Triangle packing in (sparse) tournaments: approximation and kernelization

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Abstract

Given a tournament $\mathcal{T}$ and a positive integer k, the C_3-Packing-T problem asks if there exists a least k (vertex-)disjoint directed 3-cycles in $\mathcal{T}$. This is the dual problem in tournaments of the classical minimal feedback vertex set problem. Surprisingly C_3-Packing-T did not receive a lot of attention in the literature. We show that it does not admit a PTAS unless P=NP, and so is NP-complete, even if we restrict the considered instances to sparse tournaments, that is tournaments with a feedback arc set (FAS) being a matching. Focusing on sparse tournaments we provide a $(1+\frac{6}{c-1})$ approximation algorithm for sparse tournaments having a linear representation where all the backward arcs have "length" at least c. Concerning kernelization, we show that C_3-Packing-T admits a kernel with $\mathcal{O}(m)$ vertices, where m is the size of a given feedback arc set. In particular, we derive a $\mathcal{O}(k)$ vertices kernel for C_3-Packing-T when restricted to sparse instances. On the negative size, we show that C_3-Packing-T does not admit a kernel of (total bit) size $\mathcal{O}(k^{2-\epsilon})$ unless NP $\subseteq$ coNP/Poly. The existence of a kernel in $\mathcal{O}(k)$ vertices for C_3-Packing-T remains an open question.

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1 Introduction and related work

Tournament

A tournament $\mathcal{T}$ on n vertices is an orientation of the edges of the complete undirected graph K_n. Thus, given a tournament $\mathcal{T}=(V,A)$, where $V=\{v_i, i\in[n]\}$, for each $i,j\in[n]$, either $v_iv_j\in A$ or $v_jv_i\in A$. A tournament $\mathcal{T}$ can alternatively be defined by an ordering $\sigma(\mathcal{T})=(v_1,\dots,v_n)$ of its vertices and a set of *backward arcs* $\overleftarrow{A}_\sigma(\mathcal{T})$ (which will be denoted $\overleftarrow{A}(\mathcal{T})$ as the considered ordering is not ambiguous), where each arc $a\in\overleftarrow{A}(\mathcal{T})$ is of the form $v_{i_1}v_{i_2}$ with $i_2<i_1$. Indeed, given $\sigma(\mathcal{T})$ and $\overleftarrow{A}(\mathcal{T})$, we can define $V=\{v_i, i\in[n]\}$ and $A=\overleftarrow{A}(\mathcal{T})\cup\overrightarrow{A}(\mathcal{T})$ where $\overrightarrow{A}(\mathcal{T})=\{v_{i_1}v_{i_2}:(i_1<i_2)\text{ and }v_{i_2}v_{i_1}\notin\overleftarrow{A}(\mathcal{T})\}$ is the set of forward arcs of $\mathcal{T}$ in the given ordering $\sigma(\mathcal{T})$. In the following, $(\sigma(\mathcal{T}),\overleftarrow{A}(\mathcal{T}))$ is called a *linear representation* of the tournament $\mathcal{T}$. For a backward arc $e=v_jv_i$ of $\sigma(\mathcal{T})$ the *span value* of e is $j-i-1$. Then `minspan`$(\sigma(\mathcal{T}))$ (resp. `maxspan`$(\sigma(\mathcal{T}))$) is simply the minimum (resp. maximum) of the span values of the backward arcs of $\sigma(\mathcal{T})$.

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A set $A' \subseteq A$ of arcs of $\mathcal{T}$ is a *feedback arc set* (or *FAS*) of $\mathcal{T}$ if every directed cycle of $\mathcal{T}$ contains at least one arc of A'. It is clear that for any linear representation $(\sigma(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$ of $\mathcal{T}$ the set $\overleftarrow{A}(\mathcal{T})$ is a FAS of $\mathcal{T}$. A tournament is *sparse* if it admits a FAS which is a matching. We denote by C_3-Packing-T the problem of packing the maximum number of vertex disjoint triangles in a given tournament, where a triangle is a directed 3-cycle. More formally, an input of C_3-Packing-T is a tournament $\mathcal{T}$, an output is a set (called a *triangle packing*) $S = \{t_i, i \in [|S|]\}$ where each t_i is a triangle and for any $i \neq j$ we have $V(t_i) \cap V(t_j) = \emptyset$, and the objective is to maximize $|S|$. We denote by $opt(\mathcal{T})$ the optimal value of $\mathcal{T}$. We denote by C_3-Perfect-Packing-T the decision problem associated to C_3-Packing-T where an input $\mathcal{T}$ is positive iff there is a triangle packing S such that $V(S) = V(\mathcal{T})$ (which is called a *perfect triangle packing*).

Related work

We refer the reader to Appendix where we recall the definitions of the problems mentioned bellow as well as the standard definitions about parameterized complexity and approximation. A first natural related problem is 3-Set-Packing as we can reduce C_3-Packing-T to 3-Set-Packing by creating an hyperedge for each triangle.

Classical complexity / approximation. It is known that C_3-Packing-T is polynomial if the tournament does not contain the three forbidden sub-tournaments described in [5]. From the point of view of approximability, the best approximation algorithm is the $\frac{4}{3}+\epsilon$ approximation of [7] for 3-Set-Packing, implying the same result for K_3-packing and C_3-Packing-T. Concerning negative results, it is known [10] that even K_3-packing is MAX SNP-hard on graphs with maximum degree four. We can also mention the related "dual" problem FAST and FVST that received a lot of attention with for example the NP-hardness and PTAS for FAS in [6] and [13] respectively, and the $\frac{7}{3}$ approximation and inapproximability results for FVST in [14].

Kernelization. We precise that the implicitly considered parameter here is the size of the solution. There is a $\mathcal{O}(k^2)$ vertex kernel for K_3-packing in [15], and even a $\mathcal{O}(k^2)$ vertex kernel for 3-Set-Packing in [1], which is obtained by only removing vertices of the ground set. This remark is important as it directly implies a $\mathcal{O}(k^2)$ vertex kernel for C_3-Packing-T (see Section 4). Let us now turn to negative results. There is a whole line of research dedicated to finding lower bounds on the size of polynomial kernels. The main tool involved in these bounds is the weak composition introduced in [11] (whose definition is recalled in Appendix). Weak composition allowed several almost tight lower bounds, with for example the $\mathcal{O}(k^{d-\epsilon})$ for d-Set-Packing and $\mathcal{O}(k^{d-4-\epsilon})$ for K_d-packing of [11]. These results where improved in [8] to $\mathcal{O}(k^{d-\epsilon})$ for perfect d-Set-Packing, $\mathcal{O}(k^{d-1-\epsilon})$ for K_d-packing, and leading to $\mathcal{O}(k^{2-\epsilon})$ for perfect K_3-packing. Notice that negative results for the "perfect" version of problems (where parameter $k = \frac{n}{d}$, where d is the number of vertices of the packed structure) are stronger than for the classical version where k is arbitrary. Kernel lower bound for these "perfect" versions is sometimes referred as *sparsification lower bounds*.

Our contributions

Our objective is to study the approximability and kernelization of C_3-Packing-T. On the approximation side, a natural question is a possible improvement of the $\frac{4}{3}+\epsilon$ approximation implied by 3-Set-Packing. We show that, unlike FAST, C_3-Packing-T does not admit a PTAS unless P=NP, even if the tournament is sparse. We point out that, surprisingly, we were not able to find any reference establishing a negative result for C_3-Packing-T, even for the NP-hardness. As these results show that there is not much room for improving the approximation ratio, we focus on sparse tournaments and followed a different approach by looking for a condition that would allow ratio arbitrarily close to 1. In that spirit, we provide a $(1+\frac{6}{c-1})$ approximation algorithm for sparse tournaments having a linear representation with `minspan` at least c. Concerning kernelization, we complete the panorama of sparsification lower bounds of [12] by proving that C_3-Perfect-Packing-T does not admit a kernel of (total bit) size $\mathcal{O}(n^{2-\epsilon})$ unless NP $\subseteq$ coNP/Poly. This implies that C_3-Packing-T does not admit a kernel of (total bit) size $\mathcal{O}(k^{2-\epsilon})$ unless NP $\subseteq$ coNP/Poly. We also prove that C_3-Packing-T admits a kernel of $\mathcal{O}(m)$ vertices, where m is the size of a given FAS of the instance, and that C_3-Packing-T restricted to sparse instances has a kernel in $\mathcal{O}(k)$ vertices (and so of total size bit $\mathcal{O}(k\log(k))$). The existence of a kernel in $\mathcal{O}(k)$ vertices for the general C_3-Packing-T remains our main open question.

2 Specific notations and observations

Given a linear representation $(\sigma(\mathcal{T}),\overleftarrow{A}(\mathcal{T}))$ of a tournament $\mathcal{T}$, a triangle t in $\mathcal{T}$ is a triple $t=(v_{i_1},v_{i_2},v_{i_3})$ with $i_l<i_{l+1}$ such that either $v_{i_3}v_{i_1}\in\overleftarrow{A}(\mathcal{T})$, $v_{i_3}v_{i_2}\notin\overleftarrow{A}(\mathcal{T})$ and $v_{i_2}v_{i_1}\notin\overleftarrow{A}(\mathcal{T})$ (in this case we call t a *triangle with backward arc* $v_{i_3}v_{i_1}$), or $v_{i_3}v_{i_1}\notin\overleftarrow{A}(\mathcal{T})$, $v_{i_3}v_{i_2}\in\overleftarrow{A}(\mathcal{T})$ and $v_{i_2}v_{i_1}\in\overleftarrow{A}(\mathcal{T})$ (in this case we call t a *triangle with two backward arcs* $v_{i_3}v_{i_2}$ and $v_{i_2}v_{i_1}$).

Given two tournaments $\mathcal{T}_1,\mathcal{T}_2$ defined by $\sigma(\mathcal{T}_l)$ and $\overleftarrow{A}(\mathcal{T}_l)$ we denote by $\mathcal{T}=\mathcal{T}_1\mathcal{T}_2$ the tournament called the concatenation of $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\sigma(\mathcal{T})=\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2)$ is the concatenation of the two sequences, and $\overleftarrow{A}(\mathcal{T})=\overleftarrow{A}(\mathcal{T}_1)\cup\overleftarrow{A}(\mathcal{T}_2)$. Given a tournament $\mathcal{T}$ and a subset of vertices X, we denote by $\mathcal{T}\setminus X$ the tournament $\mathcal{T}[V(\mathcal{T})\setminus X]$ induced by vertices $V(\mathcal{T})\setminus X$, and we call this operation *removing* X *from* $\mathcal{T}$. Given an arc $a=uv$ we define $h(a)=v$ as the head of a and $t(a)=u$ as the tail of a. Given a linear representation $(V(\mathcal{T}),\overleftarrow{A}(\mathcal{T}))$ and an arc $a\in\overleftarrow{A}(\mathcal{T})$, we define $s(a)=\{v:h(a)<v<t(a)\}$ as the *span* of a. Notice that the span value of a is then exactly $|s(a)|$.
Given a linear representation $(V(\mathcal{T}),\overleftarrow{A}(\mathcal{T}))$ and a vertex $v\in V(\mathcal{T})$, we define the degree of v by $d(v)=(a,b)$, where $a=|\{vu\in\overleftarrow{A}(\mathcal{T}):u<v\}|$ is called the *left degree* of v and $b=|\{uv\in\overleftarrow{A}(\mathcal{T}):u>v\}|$ is called the *right degree* of v. We also define $V_{(a,b)}=\{v\in V(\mathcal{T})|d(v)=(a,b)\}$. Given a set of pairwise distinct pairs D, we denote by C_3-Packing-T^D the problem C_3-Packing-T restricted to tournaments such that there exists a linear representation where $d(v)\in D$ for all v. Notice that when $D_M=\{(0,1),(1,0),(0,0)\}$, instances of C_3-Packing-T^{D_M} are the sparse tournaments.
Finally let us point out that it is easy to decide in polynomial time if a tournament is sparse or not, and if so, to give a linear representation whose FAS is a matching. The corresponding algorithm is detailed in Appendix in Lemma 23. Thus, in the following, when considering a sparse tournament we will assume that a linear ordering of it where backward arcs form a matching is also given.

