p_0, q_0 \in [g]$  and  $S_{(p_0, q_0)} \subseteq S$  such that  $V(S_{(p_0, q_0)}) = V(\mathcal{T}_{(p_0, q_0)})$  (or equivalently such that  $S_{(p_0, q_0)}$  is a perfect packing of  $\mathcal{T}_{(p_0, q_0)}$ ).

### Proof of the weak composition

► **Theorem 15.** For any  $\epsilon > 0$ ,  $C_3$ -PERFECT-PACKING-T (parameterized by the total number of vertices  $N$ ) does not admit a polynomial (generalized) kernelization with size bound  $\mathcal{O}(N^{2-\epsilon})$  unless  $\text{NP} \subseteq \text{coNP}/\text{Poly}$ .

**Proof.** Given  $t$  instances  $\{\mathcal{I}_l\}$  of  $C_3$ -PERFECT-PACKING-T restricted to instances of Theorem 7, we define an instance  $\mathcal{T}$  of  $C_3$ -PERFECT-PACKING-T as defined in Section 4. We recall that  $g = \sqrt{t}$ , and that for any  $l \in [t]$ ,  $|V(L_l)| = n$  and  $|V(K_l)| = m$ . Let  $N = |V(\mathcal{T})|$ . As  $N = |V(P'_{(m, g)})| + m + (g-1)m + 2mg + |V(P_{(n, g)})| + n + (g-1)n + 2ng$  and  $|V(P_{(\omega, \gamma)})| = \mathcal{O}(\omega\gamma)$  by Lemma 13, we get  $N = \mathcal{O}(g(n+m)) = \mathcal{O}(t^{\frac{1}{2+\epsilon(1)}} \max(|\mathcal{I}_l|))$ . Let us now verify that there exists  $l \in [t]$  such that  $\mathcal{I}_l$  admits a perfect packing iff  $\mathcal{T}$  admits a perfect packing. First assume that there exist  $p_0, q_0 \in [g]$  such that  $\mathcal{I}_{(p_0, q_0)}$  admits a perfect packing. By Lemma 14.4, there is a packing  $S_{P'}$  of  $P'_{(m, g)}$  such that  $V(S_{P'}) = V(P'_{(m, g)}) \setminus X'^{q_0}$ . We define a set  $S_{\tilde{M}_D}$  of  $m$  vertex disjoint triangles of the form  $(u, v, w)$  with  $u \in \tilde{L}_{q_0}, v \in \tilde{M}_D, w \in X'^{q_0}$ . Then, we define a set  $S_{M_D}$  of  $(g-1)m$  vertex disjoint triangles of the form  $(u, v, w)$  with  $u \in L \setminus L_{q_0}, v \in M_D, w \in \tilde{L} \setminus \tilde{L}_{q_0}$ . In the same way we define  $S_P, S_{\tilde{M}_G}$  and  $S_{M_G}$ . Observe that  $V(\mathcal{T}) \setminus ((S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) \cup (S_P \cup S_{\tilde{M}_G} \cup S_{M_G})) = K_{q_0} \cup L_{p_0}$ , and thus we can complete our packing into a perfect packing of  $\mathcal{T}$  as  $\mathcal{I}_{(p_0, q_0)}$  admits a perfect packing. Conversely if there exists a perfect packing  $S$  of  $\mathcal{T}$ , then by Claim 14.4 there exists  $p_0, q_0 \in [g]$  and  $S_{(p_0, q_0)} \subseteq S$  such that  $V(S_{(p_0, q_0)}) = V(\mathcal{T}_{(p_0, q_0)})$ , implying that  $\mathcal{I}_{(p_0, q_0)}$  admits a perfect packing. ◀

► **Corollary 16.** For any  $\epsilon > 0$ ,  $C_3$ -PACKING-T (parameterized by the size  $k$  of the solution) does not admit a polynomial kernel with size  $\mathcal{O}(k^{2-\epsilon})$  unless  $\text{NP} \subseteq \text{coNP}/\text{Poly}$ .

## 5 Conclusion and open questions

Concerning approximation algorithms for  $C_3$ -PACKING-T restricted to sparse instances, we have provided a  $(1 + \frac{6}{c+5})$ -approximation algorithm where  $c$  is a lower bound of the *minspan* of the instance. On the other hand, it is not hard to solve by dynamic programming  $C_3$ -PACKING-T for instances where *maxspan* is bounded above. Using these two opposite approaches it could be interesting to derive an approximation algorithm for  $C_3$ -PACKING-T with factor better than  $4/3$  even for sparse tournaments.