3 Approximation for sparse tournaments

3.1 APX-hardness for sparse tournaments

In this subsection we prove that C_3-Packing-T^{D_M} is APX-hard by providing a L-reduction (see Definition 17 in appendix) from Max 2-SAT(3), which is known to be APX-hard [2, 3].

Recall that in the MAX 2-SAT(3) problem where each clause contains exactly 2 variables and each variable appears in at most 3 clauses (and at most twice positively and once negatively).

Definition of the reduction Let $\mathcal{F}$ be an instance of MAX 2-SAT(3). In the following, we will denote by n the number of variables in $\mathcal{F}$ and m the number of clauses. Let $\{x_i, 1 \in [n]\}$ be the set of variables of $\mathcal{F}$ and $\{C_j, j \in [m]\}$ its set of clauses.

We now define a reduction f which maps an instance $\mathcal{F}$ of MAX 2-SAT(3) to an instance $\mathcal{T}$ of C_3-PACKING-T^{D_M}. For each variable x_i with $i \in [n]$, we create a tournament L_i as follows and we call it *variable gadget*. We refer the reader to Figure 1 where an example of variable gadget is depicted. Let $\sigma(L_i) = (X_i, X'_i, \overline{X_i}, \overline{X_i}', \{\beta_i\}, \{\beta'_i\}, A_i, B_i, \{\alpha_i\}, A'_i, B'_i)$. We define $C = \{X_i, X'_i, \overline{X_i}, \overline{X_i}', A_i, B_i, A'_i, B'_i\}$. All sets of C have size 4. We denote $X_i = (x_i^1, x_i^2, x_i^3, x_i^4)$, and we extend the notation in a straightforward manner to the other others sets of C. Let us now define $\overleftarrow{A}(L_i)$. For each set of C, we add a backward arc whose head is the first element and the tail is the last element (for example for X_i we add the arc $x_i^4 x_i^1$). Then, we add to $\overleftarrow{A}(L_i)$ the set $\{e_1, e_2, e_3, e_4\}$ where $e_1 = x_i^3 a_i^3$, $e_2 = x_i'^3 a_i'^3$, $e_3 = \overline{x_i^3} b_i^3$, $e_4 = \overline{x_i'^3} b_i'^3$ and the set $\{m_1, m_2\}$ where $m_1 = a_i'^2 a_i^2$, $m_2 = b_i'^2 b_i^2$ called the two *medium arcs* of the variable gadget. This completes the description of tournament L_i. Let $L = L_1 \dots L_n$ be the concatenation of the L_i.

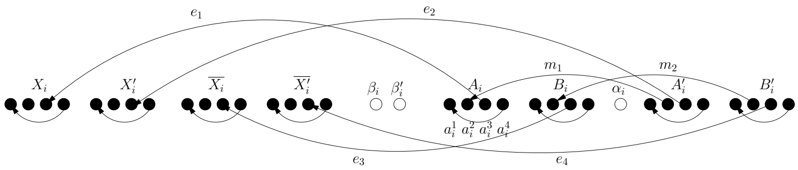

Figure 1 Example of a variable gadget L_i.

For each clause C_j with $j \in [1, m]$, we create a tournament K_j with ordering $\sigma(K_i) = (\theta_j, d_j^1, c_j^1, c_j^2, d_j^2)$ and $\overleftarrow{A}(K_i) = \{d_j^2 d_j^1\}$. We also define $K = K_1 \dots K_m$. Let us now define $\mathcal{T} = LK$. We add to $\overleftarrow{A}(\mathcal{T})$ the following backward arcs from $V(K)$ to $V(L)$. If $C_j = l_{i_1} \vee l_{i_2}$ is a clause in $\mathcal{F}$ then we add the arcs $c_j^1 v_{i_1}, c_j^2 v_{i_2}$ where v_{i_c} is the vertex in $\{x_{i_c}^2, x_{i_c}'^2, \overline{x_{i_c}^2}\}$ corresponding to l_{i_c}: if l_{i_c} is a positive occurrence of variable i_c we chose $v_{i_c} \in \{x_{i_c}^2, x_{i_c}'^2\}$, otherwise we chose $v_{i_c} = \overline{x_{i_c}^2}$. Moreover, we chose vertices v_{i_c} in such a way that for any $i \in [n]$, for each $v \in \{x_i^2, x_i'^2, \overline{x_i^2}\}$ there exists a unique arc $a \in \overleftarrow{A}(\mathcal{T})$ such that $h(a) = v$. This is always possible as each variable has at most two positive occurrences and one negative occurrence. Thus, x_i^2 represent the first positive occurrence of variable i, and $x_i'^2$ the second one. We refer the reader to Figure 2 where an example of the connection between variable and clause gadget is depicted.

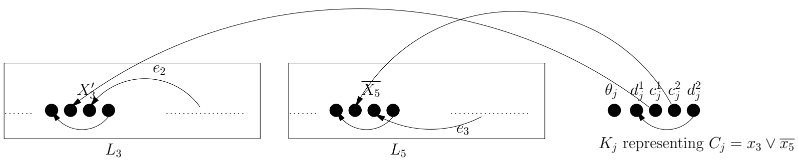

Figure 2 Example showing how a clause gadget is attached to variable gadgets.

Notice that vertices of $\overline{X'_i}$ are never linked to the clauses gadget. However, we need this set to keep the variable gadget symmetric so that setting x_i to true or false leads to the same number of triangles inside L_i. This completes the description of $\mathcal{T}$. Notice that the degree of any vertex is in $\{(0,1),(1,0),(0,0)\}$, and thus $\mathcal{T}$ is an instance of C_3-PACKING-T^{D_M}.

Let us now distinguish three different types of triangles in $\mathcal{T}$. A triangle $t=(v_1,v_2,v_3)$ of $\mathcal{T}$ is called an *outer* triangle iff $\exists j \in [m]$ such that $v_2=\theta_j$ and $v_3=c^l_j$ (implying that $v_1 \in V(L)$), *variable inner* iff $\exists i \in [n]$ such that $V(t) \subseteq V(L_i)$, and *clause inner* iff $\exists j \in [m]$ such that $V(t) \subseteq V(K_j)$. Notice that a triangle $t=(v_1,v_2,v_3)$ of $\mathcal{T}$ which is neither outer, variable or clause inner has necessarily $v_3=c^l_j$ for some j, and $v_2 \neq \theta_j$ (v_2 could be in $V(L)$ or $V(K)$). In the following definition, for any $Y \in C$ (where $C=\{X_i, X'_i, \overline{X_i}, \overline{X_i}', A_i, B_i, A'_i, B'_i\}$) with $Y=(y^1,y^2,y^3,y^4)$, we define $t^2_Y=(y^1,y^2,y^4)$ and $t^3_Y=(y^1,y^3,y^4)$. For example, $t^2_{X'_i}=(x'^1_i, x'^2_i, x'^4_i)$. For any $i \in [n]$, we define P_i and $\overline{P_i}$, two sets of vertex disjoint variable inner triangles of $V(L_i)$, by:

- $P_i=\{t^3_{X_i}, t^3_{X'_i}, t^2_{\overline{X_i}}, t^2_{\overline{X'_i}}, t^3_{A_i}, t^2_{B_i}, t^3_{A'_i}, t^2_{B'_i}, (h(e_3),\beta_i,t(e_3)), (h(e_4),\beta'_i,t(e_4)), (h(m_1),\alpha_i,t(m_1))\}$
- $\overline{P_i}=\{t^2_{X_i}, t^2_{X'_i}, t^3_{\overline{X_i}}, t^3_{\overline{X'_i}}, t^2_{A_i}, t^3_{B_i}, t^2_{A'_i}, t^3_{B'_i}, (h(e_1),\beta_i,t(e_1)), (h(e_2),\beta'_i,t(e_2)), (h(m_2),\alpha_i,t(m_2))\}$

Notice that P_i (resp. $\overline{P_i}$) uses all vertices of L_i except $\{x^2_i, x'^2_i\}$ (resp. $\{\overline{x^2_i}, \overline{x'^2_i}\}$). For any $j \in [m]$, and $x \in [2]$ we define the set of clause inner triangle of K_j, that is $Q^x_j=\{(d^1_j, c^x_j, d^2_j)\}$.

Informally, setting variable x_i to true corresponds to create the 11 triangles of P_i in L_i (as leaving vertices $\{x^2_i, x^{2'}_i\}$ available allows to create outer triangles corresponding to satisfied clauses), and setting it to false corresponds to create the 11 triangles of $\overline{P_i}$. Satisfying a clause j using its x^{th} literal (represented by a vertex $v \in V(L)$) corresponds to create triangle in Q^{3-x}_j as it leaves c^x_j available to create the triangle (v,θ_j,c^x_j). Our final objective (in Lemma 4) is to prove that satisfying k clauses is equivalent to find $11n+m+k$ vertex disjoint triangles.

Restructuration lemmas Given a solution S, let $I^L_i=\{t \in S : V(t) \subseteq V(L_i)\}$, $I^K_j=\{t \in S : V(t) \subseteq V(K_j)\}$, $I^L=\cup_{i\in[n]} I^L_i$ be the set of variable inner triangles of S, $I^K=\cup_{j\in[m]} I^K_j$ be the set of clause inner triangles of S, and $O=\{t \in S\ t \text{ is an outer triangle }\}$ be the set of outer triangles of S. Notice that *a priori* I^L, I^K, O does not necessarily form a partition of S. However, we will show in the next lemmas how to restructure S such that I^L, I^K, O becomes a partition.

▶ **Lemma 1.** *For any S we can compute in polynomial time a solution $S'=\{t'_l, l \in [k]\}$ such that $|S'| \geq |S|$ and for all $j \in [m]$ there exists $x \in [2]$ such that $I'^K_j=Q^x_j$ and*

- *either S' does not use any other vertex of K_j ($V(S') \cap V(K_j)=V(Q^x_j)$)*
- *either S' contains an outer triangle $t'_l=(v,\theta_j,c^{3-x}_j)$ with $v \in V(L)$ (implying $V(S') \cap V(K_j)=V(K_j)$)*

Proof. Consider a solution $S=\{t_l, l \in [k]\}$. Let us suppose that S does not verify the desired property. We say that $j \in [m]$ satisfies ($\star$) iff there exists $x \in [2]$ such that $I^K_j=Q^x_j$ and either S does not use any other vertex of K_j, or S contains an outer triangle $t_l=(v,\theta_j,c^{3-x}_j)$ with $v \in V(L)$.

Let us restructure S to increase the number of j satisfying ($\star$), which will be sufficient to prove the lemma. Consider the largest $j \in [m]$ which does not satisfy ($\star$). Let $c=|I^K_j|$. Notice that the only possible triangle of I^K_j contains $a=d^2_jd^1_j$, implying $c \leq 1$.

If $c=1$, let $t \in I^K_j$ and $v_0=\{c^1_j,c^2_j\} \backslash V(t)$. If $v_0 \notin V(S)$, then let us prove that $\theta_j \notin V(S)$. Indeed, by contradiction if $\theta_j \in V(S)$, let $t' \in S$ such that $\theta_j \in V(t')$. As $d(\theta_j)=(0,0)$ we necessarily have $t'=(u,\theta_j,w)$ with $w=c^{x'}_{j'}$ with $j' \geq j$, which contradicts the maximality of j. Otherwise ($v_0 \in V(S)$), then denoting by t' the triangle of S which contains v_0 we must have $t' = (u, v, v_0)$. Indeed, we cannot have (for some u', v') $t' = (v_0, u', v')$ as there is no backward arc a with $h(a) = v_0$ and we cannot have either $t' = (u', v_0, v')$ as this would imply $v' = c^{x'}_{j'}$ for $j' > j$ and again contradict the definition of j. As, again, by maximality of j we get $\theta_j \notin V(S)$ (and since $u\theta_j$ and $\theta_j v_0$ are forward arcs), we can replace t' by the triangle (u, θ_j, v_0) which is disjoint to the other triangles of S.

If $c = 0$. Notice first that by maximality of j, $d^2_j \notin V(S)$ as d^2_j could only be used in a triangle $t = (v, d^2_j, c^x_{j'})$ with $j' > j$. Let $Z = V(S) \cap \{c^1_j, c^2_j\}$. If $|Z| = 0$, then by maximality of j we get $d^1_j \notin V(S)$ and $\theta_j \notin V(S)$, and thus we add to S triangle (d^1_j, c^1_j, d^2_j). If $|Z| = 1$, let $c^x_j \in Z$ and $t \in S$ such that $c^x_j \in V(t)$. By maximality of j we necessarily have $t = (u, v, c^x_j)$ for some u, v. If $v \neq \theta_j$ then by maximality of j we have $\theta_j \notin V(S)$, and thus we swap v and θ_j in t and now suppose that $\theta_j \in V(t)$. This implies that $d^1_j \notin V(S)$ (before the swap we could have had $v = d^1_j$, but now by maximality of j we know that d^1_j is unused), and we add $(d^1_j, c^{3-x}_j, d^2_j)$ to S. It only remains now case where $|Z| = 2$. If there exists $t \in S$ with $Z \subseteq V(t)$, then $t = (u, c^1_j, c^2_j)$. Using the same arguments as above we get that $\{\theta_j, d^1_j\} \cap V(S) = \emptyset$, and thus we swap c^1_j by θ_j in t and add $(d^1_, c^1_j, d^2_j)$ to S. Otherwise, let $t_x \in S$ such that $c^x_j \in V(t_x)$ for $x \in [2]$. This implies that $t_x = (u_x, v_x, c^x_j)$. If $\theta_j \notin V(t_1) \cup V(t_2)$ then $\theta_j \notin V(S)$ and we swap v_1 with θ_j. Therefore, from now on we can suppose that $\theta_j \in V(t_x)$ for $x \in [2]$. Then, if $d^1_j \notin V(t_{3-x})$ then $d^1_j \notin V(S)$ and thus we swap v_{3-x} with d^1_j and we now assume that $d^1_j \in V(t_{3-x})$. Finally, we remove t_{3-x} from S and add instead $(d^1_j, c^{3-x}_j, d^2_j)$. ◀

▶ **Corollary 2.** *For any S we can compute in polynomial time a solution S' such that $|S'| \geq |S|$, and S' only contains outer, variable inner, and clause inner triangles. Indeed, in the solution S' of Lemma 1, given any $t \in S'$, either $V(t)$ intersects $V(K_j)$ for some j and then t is an outer or a clause inner triangle, or $V(t) \subseteq V(L_i)$ for $i \in [n]$ as there is no backward arc uv with $u \in V(L_{i_1})$ and $v \in V(L_{i_2})$ with $i_1 \neq i_2$.*

▶ **Lemma 3.** *For any S we can compute in polynomial time a solution S' such that $|S'| \geq |S|$, S' satisfies Lemma 1, and for every $i \in [n]$, $I'^L_i = P_i$ or $I'^L_i = \overline{P_i}$.*

Proof. Let S_0 be an arbitrary solution, and S be the solution obtained from S_0 after applying Lemma 1. By Corollary 2, we partition S into $S = I^L \cup I^K \cup O$. Let us say that $i \in [n]$ satisfies ($\star$) if $I^L_i = P_i$ or $I^L_i = \overline{P_i}$. Let us suppose that S does not verify the desired property, and show how to restructure S to increase the number of i satisfying ($\star$) while still satisfying Lemma 1, which will prove the lemma.

Let $Lft_i = X_i \cup X'_i \cup \overline{X_i} \cup \overline{X'_i}$ and $Rgt_i = A_i \cup B_i \cup \{\alpha_i\} \cup A'_i \cup B'_i$ be two subset of vertices of $V(L_i)$. Given any solution $\tilde{S}$ satisfying Lemma 1, we define the following sets. Let $\tilde{S}^{Lft_i} = \{t \in \tilde{I}^L_i : V(t) \subseteq Lft_i\}$, $\tilde{S}^{Rgt_i} = \{t \in \tilde{I}^L_i : V(t) \subseteq Rgt_i\}$, and $\tilde{S}^{Lft_i Rgt_i} = \{t \in \tilde{I}^L_i : V(t) \cap Lft_i \neq \emptyset \text{ and } V(t) \cap Rgt_i \neq \emptyset\}$. Observe that these three sets define a partition of $\tilde{I}^L_i$, and that triangles of $\tilde{S}^{Lft_i}$ are even included in W with $W \in \{X_i, X'_i, \overline{X_i}, \overline{X_i}'\}$. Let $\tilde{S}^{O_i} = \{t \in \tilde{O} : V(t) \cap V(L_i) \neq \emptyset\}$ be the set of outer triangles of $\tilde{S}$ intersecting L_i. We also define $g_i(\tilde{S}) = (|\tilde{S}^{Lft_i}|, |\tilde{S}^{Lft_i Rgt_i}|, |\tilde{S}^{Rgt_i}|, |\tilde{S}^{O_i}|)$ and $h_i(\tilde{S}) = |\tilde{S}^{Lft_i}| + |\tilde{S}^{Lft_i Rgt_i}| + |\tilde{S}^{Rgt_i}| + |\tilde{S}^{O_i}| = |\tilde{I}^L_i \cup \tilde{S}^{O_i}|$.

Our objective is to restructure S into a solution S' with $S' = (S \setminus (I^L_i \cup S^{O_i})) \cup (I'^L_i \cup S'^{O_i})$. We will define I'^L_i and S'^{O_i} verifying the following properties ($\triangle$):

$\triangle_1$ **:** $I'^L_i = P_i$ or $I'^L_i = \overline{P_i}$,
$\triangle_2$ **:** $S'^{O_i} \subseteq S^{O_i}$,
$\triangle_3$ **:** $|(I'^L_i \cup S'^{O_i})| \geq |(I^L_i \cup S^{O_i})|$ (which is equivalent to $h_i(S') \geq h_i(S)$),
$\triangle_4$ **:** triangles of $I'^L_i \cup S'^{O_i}$ are vertex disjoint.

Notice that $\triangle_2$ and $\triangle_4$ imply that all triangles of S' are still vertex disjoint. Indeed, as S satisfies Lemma 1, the only triangles of S intersecting L_i are $I_i^L \cup S^{O_i}$, and thus replacing them with $I_i^{'L} \cup S^{'O_i}$ satisfying the above property implies that all triangles of S' are vertex disjoint. Moreover, S' will still satisfy Lemma 1 even with $S^{'O_i} \subseteq S^{O_i}$ as removing outer triangles cannot violate property of Lemma 1. Finally $\triangle_3$ implies that $|S'| \geq |S|$. Thus, defining $I_i^{'L}$ and $S^{'O_i}$ satisfying $(\triangle)$ will be sufficient to prove the lemma. Let us now state some useful properties.

p_1 : $|S^{Lft_i}| \leq 4$

p_2 : $|S^{Lft_iRgt_i}| \leq 4$ as for any $t \in S^{Lft_iRgt_i}$ there exists $l \in [4]$ such that $V(t) \supseteq V(e_l)$.

p_3 : $|S^{Rgt_i}| \leq 5$ (as $|V(S^{Rgt_i})| = 17$). Let $Z = V(S^{Lft_iRgt_i}) \cap Rgt_i$. Let us also prove that if $Z \supseteq \{a_i^3, b_i^3\}$, $Z \supseteq \{a_i^{'3}, b_i^{'3}\}$, $Z \supseteq \{a_i^3, b_i^{'3}\}$ or $Z \supseteq \{a_i^{'3}, b_i^3\}$ then $|S^{Rgt_i}| \leq 4$. For any $W \in \{A_i, B_i, A'_i, B'_i\}$, let s_W be the unique arc a of $\mathcal{T}$ such that $V(a) \subseteq W$ and let m_W be the unique medium arc a such that $V(a) \cap W \neq \emptyset$. Let us call the $\{s_W\}$ the four small arcs of the tournament induced by Rgt_i. Let $\overleftarrow{A}(S^{Rgt_i}) = \{a \in \overleftarrow{A}(L_i) : \exists t \in S^{Rgt_i}$ such that $V(a) \subseteq V(t)\}$ be the set of backward arcs used by S^{Rgt_i}. Observe that arcs of $\overleftarrow{A}(S^{Rgt_i})$ are small or medium arcs. Let us bound $|\overleftarrow{A}(S^{Rgt_i})| = |S^{Rgt_i}|$. Notice that for any $W \in \{A_i, B_i, A'_i, B'_i\}$, $W \cap Z \neq \emptyset$ implies that $\overleftarrow{A}(S^{Rgt_i})$ cannot contain both s_W and m_W. If S^{Rgt_i} contains the 4 small arcs then by previous remark S^{Rgt_i} cannot contain any medium arc, and thus $|S^{Rgt_i}| \leq 4$. If S^{Rgt_i} contains 3 small arcs then it can only contain one medium arc, implying $|S^{Rgt_i}| \leq 4$. Obviously, if $|S^{Rgt_i}|$ contains 2 or less small arcs then $|S^{Rgt_i}| \leq 4$.

p_4 : property p_3 implies that if $|S^{Lft_iRgt_i}| \geq 3$, or if $|S^{Lft_iRgt_i}| = 2$ and triangles of $S^{Lft_iRgt_i}$ contain $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_3\}$ or $\{e_2, e_4\}$, then $|S^{Rgt_i}| \leq 4$ (where triangles of $S^{Lft_iRgt_i}$ contains $\{e_i, e_j\}$ means that there exist t_1, t_2 in $S^{Lft_iRgt_i}$ such that $V(t_1) \supseteq V(e_i)$ and $V(t_2) \supseteq V(e_j)$).

p_5 : $|S^{O_i}| \leq 3$. Moreover, if $|S^{Lft_i}| = 4$ then $|S^{O_i}| \leq 4 - |S^{Lft_iRgt_i}|$, and if $|S^{Lft_i}| = 3$ and $|S^{Lft_iRgt_i}| = 4$ then $|S^{O_i}| \leq 1$. The last two inequalities come from the fact that for any $W \in \{X_i, X'_i, \overline{X_i}, \overline{X'_i}\}$, we cannot have both $t_1 \in S^{O_i}$, $t_2 \in S^{Lft_iRgt_i}$ and $t_3 \in S^{Lft_i}$ with $V(t_i) \cap W \neq \emptyset$.

Notice that if a solution S' satisfies $I_i^{'L} = P_i$ or $I_i^{'L} = \overline{P_i}$ then $g_i(S') = (4, 2, 5, z)$ where $z \in [2]$, and $h_i(S') = 11 + z$. In the following we write $(u_1^1, u_2^1, u_3^1, u_4^1) \leq (u_1^2, u_2^2, u_3^2, u_4^2)$ iff $u_i^1 \leq u_i^2$ for any $i \in [4]$. Let us describe informally the following argument which will be used several times. Let $z = |S^{O_i}|$. If $z \leq 1$ or if $z = 2$ but the two corresponding outer triangles do not use one vertex in $X_i \cup X'_i$ and one vertex in $\overline{X_i}$, then we will able to "save" all these outer triangles (while creating the optimal number of variable inner triangles in L_i), meaning that $S^{'O_i} = S^{O_i}$, as either P_i or $\overline{P_i}$ will leave vertices of $S^{O_i} \cap Lft_i$ available for outer triangles. Let us proceed by case analysis according to the value $|S^{Lft_iRgt_i}|$. Remember that $|S^{Lft_iRgt_i}| \leq 4$ according to p_2.