Concerning FPT algorithms, the approach we used for sparse tournament (reducing to the case where  $m = \mathcal{O}(k)$  and apply the  $\mathcal{O}(m)$  vertices kernel) cannot work the general case. Indeed, if we were able to sparsify the initial input such that  $m' = \mathcal{O}(k^{2-\epsilon})$ , applying the kernel in  $\mathcal{O}(m')$  would lead to a tournament of total bit size (by encoding the two endpoint of each arc  $\mathcal{O}(m' \log(m')) = \mathcal{O}(k^{2-\epsilon})$ , contradicting Corollary 16. Thus the situation for  $C_3$ -PACKING-T could be as in vertex cover where there exists a kernel in  $\mathcal{O}(k)$  vertices, derived from [16], but the resulting instance cannot have  $\mathcal{O}(k^{2-\epsilon})$  edges [8]. So it is challenging question to provide a kernel in  $\mathcal{O}(k)$  vertices for the general  $C_3$ -PACKING-T problem.

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## A Definitions

### Approximation

► **Definition 17** ([17]). Let  $\Pi$  and  $\Pi'$  be two optimization (maximization or minimization) problems. We say that  $\Pi$   $L$ -reduces to  $\Pi'$  if there are two polynomial-time algorithms  $f, g$ , and constants  $\alpha, \beta > 0$  such that for each instance  $I$  of  $\Pi$

- (a) Algorithm  $f$  produces an instance  $I' = f(I)$  of  $\Pi'$  such that the optima of  $I$  and  $I'$ ,  $OPT(I)$  and  $OPT(I')$ , respectively, satisfy  $OPT(I') \leq \alpha OPT(I)$
- (b) Given any solution of  $I'$  with cost  $c$ , algorithm  $g$  produces a solution of  $I$  with cost  $c$  such that  $|c - OPT(I)| \leq \beta |c - OPT(I')|$ .

► **Definition 18.** Let  $A$  be an algorithm of a maximization (resp. minimization) problem  $\Pi$ . For  $\rho \geq 1$ , we say that  $A$  is a  $\rho$ -approximation of  $\Pi$  iff for any instance  $I$  of  $\Pi$ ,  $A_I \geq OPT(I)/\rho$  (resp.  $A_I \leq \rho OPT(I)$ ) where  $A_I$  is the value of the solution  $A(I)$  and  $OPT(I)$  the value of a optimal solution of  $I$ .

► **Definition 19.** Let  $\Pi$  be a NP-optimization problem. The problem  $\Pi$  is in APX if there exists a constant  $\rho > 1$  such that  $\Pi$  admits a  $\rho$ -approximation algorithm.

► **Definition 20.** Let  $\Pi$  be a NP-optimization problem. The problem  $\Pi$  admits a PTAS if for any  $\epsilon > 0$ , there exists a polynomial  $(1 + \epsilon)$ -approximation of  $\Pi$ .

### Parameterized complexity

We refer the reader to [9] for more details on parameterized complexity and kernelization, and we recall here only some basic definitions. A *parameterized problem* is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ . For an instance  $I = (x, k) \in \Sigma^* \times \mathbb{N}$ , the integer  $k$  is called the *parameter*.

A parameterized problem is *fixed-parameter tractable* (FPT) if there exists an algorithm  $A$ , a computable function  $f$ , and a constant  $c$  such that given an instance  $I = (x, k)$ ,  $A$  (called an FPT algorithm) correctly decides whether  $I \in L$  in time bounded by  $f(k) \cdot |I|^c$ , where  $|I|$  denotes the size of  $I$ . Given a computable function  $g$ , a *kernelization algorithm* (or simply a *kernel*) for a parameterized problem  $L$  of *size*  $g$  is an algorithm  $A$  that given any instance  $I = (x, k)$  of  $L$ , runs in polynomial time and returns an equivalent instance  $I' = (x', k')$  with  $|I'| + k' \leq g(k)$ . It is well-known that the existence of an FPT algorithm is equivalent to the existence of a kernel (whose size may be exponential), implying that problems admitting a polynomial kernel form a natural subclass of FPT. Among the wide literature on polynomial kernelization, we only recall in the notion of weak composition used to lower bound the size of a kernel.

► **Definition 21** (Definition as written in [12]). Let  $L \subseteq \Sigma^*$  be a language,  $R$  be a polynomial equivalence relation on  $\Sigma^*$ , let  $Q \subseteq \Sigma^* \times \mathbb{N}$  be a parameterized problem, and let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function. An *or-cross-composition* of  $L$  into  $Q$  (with respect to  $R$ ) of cost  $f(t)$  is an algorithm that, given  $t$  instances  $x_i \in \Sigma^*$  of  $L$  belonging to the same equivalence class of  $R$ , takes time polynomial in  $\sum_{i \in [t]} |x_i|$  and outputs an instance  $(y, k) \in \Sigma^* \times \mathbb{N}$  such that:

1. the parameter  $k$  is bounded by  $\mathcal{O}(f(t) \max_i |x_i|^c)$ , where  $c$  is some constant independent of  $t$ , and
2.  $(y, k) \in Q$  if and only if there is an  $i \in [t]$  such that  $x_i \in L$ .