Case 1: $|S^{Lft_iRgt_i}| \leq 1$. According to p_1, p_3 we get $g_i(S) \leq (4, 1, 5, z)$ where $z \in [3]$. In this case, $S^{'O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$ and $I_i^{'L} = P_i$ verify $(\triangle)$. In particular, we have $h_i(S') \geq h_i(S)$ as $g_i(S') \geq (4, 2, 5, z - 1)$.

Case 2: $|S^{Lft_iRgt_i}| = 2$. Let $g_i(S) = (x, 2, y, z)$. If $x \leq 3$, then $g_i(S) \leq (3, 2, 5, z)$ by p_3 and we set $S^{'O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$ and $I_i^{'L} = P_i$. This satisfies $(\triangle)$ as in particular we have $h_i(S') \geq h_i(S)$ as $g_i(S') \geq (4, 2, 5, z - 1)$. Let us now turn to case where $x = 4$. Let $S^{Lft_iRgt_i} = \{t_1, t_2\}$. Let us first suppose that triangles of $S^{Lft_iRgt_i}$ contain $\{e_i, e_j\}$ with $\{e_i, e_j\} \in \{\{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}\}$. By p_4 we get $y \leq 4$, implying $g_i(S) \leq (4, 2, 4, z)$. In this case, $S^{'O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$ and $I_i^{'L} = P_i$ verify $(\triangle)$.

In particular, we have $h_i(S') \geq h_i(S)$ as $g_i(S') = (4, 2, 5, z-1)$. Let us suppose now that t_1 contains e_1 and t_2 contains e_2 (case (2a)), or t_1 contains e_3 and t_2 contains e_4 (case (2b)). In both cases we have $g_i(S) \leq (4, 2, 5, z)$ where $z \in [2]$ by p_5. More precisely, p_5 implies that $\{W \in \{X_i, X'_i, \overline{X_i}, \overline{X'_i}\} : W \cap V(S^{O_i})\} \neq \emptyset$ is included in $\{X, X'_i\}$ (case 2b) or in $\overline{X_i}$ (case 2a). Thus, in case (2a) we define $S'^{O_i} = S^{O_i}$ and $I'^{L}_i = \overline{P_i}$. In case (2b) we define $S'^{O_i} = S^{O_i}$ and $I'^{L}_i = P_i$. In both cases these sets verify ($\triangle$) as in particular $g_i(S') = (4, 2, 5, z)$.

Case 3: $|S^{Lft_iRgt_i}| = 3$. In this case $g_i(S) \leq (x, 3, 4, z)$ by p_4. If $x \leq 3$, the sets $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$ and $I'^{L}_i = P_i$ verify ($\triangle$). In particular, we have $h_i(S') \geq h_i(S)$ as $g_i(S') \geq (4, 2, 5, z-1)$. If $x = 4$ then $z \leq 1$ by p_5. Thus, we define $I'^{L}_i = P_i$ if $V(S^{O_i}) \cap (X_i \cup X'_i) \neq \emptyset$, and $I'^{L}_i = \overline{P_i}$ otherwise, and $S'^{O_i} = S^{O_i}$. These sets satisfy ($\triangle$) as in particular $g_i(S') = (4, 2, 5, z)$.

Case 4: $|S^{Lft_iRgt_i}| = 4$. Let $g_i(S) = (x, 4, y, z)$. If $x = 4$ then $z \leq 0$ by p_5 and $y \leq 3$ as $x + 4 + y \leq \frac{|V(L_i)|}{3}$.

Thus, we set $S'^{O_i} = S^{O_i} = \emptyset$, $I'^{L}_i = P_i$ (which is arbitrary in this case), and we have property ($\triangle$) as $g_i(S') \geq (4, 2, 5, 0)$. If $x = 3$ (this case is depicted Figure 3) then $y \leq 4$ by p_3 and $z \leq 1$ by p_5, implying $g_i(S) = (3, 4, 4, z)$. Thus, we define $I'^{L}_i = P_i$ if $V(S^{O_i}) \cap (X_i \cup X'_i) \neq \emptyset$, and $I'^{L}_i = \overline{P_i}$ otherwise, and $S'^{O_i} = S^{O_i}$. These sets satisfy ($\triangle$) as in particular $g_i(S') = (4, 2, 5, z)$. Finally, if $x \leq 2$ then $g_i(S) \leq (2, 4, 4, z)$ by p_3. In this case, $S'^{O_i} = S^{O_i} \setminus \{t \in S : V(t) \ni \overline{x_i^2}\}$ and $I'^{L}_i = P_i$ verify ($\triangle$). In particular, we have $h_i(S') \geq h_i(S)$ as $g_i(S') \geq (4, 2, 5, z-1)$.

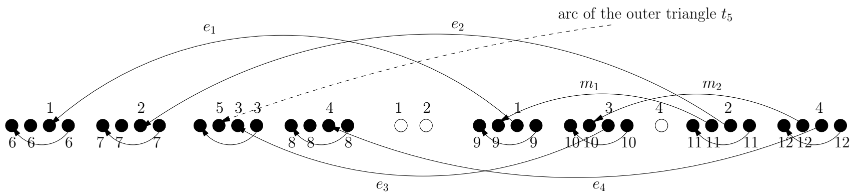

Figure 3 Example showing a "bad shaped" solution of case 4 with $g_i(S) = (3, 4, 4, 1)$. We have $S^{Lft_iRgt_i} = \{t_1, t_2, t_3, t_4\}$, $S^{O_i} = \{t_5\}$, $S^{Lft_i} = \{t_6, t_7, t_8\}$ and $S^{Rgt_i} = \{t_9, t_{10}, t_{11}, t_{12}\}$. The three vertices of triangle t_l are annotated with label l.

◀

Proof of the L-reduction We are now ready to prove the main lemma (recall that f is the reduction from Max 2-SAT(3) to C_3-Packing-T^{D_M} described in Section 3.1), and also the main theorem of the section.

▶ **Lemma 4.** *Let $\mathcal{F}$ be an instance of* Max 2-SAT(3)*. For any k, there exists an assignment a of $\mathcal{F}$ satisfying at least k clauses if and only if there exists a solution S of $f(\mathcal{F})$ with $|S| \geq 11n + m + k$, where n and m are respectively is the number of variables and clauses in $\mathcal{F}$. Moreover, in the $\Leftarrow$ direction, assignment a can be computed from S in polynomial time.*

Proof. For any $i \in [n]$, let $A_i = P_i$ if x_i is set to true in a, and $A_i = \overline{P_i}$ otherwise. We first add to S the set $\cup_{i \in [n]} A_i$. Then, let $\{C_{j_l}, l \in [k]\}$ be k clauses satisfied by a. For any $l \in [k]$, let i_l be the index of a literal satisfying C_{j_l}, let $x \in [2]$ such that $c^x_{j_l}$ corresponds to this literal, and let $Z_l = \{x^2_{i_l}, x'^2_{i_l}\}$ if literal i_l is positive, and $Z_l = \{\overline{x^2_{i_l}}\}$ otherwise. For any $j \in [m]$, if $j = i_l$ for some l (meaning that j corresponds to a satisfied clause), we add to S the triangle in Q^{3-x}_j, and otherwise we arbitrarily add the triangle Q^1_j. Finally, for any

$l \in [k]$ we add to S triangle $t_l = (y_l, \theta_{j_l}, c^x_{j_l})$ where $y_l \in Z_l$ is such that y_l is not already used in another triangle. Notice that such an y_l always exists as triangles of $A_i, i \in [n]$ do not intersect Z_l (by definition of the A_i), and as there are at most two clauses that are true due to positive literal, and one clause that is true due to a negative literal. Thus, S has $11n + m + k$ vertex disjoint triangles.

Conversely, let S a solution of $f(\mathcal{F})$ with $|S| \geq 11n+m+k$. By Lemma 3 we can construct in polynomial time a solution S' from S such that $|S'| \geq |S|$, S' only contains outer, variable or clause inner triangles, for each $j \in [m]$ there exists $x \in [2]$ such that $I'^K_j = Q^x_j$, and for each $i \in [n], I'^L_i = P_i$ or $I'^L_i = \overline{P_i}$. This implies that the $k' \geq k$ remaining triangles must be outer triangles. Let $\{t'_l, l \in [k']\}$ be these k' outer triangles with $t'_l = (y_l, \theta_{j_l}, c^{x_l}_{j_l})$ Let us define the following assignation a: for each $i \in [n]$, we set x_i to true if $I'^L_i = P_i$, and false otherwise. This implies that a satisfies at least clauses $\{C_{j_l}, l \in [k']\}$. ◀

▶ **Theorem 5.** *C_3-PACKING-T^{D_M} is* APX*-hard, and thus does not admit a* PTAS *unless $P =$* NP.

Proof. Let us check that Lemma 4 implies a L-reduction (whose definition is recalled in Definition 17 of appendix). Let OPT_1 (resp. OPT_2) be the optimal value of $\mathcal{F}$ (resp. $f(\mathcal{F})$). Notice that Lemma 4 implies that $OPT_2 = OPT_1 + 11n + m$. It is known that $OPT_1 \geq \frac{3}{4}m$ (where m is the number of clauses of $\mathcal{F}$). As $n \leq m$ (each variable has at least one positive and one negative occurrence), we get $OPT_2 = OPT_1 + 11n + m \leq \alpha OPT_1$ for an appropriate constant α, and thus point (a) of the definition is verified. Then, given a solution S' of $f(\mathcal{F})$, according to Lemma 4 we can construct in polynomial time an assignment a satisfying $c(a)$ clauses with $c(a) \geq S' - 11n - m$. Thus, the inequality (b) of Definition 17 with $\beta = 1$ becomes $OPT_1 - c(a) \leq OPT_2 - S' = OPT_1 + 11n + m - S'$, which is true. ◀

Reduction of Theorem 5 does not imply the NP-hardness of C_3-PERFECT-PACKING-T as there remain some unused vertices. However, it is straightforward to adapt the reduction by adding backward arcs whose head (resp. tail) are before (resp. after) $\mathcal{T}$ to consume the remaining vertices. This leads to the following result.

▶ **Theorem 6.** *C_3-PERFECT-PACKING-T^{D_M} is* NP*-hard.*

Proof. Let $(\mathcal{F}, k)$ be an instance of the decision problem of $MAX-2-SAT(3)$ and let $\mathcal{T} = f(\mathcal{F})$ be the tournament defined in Section 3.1. Recall that we have $\mathcal{T} = LK$. Let $N = |V(T)| = 35n + 5m$, $x^* = 33n + 3m + 3k$ and $n' = N - x^*$. We now define $\mathcal{T}'$ by adding $2n'$ new vertices in $\mathcal{T}$ as follows: $V(\mathcal{T}') = R_1 V(\mathcal{T}) R_2$ with $R_i = \{r^l_i, l \in [n']\}$. We add to $\overleftarrow{A}(\mathcal{T}')$ the set of arcs $R = \{(r^l_2 r^l_1), l \in [n']\}$ which are called the dummy arcs. We say that a triangle $t = (u, v, w)$ is dummy iff $(wu) in R$ and $v \in V(\mathcal{T})$. Let us prove that there are at least k clauses satisfiable in $\mathcal{F}$ iff there exists a perfect packing in $\mathcal{T}'$.

$\Rightarrow$

Given an assignement satisfying k clause we define a solution S with $V(S) \subseteq V(\mathcal{T})$ as in Lemma 4 (triangles of P_i or $\overline{P_i}$ for each $i \in [n]$, a triangle Q^x_j for each $j \in [m]$, and an outer triangle t_l with $l \in [k]$ for each satisfied clause. We have $|S| = 11n + m + k$. This implies that $|V(\mathcal{T}) \setminus V(S)| = n'$, and thus we use n' remaining vertices of $V(\mathcal{T})$ by adding to S n' dummy triangles.

$\Leftarrow$

Let S' be a perfect packing of $\mathcal{T}'$. Let $S = \{t \in S' : V(t) \subseteq V(\mathcal{T})\}$. Let $X = V(\mathcal{T}) \setminus V(S)$. As S' is a perfect packing of $\mathcal{T}'$, vertices of X must be used by $|X|$ dummy triangles of S', implying $|X| \leq n'$ and $|S| \geq 11n + m + k$. As S is set of vertex disjoint triangles of $\mathcal{T}$ of size at least $11n + m + k$, this implies by Lemma 4 that at least k clauses are satisfiable in $\mathcal{F}$.

◀

To establish the kernel lower bound of Section 4, we also need the NP-hardness of C_3-Perfect-Packing-T where instances have a slightly simpler structure (to the price of losing the property that there exists a FAS which is a matching).

▶ **Theorem 7.** C_3-Perfect-Packing-T *remains* NP*-hard even restricted to tournament* $\mathcal{T}$ *admitting the following linear ordering.*

- $\mathcal{T} = LK$ *where* L *and* K *are two tournaments*
- *tournaments* L *and* K *are "fixed":*
  - $K = K_1 \dots K_m$ *for some* m*, where for each* $j \in [m]$ *we have* $V(K_j) = (\theta_j, c_j)$
  - $L = R_1 L_1 \dots L_n R_2$*, where each* L_i *has is a copy of the variable gadget of Section 3.1,* $R_i = \{r_i^l, l \in [n']\}$ *where* $n' = 2n - m$*, and in addition* $\overleftarrow{L}$ *also contains* $R = \{(r_2^l r_1^l), l \in [n']\}$ *which are called the dummy arcs.*

Proof. We adapt the reduction of Section 3.1, reducing now from 3-SAT(3) instead of MAX 2-SAT(3). Given $\mathcal{F}$ be an instance of 3-SAT(3) with n variables $\{x_i\}$ nd m clauses $\{C_j\}$. For each variable x_i with $i \in [n]$, we create a tournament L_i exactly as in Section 3.1 and we define $L = L_1 \dots L_n$. For each clause C_j with $j \in [m]$, we create a tournament K_j with $V(K_j) = (\theta_j, c_j)$, and we define $K = K_1 \dots K_m$. Let us now define $\mathcal{T} = LK$. Now, we add to $\overleftarrow{A}(\mathcal{T})$ the following backward arcs from $V(K)$ to $V(L)$ (again, we follow the construction of Section 3.1 except that now each c_j has degree $(3, 0)$). If $C_j = l_{i_1} \vee l_{i_2} \vee l_{i_3}$ is a clause in $\mathcal{F}$ then we add the arcs $c_j v_{i_1}, c_j v_{i_2}, c_j v_{i_3}$ where v_{i_c} is the vertex in $\{x_{i_c}^2, x_{i_c}'^2, \overline{x_{i_c}^2}\}$ corresponding to l_{i_c}: if l_{i_c} is a positive occurrence of variable i_c we chose $v_{i_c} \in \{x_{i_c}^2, x_{i_c}'^2\}$, otherwise we chose $v_{i_c} = \overline{x_{i_c}^2}$. Moreover, we chose vertices v_{i_c} in such a way that for any $i \in [n]$, for each $v \in \{x_i^2, x_i'^2, \overline{x_i^2}\}$ there exists a unique arc $a \in \overleftarrow{A}(\mathcal{T})$ such that $h(a) = v$. This is always possible as each variable has at most 2 positive occurrences and 1 negative one.

Finally, we add $2n'$ new vertices in $\mathcal{T}$ as follows: $V(T) = R_1 V(L) R_2 V(K)$, $R_i = \{r_i^l, l \in [n']\}$ where $n' = 2n - m$. We add to $\overleftarrow{A}(\mathcal{T})$ the set of arcs $R = \{(r_2^l r_1^l), l \in [n']\}$ which are called the dummy arcs. Notice that $\mathcal{T}$ satisfies the claimed structure (defining the left part as $R_1 L R_2$ and not only L). We define an outer and variable inner triangle as in Section 3 (there are no more clause inner triangle), and in addition we say that a triangle $t = (u, v, w)$ is dummy iff $(wu) \in R$ and $v \in V(L)$. Let us prove that there is an assignment satisfying the m clauses of $\mathcal{F}$ iff $\mathcal{T}$ has a perfect packing.

$\Rightarrow$

Given an assignment satisfying the m clauses we define a solution S containing only outer, variable inner and dummy triangles. The variable inner triangle are defined as in Lemma 4 (triangles of P_i or $\overline{P_i}$ for each $i \in [n]$). For each clause $j \in [m]$ satisfied by a literal l_{i_x} we create an outer triangle (v_{i_x}, θ_j, c_j). It remains now $2n - m = n'$ vertices of L, that we use by adding n' dummy triangles to S.

$\Leftarrow$

Let S be a perfect packing of $\mathcal{T}'$. Notice that restructuration lemmas of Section 3 do not directly remain true because of the dummy arcs. However, we can adapt in a straightforward manner arguments of these lemmas, using the fact that S is even a perfect packing. Given a solution S, we define as in Section 3 set $I_i^L = \{t \in S : V(t) \subseteq V(L_i)$, $I^L = \cup_{i \in [n]} I_i^L$, $O = \{t \in S\ t$ is an outer triangle $\}$, and $D = \{t \in S\ t$ is a dummy triangle $\}$. Again, we do not claim (at this point) that S does not contain other triangles. Given any perfect packing S of $\mathcal{T}$, we can prove the following properties.

- S must contain exactly m outer triangles ($|O| = m$). Indeed, for any j from m to 1, the only way to use θ_j is to create an outer triangle (u_j, θ_j, c_j). This implies that triangles of O consume exactly m disjoint vertices in L.
- for any $i \in [n]$, we must have $|I_i^L| = 11$. Indeed, let x be the number of vertices of L used in S (as S is a perfect packing we know that $x = |L| = 35n$). The only triangles of S that can use a vertex of L are the outer, the variable inner and the dummy triangles, implying $x \leq (\sum_{i\in[n]} |I_i^L|) + m + n'$ as $|D| \leq n'$. As $|V(L_i)| = 35$ we have $|I_i^L| \leq 11$ and thus we must have $|I_i^L| = 11$ for any i.

Let us now consider the tournament $\mathcal{T}_0 = \mathcal{T}[V(\mathcal{T}) \setminus V(R)]$ without the dummy arcs, and $S_0 = \{t \in S : V(t) \subseteq V(\mathcal{T}_0)\}$. We adapt in a straightforward way the notion of variable inner and outer triangle in $\mathcal{T}_0$. Observe that the variable inner and outer triangles of S and S_0 are the same, and thus are both denoted respectively I_i^L and S^{O_i}. In particular, S_0 still contains m outer triangle of $\mathcal{T}_0$. Now we simply apply proof of Lemma 3 on S_0. More precisely, Lemma 3 restructures S_0 into a solution S'_0 with $S'_0 = (S_0 \setminus (I_i^L \cup S^{O_i})) \cup (I_i'^L \cup S'^{O_i})$, where $I_i'^L$ and S'^{O_i} satisfy properties ($\triangle$). In particular, as $|I_i^L| = |I_i'^L| = 11$, $\triangle_3$ implies that $|S_0'^{O_i}| \geq |S_0^{O_i}|$, and thus that $|S_0'^{O}| \geq |S_0^{O}| = m$. Thus, S'_0 satisfies $I_i^L = P_i$ or $I_i^L = \overline{P_i}$ for any i, and has m outer triangles. We can now define as in Lemma 4 from S'_0 an assignment satisfying the m clauses.

◀

3.2 $(1 + \frac{6}{c-1})$-approximation when backward arcs have large minspan

Given a set of pairwise distinct pairs D and an integer c, we denote by C_3-Packing-$\mathrm{T}^D_{\geq c}$ the problem C_3-Packing-T^D restricted to tournaments such that there exists a linear representation of minspan at least c and where $d(v) \in D$ for all v. In all this section we consider an instance $\mathcal{T}$ of C_3-Packing-$\mathrm{T}^{D_M}_{\geq c}$ with a given linear ordering $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$ of minspan at least c and whose degrees belong to D_M. The motivation for studying the approximability of this special case comes from the situation of MAX-SAT(c) where the approximability becomes easier as c grows, as the derandomized uniform assignment provides a $\frac{2^c}{2^c-1}$ approximation algorithm. Somehow, one could claim that MAX-SAT(c) becomes easy to approximate for large c as there many ways to satisfy a given clause. As the same intuition applies for tournament admitting an ordering with large minspan (as there are $c-1$ different ways to use a given backward in a triangle), our objective was to find a polynomial approximation algorithm whose ratio tends to 1 when c increases.

Let us now define algorithm Φ. We define a bipartite graph $G = (V_1, V_2, E)$ with $V_1 = \{v_a^1 : a \in \overleftarrow{A}(\mathcal{T})\}$ and $V_2 = \{v_l^2 : v_l \in V_{(0,0)}\}$. Thus, to each backward arc we associate a vertex in V_1 and to each vertex v_l with $d(v_l) = (0,0)$ we associate a vertex in V_2. Then, $\{v_a^1, v_l^2\} \in E$ iff $(h(a), v_l, t(a))$ is a triangle in $\mathcal{T}$.

In phase 1, Φ computes a maximum matching $M = \{e_l, l \in [|M|]\}$ in G. For every $e_l = \{v_{ij}^1, v_l^2\} \in M$ create a triangle $t_l^1 = (v_j, v_l, v_i)$. Let $S^1 = \{t_l^1, l \in [|M|]\}$. Notice that triangles of S^1 are vertex disjoint. Let us now turn to phase 2. Let $\mathcal{T}^2$ be the tournament $\mathcal{T}$ where we removed all vertices $V(S^1)$. Let $(V(\mathcal{T}^2), \overleftarrow{A}(\mathcal{T}^2))$ be the linear ordering of $\mathcal{T}^2$ obtained by removing $V(S^1)$ in $(V(\mathcal{T}), \overleftarrow{A}(\mathcal{T}))$. We say that three distinct backward edges $\{a_1, a_2, a_3\} \subseteq \overleftarrow{A}(\mathcal{T}^2)$ can be packed into triangles t_1 and t_2 iff $V(\{t_1, t_2\}) = V(\{a_1, a_2, a_3\})$ and the t_i are vertex disjoint. For example, if $h(a_1) < h(a_2) < t(a_1) < h(a_3) < t(a_2) < t(a_3)$, then $\{a_1, a_2, a_3\}$ can be packed into $(h(a_1), h(a_2), t(a_1))$ and $(h(a_3), t(a_2), t(a_3))$ (recall that when $\overleftarrow{A}(\mathcal{T})$ form a matching, (u, v, w) is triangle iff $wu \in \overleftarrow{A}(\mathcal{T})$ and $u < v < w$), and if $h(a_1) < h(a_2) < t(a_2) < h(a_3) < t(a_3) < t(a_1)$, then $\{a_1, a_2, a_3\}$ cannot be packed into two triangles. In phase 2, while it is possible, Φ finds a triplet of arcs of $Y \subseteq \overleftarrow{A}(\mathcal{T}^2)$ that can be packed into triangles, create the two corresponding triangles, and remove $V(Y)$. Let S^2 be the triangle created in phase 2 and let $S = S^1 \cup S^2$.

▶ Observation 8. For any $a \in \overleftarrow{A}(\mathcal{T})$, either $V(a) \subseteq V(S)$ or $V(a) \cap V(S) = \emptyset$. Equivalently, no backward arc has one endpoint in $V(S)$ and the other outside $V(S)$.