► **Theorem 22** ([4]). Let  $L \subseteq \Sigma^*$  be a language, let  $Q \subseteq \Sigma^* \times \mathbb{N}$  be a parameterized problem, and let  $d, \epsilon$  be positive reals. If  $L$  is NP-hard under Karp reductions, has an or-cross-composition into  $Q$  with cost  $f(t) = t^{1/d+o(1)}$ , where  $t$  denotes the number of instances,

and  $Q$  has a polynomial (generalized) kernelization with size bound  $\mathcal{O}(k^{d-\epsilon})$ , then  $\text{NP} \subseteq \text{coNP}/\text{Poly}$ .

## B Problems

### ► Problem 1. (FVS)

**Input:** A directed graph  $D = (V, A)$ .

**Output:** A set of vertices  $X \subseteq V$  such that  $D[V \setminus X]$  is acyclic.

**Optimisation:** Minimise  $|X|$ .

The problem is called FVST if the input is a tournament.

### ► Problem 2. ( $d$ -SET PACKING)

**Input:** An integer  $d \geq 3$  and a  $d$ -uniform hypergraph  $G = (V, H)$ .

**Output:** A subset of hyperedges  $X = \{X_i, i \in [k]\}$  with  $X_i \in H$  such that for every  $i \neq j$ ,  $X_i \cap X_j = \emptyset$ .

**Optimisation:** Maximise  $k$ .

### ► Problem 3. (PERFECT $d$ -SET PACKING)

**Input:** An integer  $d \geq 3$  and a  $d$ -uniform hypergraph  $G = (V, H)$ .

**Question:** Is there a subset of hyperedges  $X = \{X_i, i \in [k]\}$  with  $X_i \in H$  such that for every  $i \neq j$ ,  $X_i \cap X_j = \emptyset$  and  $\bigcup_{i \in [k]} X_i = V$ ?

### ► Problem 4. ( $H$ -PACKING)

**Input:** A graph  $G = (V, E)$  and a subgraph  $H$ .

**Output:** A collection of subgraphs  $X = \{H_i, i \in [k]\}$  such that for every  $i$ ,  $H_i$  is isomorphic to  $H$  and for every  $j \neq i$ ,  $V(H_i) \cap V(H_j) = \emptyset$ .

**Optimisation:** Maximise  $k$ .

### ► Problem 5. (PERFECT $H$ -PACKING)

**Input:** A graph  $G = (V, E)$  and a subgraph  $H$ .

**Question:** Is there a collection of subgraphs  $X = \{H_i, i \in [k]\}$  such that for every  $i$ ,  $H_i$  is isomorphic to  $H$ , for every  $j \neq i$ ,  $V(H_i) \cap V(H_j) = \emptyset$  and  $\bigcup_{i \in [k]} H_i = V$ ?

## C Polynomial detection of sparse tournaments

► **Lemma 23.** *In polynomial time, we can decide if a tournament is sparse or not, and if so, to give a linear representation whose FAS is a matching*

**Proof.** Indeed if a tournament  $\mathcal{T}$  is sparse we can detect the first vertex (or vertices) of a linear representation  $\sigma(\mathcal{T})$  of  $\mathcal{T}$  where  $\overleftarrow{A}(\mathcal{T})$  is a matching. If  $\mathcal{T}$  has a vertex  $x$  of indegree 0 then  $x$  must be the first or the second vertex of  $\sigma(\mathcal{T})$ , and we can always suppose that  $x$  is the first vertex of  $\sigma(\mathcal{T})$ . Otherwise, we look at  $Z$  the set of vertices of  $\mathcal{T}$  with indegree 1. As  $\mathcal{T}$  is a tournament we have  $|Z| \leq 3$  and if  $Z = \emptyset$  then  $\mathcal{T}$  is not a sparse tournament. If  $|Z| = 1$ , then the only element of  $Z$  must be the first vertex of  $\sigma(\mathcal{T})$ . If  $|Z| = 2$  with  $Z = \{x, y\}$  such that  $xy$  is an arc of  $\mathcal{T}$ , then  $x$  must be the first element of  $\sigma(\mathcal{T})$  and  $y$  its second element. Finally, if  $|Z| = 3$  with  $Z = \{x, y, z\}$  then  $xyz$  must be a triangle of  $\mathcal{T}$  and must be placed at the beginning of  $\sigma(\mathcal{T})$ . So repeating inductively these arguments we obtain in polynomial time in  $|\mathcal{T}|$  either  $\sigma(\mathcal{T})$  such that  $\overleftarrow{A}(\mathcal{T})$  is a matching or a certificate that  $\mathcal{T}$  is not sparse. ◀