According to Observation 8, we can partition $\overleftarrow{A}(\mathcal{T}) = \overleftarrow{A}_0 \cup \overleftarrow{A}_1 \cup \overleftarrow{A}_2$, where for $i \in \{1,2\}$, $\overleftarrow{A}^i = \{a \in \overleftarrow{A}(\mathcal{T}) : V(a) \subseteq V(S^i)$ is the set of arcs used in phase i, and $\overleftarrow{A}_0 =_{def} \{b_i, i \in [x]\}$ are the remaining unused arcs. Let $\overleftarrow{A}_\Phi = \overleftarrow{A}_1 \cup \overleftarrow{A}_2$, $m_i = |\overleftarrow{A}_i|$, $m = m_0 + m_1 + m_2$ and $m_\Phi = m_1 + m_2$ the number of arcs (entirely) consumed by Φ. To prove the $1 + f(\frac{6}{c-1})$ desired approximation ratio, we will first prove in Lemma 9 that Φ uses at most all the arcs ($m_A \geq (1-\epsilon(c))m$), and in Theorem 10 that the number of triangles made with these arcs is "optimal". Notice that the latter condition is mandatory as if Φ used its m_Φ arcs to only create $\frac{2}{3}(m_\Phi)$ triangles in phase 2 instead of creating $m' \approx m_\Phi$ triangle with m' backward arcs and m' vertices of degree $(0,0)$, we would have a $\frac{3}{2}$ approximation ratio.

▶ **Lemma 9.** *For any $c \geq 2$, $m_\Phi \geq (1 - \frac{6}{c+5})m$*

Proof. In all this proof, the span $s(a)$ is always considered in the initial input $\mathcal{T}$, and not in $\mathcal{T}^2$. For any $i \in [x]$, let us associate to each $b_i \in \overleftarrow{A}_0$ a set $B_i \subseteq \overleftarrow{A}_\Phi$ defined as follows (see Figure 4 for an example). Let $b_j \in \overleftarrow{A}_0$ such that $s(b_j) \subseteq s(b_i)$ and there does not exist a $b_k \in \overleftarrow{A}_0$ such that $s(b_k)$ included in $s(b_j)$ (we may have $b_j = b_i$). Let $Z = V(\overleftarrow{A}_0) \cap s(b_j)$. Notice that $|Z| \leq 1$, meaning that there is at most one endpoint of a $b_l, l \neq j$ in $s(b_j)$, as otherwise we would be three arcs in $\overleftarrow{A}_0$ that could be packed in two triangles. If there exists $a \in \overleftarrow{A}_\Phi$ with $s(a) \subseteq s(b_j)$ we define $a_0 = a$, and otherwise we define $a_0 = b_j$. Now, let $v \in s(a_0) \setminus Z$. Observe that $V(\mathcal{T})$ is partitioned into $V(\overleftarrow{A}_0) \cup V(\overleftarrow{A}_\Phi) \cup V_{(0,0)}$. If $v \in V_{(0,0)}$, then there exists $t^1_l = (u, v, w)$ with $wu \in \overleftarrow{A}_1$ (as otherwise the matching in phase 1 would not be maximal and we could add b_j and v), and we add wu to B_i. Otherwise, $v \in V(a)$ with $a \in \overleftarrow{A}_\Phi$ (this arcs could have been used in phase 1 or phase 2), and we add a to B_i. Notice that as a_0 does not properly contains another arc of $\overleftarrow{A}_\Phi$, all the added arcs are pairwise distinct, and thus $|B_i| = |s(a_0) \setminus Z| \geq c - 1$.

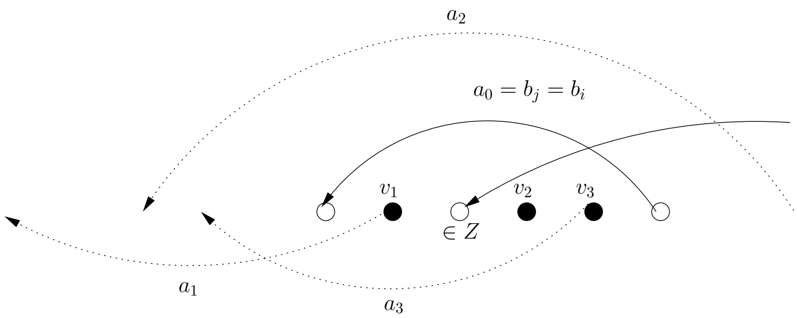

Figure 4 On this example white vertices represent $V(\mathcal{T}) \setminus V(S)$ (vertices not used by Φ), and black ones represent $V(S)$. In this case we have $B_i = \{a_l, l \in [3]\}$. Indeed, each $v_l \in s(a_0) \setminus Z$, for $l \in [3]$, brings a_l in B_i. In particular $v_2 \in V_{(0,0)}$ and was used with a_2 to create a triangle in phase 1.

Given $a \in \overleftarrow{A}_\Phi$, let $B(a) = \{B_i, a \in B_i\}$. Let us prove that $|B(a)| \leq 6$ for any $a \in \overleftarrow{A}_\Phi$. For any $v \in V(S)$, let $d_B(v) = |\{b_i : v \in s(b_i)\}|$. Observe that $d_B(v) \leq 2$, as otherwise any

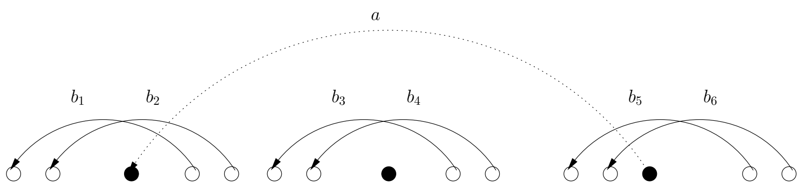

Figure 5 Example where $|B(a)| = 6$ for $a \in \overleftarrow{A}_\Phi$, where $B(a) = \{b_l, l \in [6]\}$.

triplet of arcs containing v in their span could be packed into two triangles (there are only 6 cases to check according to the 3! possible ordering of the tail of these 3 arcs). For any $a \in \overleftarrow{A}_1$, let $V'(a) = V(t^a)$ where $t^a \in S$ is the triangle containing a, and for any $a \in A_2$, let $V'(a) = V(a)$. Observe that by definition of the B_i, $a \in B_i$ implies that b_i contributes to the degree $d_B(v)$ for a $v \in V'(a)$. As in particular $d_B(v)$ for any $v \in V'(a)$, this implies by pigeonhole principle that $|B(a)| \leq 6$ (notice that this bound is tight as depicted Figure 5). Thus, if we consider the bipartite graph with vertex set $(\overleftarrow{A}_0, \overleftarrow{A}_\Phi)$ and an edge between $b_i \in \overleftarrow{A}_0$ and $a \in \overleftarrow{A}_\Phi$ iff $a \in B_i$, the number of edges x of this graph satisfies $|\overleftarrow{A}_0|(c-1) \leq x \leq 6|\overleftarrow{A}_\Phi|$, implying the desired inequality as $m_\Phi = m - m_0$. ◀

▶ **Theorem 10.** *For any $c \geq 2$, Φ is a polynomial $(1 + \frac{6}{c-1})$ approximation algorithm for C_3-*Packing*-$\mathrm{T}^{D_M}_{\geq c}$.*

Proof. Let OPT be an optimal solution. Let us define set $OPT_i \subseteq OPT$ and $\overleftarrow{A}^*_i \subseteq \overleftarrow{A}(\mathcal{T})$ as follows. Let $t = (u, v, w) \in OPT$. As the FAS of the instance is a matching, we know that $wu \in \overleftarrow{A}(\mathcal{T})$ as we cannot have a triangle with two backward arcs. If $d(v) = (0, 0)$ then we add t to OPT_1 and wu to $\overleftarrow{A}^*_1$. Otherwise, let v' be the other endpoint of the unique arc a containing v. If $v' \notin V(OPT)$, then we add t to OPT_3 and $\{wu, a\}$ to $\overleftarrow{A}^*_3$. Otherwise, let $t' \in OPT$ such that $v' \in V(t')$. As the FAS of the instance is a matching we know that v' is the middle point of t', or more formally that $t' = (u', v', w')$ with $u'w' \in \overleftarrow{A}(\mathcal{T})$. We add $\{t, t'\}$ to OPT_2 and $\{wu, a, w'u'\}$ to $\overleftarrow{A}^*_2$. Notice that the OPT_i form a partition of OPT, and that the $\overleftarrow{A}^*_i$ have pairwise empty intersection, implying $|\overleftarrow{A}^*_1| + |\overleftarrow{A}^*_2| + |\overleftarrow{A}^*_3| \leq m$. Notice also that as triangles of OPT_1 correspond to a matching of size $|OPT_1|$ in the bipartite graph defined in phase 1 of algorithm Φ, we have $|OPT_1| = |\overleftarrow{A}^*_1| \leq |\overleftarrow{A}_1|$.

Putting pieces together we get (recall that S is the solution computed by Φ): $|OPT| = |OPT_1| + |OPT_2| + |OPT_3| = |\overleftarrow{A}^*_1| + \frac{2}{3}|\overleftarrow{A}^*_2| + \frac{1}{2}|\overleftarrow{A}^*_3| \leq |\overleftarrow{A}^*_1| + \frac{2}{3}(|\overleftarrow{A}^*_2| + |\overleftarrow{A}^*_3|) \leq |\overleftarrow{A}^*_1| + \frac{2}{3}(m - |\overleftarrow{A}^*_1|) \leq \frac{1}{3}|\overleftarrow{A}_1| + \frac{2}{3}m$ and $|S| = |S^1| + |S^2| = |\overleftarrow{A}_1| + \frac{2}{3}|\overleftarrow{A}_2| \geq |\overleftarrow{A}_1| + \frac{2}{3}((1 - \frac{6}{c+5})m - |\overleftarrow{A}_1|) = \frac{1}{3}|\overleftarrow{A}_1| + \frac{2}{3}(1 - \frac{6}{c+5})m$ which implies the desired ratio. ◀

4 Kernelization

In all this section we consider the decision problem C_3-Packing-T parameterized by the size of the solution. Thus, an input is a pair $I = (\mathcal{T}, k)$ and we say that I is positive iff there exists a set of k vertex disjoint triangles in $\mathcal{T}$.

4.1 Positive results for sparse instances

Observe first that the kernel in $\mathcal{O}(k^2)$ vertices for 3-Set Packing of [1] directly implies a kernel in $\mathcal{O}(k^2)$ vertices for C_3-Packing-T. Indeed, given an instance $(\mathcal{T} = (V, A), k)$ of

C_3-Packing-T, we create an instance $(I' = (V, C), k)$ of 3-Set Packing by creating an hyperedge $c \in C$ for each triangle of $\mathcal{T}$. Then, as the kernel of [1] only removes vertices, it outputs an induced instance $(\overline{I'} = I'[V'], k')$ of I with $V' \subseteq V$, and thus this induced instance can be interpreted as a subtournament, and the corresponding instance $(\mathcal{T}[V'], k')$ is an equivalent tournament with $\mathcal{O}(k^2)$ vertices.

As shown in the next theorem, as we could expect it is also possible to have kernel bounded by the number of backward arcs.

▶ **Theorem 11.** *C_3-Packing-T admits a polynomial kernel with $\mathcal{O}(m)$ vertices, where m is the number of arcs in a given FAS of the input.*

Proof. Let $I = (\mathcal{T}, k)$ be an input of the decision problem associated to C_3-Packing-T. Observe first that we can build in polynomial time a linear ordering $\sigma(\mathcal{T})$ whose backward arcs $\overleftarrow{A}(\mathcal{T})$ correspond to the given FAS. We will obtain the kernel by removing useless vertices of degree $(0, 0)$. Let us define a bipartite graph $G = (V_1, V_2, E)$ with $V_1 = \{v_a^1 : a \in \overleftarrow{A}(\mathcal{T})\}$ and $V_2 = \{v_l^2 : v_l \in V_{(0,0)}\}$. Thus, to each backward arc we associate a vertex in V_1 and to each vertex v_l with $d(v_l) = (0, 0)$ we associate a vertex in V_2. Then, $\{v_a^1, v_l^2\} \in E$ iff $(h(a), v_l, t(a))$ is a triangle in $\mathcal{T}$. By Hall's theorem, we can in polynomial time partition V_1 and V_2 into $V_1 = A_1 \cup A_2$, $V_2 = B_0 \cup B_1 \cup B_2$ such that $N(A_2) \subseteq B_2$, $|B_2| \leq |A_2|$, and there is a perfect matching between vertices of A_1 and B_1 (B_0 is simply defined by $B_0 = V_2 \setminus (B_1 \cup B_2)$).

For any $i, 0 \leq i \leq 2$, let $X_i = \{v_l \in V_{(0,0)} : v_l^2 \in B_i\}$ be the vertices of $\mathcal{T}$ corresponding to B_i. Let $V_{\neq(0,0)} = V(\mathcal{T}) \setminus V_{(0,0)}$. Notice that $|V_{\neq(0,0)}| \leq 2m$. We define $\mathcal{T}' = \mathcal{T}[V_{\neq(0,0)} \cup X_1 \cup X_2]$ the sub-tournament obtained from $\mathcal{T}$ by removing vertices of X_0, and $I' = (\mathcal{T}', k)$. We point out that this definition of $\mathcal{T}'$ is similar to the final step of the kernel of [1] as our partition of V_1 and V_2 (more precisely $(A_1, B_0 \cup B_1)$) corresponds in fact to the crown decomposition of [1]. Observe that $|V(\mathcal{T}')| \leq 2m + |A_1| + |A_2| \leq 3m$, implying the desired bound of the number of vertices of the kernel.

It remains to prove that I and I' are equivalent. Let $k \in \mathbb{N}$, and let us prove that there exists a solution S of $\mathcal{T}$ with $|S| \geq k$ iff there exists a solution S' of $\mathcal{T}'$ with $|S'| \geq k$. Observe that the $\Leftarrow$ direction is obvious as $\mathcal{T}'$ is a subtournament of $\mathcal{T}$. Let us now prove the $\Rightarrow$ direction. Let S be a solution of $\mathcal{T}$ with $|S| \geq k$. Let $S = S_{(0,0)} \cup S_1$ with $S_{(0,0)} = \{t \in S : t = (h(a), v, t(a)) \text{ with } v \in V_{(0,0)}, a \in \overleftarrow{A}(\mathcal{T})\}$ and $S_1 = S \setminus S_{(0,0)}$. Observe that $V(S_1) \cap V_{(0,0)} = \emptyset$, implying $V(S_1) \subseteq V_{\neq(0,0)}$. For any $i \in [2]$, let $S^i_{(0,0)} = \{t \in S_{(0,0)} : t = (h(a), v, t(a)) \text{ with } v \in V_{(0,0)}, v_a^1 \in A_i\}$ be a partition of $S_{(0,0)}$. We define $S' = S_1 \cup S^2_{(0,0)} \cup S'^1_{(0,0)}$, where $S'^1_{(0,0)}$ is defined as follows. For any $v_a^1 \in A_1$, let $v^2_{\mu(a)} \in B_1$ be the vertex associated to v_a^1 in the (A_1, B_1) matching. To any triangle $t = (h(a), v, t(a)) \in S^1_{(0,0)}$ we associate a triangle $f(t) = (h(a), v_{\mu(a)}, t(a)) \in S'^1_{(0,0)}$, where by definition $v_{\mu(a)} \in X_1$. For the sake of uniformity we also say that any $t \in S_1 \cup S^2_{(0,0)}$ is associated to $f(t) = t$.

Let us now verify that triangles of S' are vertex disjoint by verifying that triangles of $S'^1_{(0,0)}$ do not intersect another triangle of S'. Let $f(t) = (h(a), v_{\mu(a)}, t(a)) \in S'^1_{(0,0)}$. Observe that $h(a)$ and $t(a)$ cannot belong to any other triangle $f(t')$ of S' as for any $f(t'') \in S'$, $V(f(t'')) \cap V_{\neq(0,0)} = V(t'') \cap V_{\neq(0,0)}$ (remember that we use the same notation $V_{\neq(0,0)}$ to denote vertices of degree $(0, 0)$ in $\mathcal{T}$ and $\mathcal{T}'$). Let us now consider $v_{\mu(a)}$. For any $f(t') \in S_1$, as $V(f(t')) \cap V_{(0,0)} = \emptyset$ we have $v_{\mu(a)} \notin V(f(t'))$. For any $f(t') = (h(a'), v_l, t(a')) \in S^2_{(0,0)}$, we know by definition that $v^1_{a'} \in A_2$, implying that $v_l^2 \in B_2$ (and $v_l \in X_2$) as $N(A_2) \subseteq B_2$ and thus that $v_l \neq v_{\mu(a)}$. Finally, for any $f(t') = (h(a'), v_l, t(a')) \in S'^1_{(0,0)}$, we know that $v_l = v_{\mu(a')}$, where $a \neq a'$, leading to $v_l \neq v_{\mu(a)}$ as μ is a matching. ◀

Using the previous result we can provide a $\mathcal{O}(k)$ vertices kernel for C_3-Packing-T restricted to sparse tournaments.

▶ **Theorem 12.** *C_3-Packing-T restricted to sparse tournaments admits a polynomial kernel with $\mathcal{O}(k)$ vertices, where k is the size of the solution.*

Proof. Let $I = (\mathcal{T}, k)$ be an input of the decision problem associated to C_3-Packing-T such that $\mathcal{T}$ is a sparse tournament. We say that an arc a is a *consecutive backward arc* of $\sigma(\mathcal{T})$ if it is a backward arc of $\mathcal{T}$ and $a = v_{i+1}v_i$ with v_i and v_{i+1} being consecutive in $\sigma(\mathcal{T})$. If $\mathcal{T}$ admits a consecutive backward arc v_iv_{i+1} then we can exchange v_i and v_{i+1} in $\mathcal{T}$. The backward arcs of the new linear ordering is exactly $\overleftarrow{A}(\mathcal{T}) \setminus v_{i+1}v_i$ and so is still a matching. Hence we can assume that $\mathcal{T}$ does not contain any consecutive backward arc. Now if $|\overleftarrow{A}(\mathcal{T})| < 5k$ then by Theorem 11 we have a kernel with $\mathcal{O}(k)$ vertices. Otherwise, if $|\overleftarrow{A}(\mathcal{T})| \geq 5k$ we will prove that T is a yes instance of C_3-Packing-T. Indeed we can greedily produce a family of k vertex disjoint triangles in T. For that consider a backward arc v_jv_i of $\mathcal{T}$, with $i < j$. As v_jv_i is not consecutive there exists l with $i < l < j$ and we select the triangle $v_iv_jv_l$ and remove the vertices v_i, v_l and v_j from $\mathcal{T}$. Denote by $\mathcal{T}'$ the resulting tournament and let $\sigma(\mathcal{T}')$ be the order induced by $\sigma(\mathcal{T})$ on $\mathcal{T}'$. So we loose at most 2 backward arcs in $\sigma(\mathcal{T}')$ (v_jv_i and a possible backward arc containing v_l) and create at most 3 consecutive backward arcs by the removing of v_i, v_l and v_j. Reducing these consecutive backward arcs as previously, we can assume that $\sigma(\mathcal{T}')$ does not contain any consecutive backward arc and satisfies $|\overleftarrow{A}(\mathcal{T}')| \geq |\overleftarrow{A}(\mathcal{T})| - 5 \geq 5(k-1)$. Finally repeating inductively this arguments, we obtain the desired family of k vertex-disjoint triangles in $\mathcal{T}$, and $\mathcal{T}$ is a yes instance of C_3-Packing-T. ◀

4.2 No (generalised) kernel in $\mathcal{O}(k^{2-\epsilon})$

In this section we provide an OR-cross composition (see Definition 21 in Appendix) from C_3-Perfect-Packing-T restricted to instances of Theorem 7 to C_3-Perfect-Packing-T parameterized by the total number of vertices.

Definition of the instance selector The next lemma build a special tournament, called an *instance selector* that will be useful to design the cross composition.

▶ **Lemma 13.** *For any $\gamma = 2^{\gamma'}$ and ω we can construct in polynomial time (in γ and ω) a tournament $P_{\omega,\gamma}$ such that*

- *there exists γ subsets of ω vertices $X^i = \{x^i_j : j \in [\omega]\}$, that we call the distinguished set of vertices, such that*
  - *the X^i have pairwise empty intersection*
  - *for any $i \in [\gamma]$, there exists a packing S of triangles of $P_{\omega,\gamma}$ such that $V(P_{\omega,\gamma}) \setminus V(S) = X^i$ (using this packing of $P_{\omega,\gamma}$ corresponds to select instance i)*
  - *for any packing S of triangles of $P_{\omega,\gamma}$ with $|V(S)| = |V(P_{\omega,\gamma})| - \omega$ there exists $i \in [\gamma]$ such that $V(P_{\omega,\gamma}) \setminus V(S) \subseteq X^i$*
- *$|V(P_{\omega,\gamma})| = \mathcal{O}(\omega\gamma)$.*

Proof. Let us first describe vertices of $P_{\omega,\gamma}$. For any $i \in [\gamma-1]_0$ (where $[x]_0$ denotes $\{0, \dots, x\}$) let $X^i = \{x^i_j : j \in [\omega]\}$, and let $X = \cup_{i \in [\gamma-1]_0} X^i$. For any $l \in [\gamma'-1]_0$, let $V^l = \{v^l_k, k \in [|V^l|]\}$ be the vertices of level l with $|V^l| = \omega\gamma/2^l + 2$, and $V = \cup_{l \in [\gamma'-1]_0} V^l$. Finally, we add a set $\alpha = \{\alpha^l : l \in [\gamma'-1]_0\}$ containing one dummy vertex for each level and finally set $V(P_{\omega,\gamma}) = X \cup V \cup \alpha$. Observe that $|V(P_{\omega,\gamma})| = \omega\gamma + \sum_{l \in [\gamma'-1]_0}(|V^l| + 1) =$

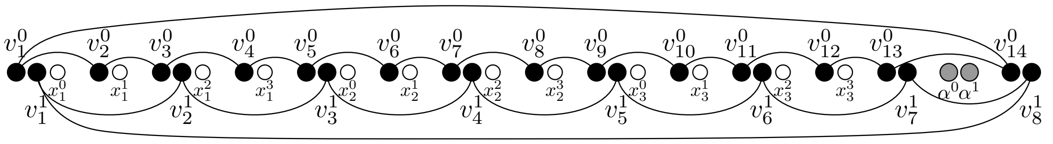

Figure 6 An example of the instance selector, where $\omega = 3$ and $\gamma = 4$. All depicted arcs are backward arcs.

$\mathcal{O}(\omega\gamma)$. Let us now describe $\sigma(P_{\omega,\gamma})$ and $\overleftarrow{A}(P_{\omega,\gamma})$ recursively. Let $P^0_{\omega,\gamma}$ be the tournament such that $\sigma(P^0_{\omega,\gamma}) = (v^0_1, x^0_1, v^0_2, x^1_1, \dots, v^0_\gamma, x^{\gamma-1}_1)$ $(v^0_{\gamma+1}, x^0_2, \dots, v^0_{2\gamma}, x^{\gamma-1}_2)$ $\dots$ $(v^0_{(\omega-1)\gamma+1}, x^0_\omega, \dots, v^0_{\omega\gamma}, x^{\gamma-1}_\omega)$ $(v^0_{\omega\gamma+1}, \alpha^1, v^0_{\omega\gamma+2})$ and $\overleftarrow{A}(P^0_{\omega,\gamma}) = Z^0_P$ where $Z^0_P = A^0_P \cup A'^0_P$ with $A^0_P = \{v^0_{k+1}v^0_k : k \in [|V^0|-2]\}$ and $A'^0_P = \{v^0_{|V^0|}v^0_{|V^0|-1}, v^0_{|V^0|}v^0_1\}$.

Then, given a tournament $P^l_{\omega,\gamma}$ with $0 \le l < \gamma' - 1$, we construct the tournament $P^{l+1}_{\omega,\gamma}$ such that the vertices of $P^{l+1}_{\omega,\gamma}$ are those of $P^l_{\omega,\gamma}$ to which are added the set V^{l+1}. For $j \in [|V^{l+1}| - 2]$, we add the vertex v^{l+1}_j of V^{l+1} just after the vertex v^l_{2j-1} in the order of $P^{l+1}_{\omega,\gamma}$, and we for $i \in \{0, 1\}$ we add vertex $v^{l+1}_{|V^{l+1}|-i}$ just after $v^l_{|V^l|-i}$. Similarly, we add the vertex α^{l+1} just after the vertex α^l. The backward arcs of $P^{l+1}_{\omega,\gamma}$ are defined by: $\overleftarrow{A}(P^{l+1}_{\omega,\gamma}) = \overleftarrow{A}(P^l_{\omega,\gamma}) \cup Z^{l+1}_P$ where $Z^{l+1}_P = A^{l+1}_P \cup A'^{l+1}_P$ are called the *arcs of level* l, with $A^{l+1}_P = \{v^{l+1}_{k+1}v^{l+1}_k : k \in [|V^{l+1}| - 2]]\}$ and $A'^{l+1}_P = \{v^{l+1}_{|V^{l+1}|}v^{l+1}_{|V^{l+1}|-1}, v^{l+1}_{|V^{l+1}|}v^{l+1}_1\}$. We can now define our gadget tournament $P_{\omega,\gamma}$ as the tournament corresponding to $P^{\gamma'-1}_{\omega,\gamma}$. We refer the reader to Figure 6 where an example of the gadget is depicted, where $\omega = 3$ and $\gamma = 4$.

In all the following given $i \in [\gamma - 1]_0$ we call the last x bits (resp. the x^{th} bit) i its x right most (resp. the x^{th}, starting from the right) bits in the binary representation of i. Let us now state the following observations.

$\triangle_1$ The vertices of X have degree $(0, 0)$ in $P_{\omega,\gamma}$.

$\triangle_2$ For any $l \in [\gamma' - 1]_0$, the extremities of the arcs of level l are exactly V^l $(V(Z^l_P) = V^l)$ and the arcs of Z^l_P induce an even circuit on V^l.

$\triangle_3$ For any $a \in A^l_P$, the span of a contains 2^l consecutive vertices of X, more precisely $s(a) \cap X = \{x^i_j, \dots, x^{i+2^l-1}_j\}$ for $j \in [m]$ and i such that the $l-1$ last bits of i are equal to 0.

$\triangle_4$ There is a unique partition $Z^l_P = Z^{l,0}_P \cup Z^{l,1}_P$ such that $|Z^{l,0}_P| = |Z^{l,1}_P| = \mu^l$, the size of a maximum matching of backward arcs in $P_{\omega,\gamma}[V^l]$, such that each $Z^{l,x}_P$ is a matching (for any $a, a' \in Z^{l,x}_P, V(a) \cap V(a') = \emptyset$), and such that $\cup_{a \in Z^{l,x}_P \setminus A'^l_P} s(a) \cap X$ is the set of all vertices x^i_j of X whose l^{th} bit of i is x.

Now let us first prove that for any $i \in [\gamma - 1]_0$, there exists an packing S of $P_{\omega,\gamma}$ such that $V(P_{\omega,\gamma}) \setminus V(S) = X^i$. Let $(x_{\gamma'-1} \dots x_0)$ be the binary representation of i. Let us define recursively $S = \cup_{l \in [\gamma'-1]_0} S_l$ in the following way. We maintain the invariant that for any l, the remaining vertices of X after defining $\cup_{z \in [l]_0} S_z$ are all the vertices of X having their l last bits equal to $(x_{l-1}, \dots, x_0)$. We define S_l as the $\mu^l - 1$ triangles $\{(h(a), x_a, t(a), a \in Z^{l,1-x_l}_P) \setminus A'^l_P\}$ such that x_a is the unique remaining vertex of X in $s(a)$ (by $\triangle_3$ and our invariant of the $S_{\le l}$, there remains exactly one vertex in $s(a)$, and by $\triangle_4$ these $\mu^l - 1$ triangles consume all remaining vertices of X whose l^{th} bit is $1 - x_l$), and a last triangle using an arc in A'^l_P with $t = (v^l_{|V^0|}, \alpha^l, v^l_{|V^0|-1})$ if $x_l = 1$ and $t = (v^l_0, \alpha^l, v^0_{|V^0|})$ otherwise. Thus, by our invariant, the remaining vertices of X after defining S are exactly X^i. As S also consumes α and V we have $V(P_{\omega,\gamma}) \setminus V(S) = X^i$. Notice that this definition of S shows that $|V(P_{\omega,\gamma})| - m = |V(S)| = 3\sum_{l \in [\gamma'-1]_0} \mu^l$

Let us now prove that for any packing S of $P_{\omega,\gamma}$ with $|V(S)| = |V(P_{\omega,\gamma})| - m =$

$3\sum_{l\in[\gamma'-1]_0}\mu^l$, there exists $i\in[\gamma]$ such that $V(P_{\omega,\gamma})\setminus V(S)\subseteq X^i$. Let $t_1,\dots,t_\mu$ be the triangles of S. For any t_k of S, we associate one backward arc a_k of t_k (if there are two backward arcs, we pick one arbitrarily). Let $Z=\{a_k : k\in[|S|]\}$ and for every $l\in|\gamma'-1]_0$ let $Z^l=\{a_k\in A : V(a_k)\subset V^l\}$ the set of the backward arcs which are between two vertices of level l. Notice that the Z^l 's form a partition of Z. For any $l\in[\gamma'-1]_0$, we have $|Z^l|\leq\mu^l$ as two arcs of Z^l correspond to two different triangles of S, implying that Z^l is a matching. Furthermore, as $|S|=|Z|=\sum_{l\in[\gamma'-1]_0}|Z^l|=\mu=\sum_{l\in[\gamma']}\mu^l$, we get the equality $|Z^l|=\mu^l$ for any $l\in[\gamma'-1]_0$. This implies that for each Z^l there exists x such that $Z^l=Z_P^{l,x}$, implying also that $V(Z^l)=V^l$, and $V(Z)=V$.

Let $A^l=Z^l\setminus A'^l_P$, $S^l=\{t_k\in S : a_k\in A^l\}$. We can now prove by induction that all the remaining vertices $R_l=X\setminus V(\cup_{x\in[l]_0}S^l)$ have the same l last bits. Notice that since all vertices of V are already used, and as triangles of S^l cannot use a dummy vertex in α, all triangles of S^l must be of the from $(h(a_k),x,t(a_k))$ with $x\in X$. As $A^l=Z_P^{l,x}\setminus A'^l_P$, by $\triangle_4$ we know that $\cup_{a\in A^l}s(a)\cap X$ contains all the remaining vertices of X, and thus of R_{l-1}, whose l^{th} bit is x. Moreover, by $\triangle_3$ we know that for any $a\in A^l$ we have $|R_{l-1}\cap s(a)|\leq 1$ because as $a\in A^l_P$ we know exactly the structure of $s(a)\cap X$, and if there remain two vertices in $s(a)\cap X$ then their last $l-1$ last bits would be different. Thus, as triangles of S^l remove on vertex in the span of each $a\in A^l$, they remove all vertices of R_{l-1} whose l^{th} bit is x, implying the desired result.

◀

Definition of the reduction We suppose given a family of t instances $F=\{\mathcal{I}_l, l\in[t]\}$ of C_3-Perfect-Packing-T restricted to instances of Theorem 7 where $\mathcal{I}_l$ asks if there is a perfect packing in $\mathcal{T}_l=L_lK_l$. We chose our equivalence relation in Definition 21 such that there exist n and m such that for any $l\in[t]$ we have $|V(L_l)|=n$ and $|V(K_l)|=m$. We can also copy some of the t instances such that t is a square number and $g=\sqrt{t}$ is a power of two. We reorganize our instances into $F=\{\mathcal{I}_{(p,q)} : 1\leq p,q\leq g\}$ where $\mathcal{I}_{(p,q)}$ asks if there is a perfect packing in $\mathcal{T}_{(p,q)}=L_pK_q$. Remember that according to Theorem 7, all the L_p are equals, and all the K_q are equals. We point out that the idea of using a problem on "bipartite" instances to allow encoding t instances on a "meta" bipartite graph $G=(A,B)$ (with $A=\{A_i, i\in\sqrt{t}\}$, $B=\{B_i, i\in\sqrt{t}\}$) such that each instance p,q is encoded in the graph induced by $G[A_i\cup B_i]$ comes from [8]. We refer the reader to Figure 7 which represents the different parts of the tournament. We define a tournament $G=LM_G\tilde{L}\tilde{M}_GP_{(n,g)}$, where $L=L_1\dots L_g$, $\tilde{M}_G$ is a set of n vertices of degree $(0,0)$, M_G is a set of $(g-1)n$ vertices of degree $(0,0)$, $\tilde{L}=\tilde{L}_1\dots\tilde{L}_g$ where each $\tilde{L}_p$ is a set of n vertices, and $P_{(n,g)}$ is a copy of the instance selector of Lemma 13. Then, for every $p\in[g]$ we add to G all the possible n^2 backward arcs going from $\tilde{L}_p$ to L_p. Finally, for every distinguished set X^p of $P_{(n,g)}$ (see in Lemma 13), we add all the possible n^2 backward arcs from X^p to $\tilde{L}_p$.

Now, in a symmetric way we define a tournament $D=KM_D\tilde{K}\tilde{M}_DP'_{(m,g)}$, where $K=K_1\dots K_g$, $\tilde{M}_D$ is a set of m vertices of degree $(0,0)$, M_D is a set of $(g-1)m$ vertices of degree $(0,0)$, $\tilde{K}=\tilde{K}_1\dots\tilde{K}_g$ where each $\tilde{K}_q$ is a set of m vertices, and $P'_{(m,g)}$ is a copy of the instance selector of Lemma 13. Then, for every $q\in[g]$ we add to G all the m^2 possible backward arcs going from $\tilde{K}_p$ to K_p. Finally, for every distinguished set X'^q of $P'_{(m,g)}$ we add all the possible m^2 backward arcs from X'^q to $\tilde{K}_q$. Finally, we define $\mathcal{T}=GD$. Let us add some backward arcs from D to G. For any p and q with $1\leq p,q\leq g$, we add backward arcs from K_q to L_p such that $\mathcal{T}[K_qL_p]$ corresponds to $\mathcal{T}_{(p,q)}$. Notice that this is possible as for any fixed p, all the $\mathcal{T}_{(p,q)}, q\in[g]$ have the same left part L_p, and the same goes for any fixed right part.

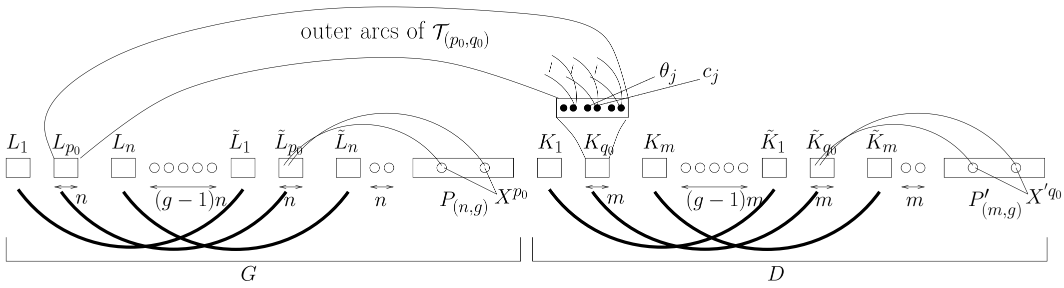

Figure 7 A example of the weak composition. All depicted arcs are backward arcs. Bold arcs represents the n^2 (or m^2) possible arcs between the two groups.

Restructuration lemmas Given a set of triangles S we define $S_{\subseteq P'} = \{t \in S | V(t) \subseteq P'_{(m,g)}\}$, $S_{\subseteq P} = \{t \in S : V(t) \subseteq P_{(n,g)}\}$, $S_{\tilde{M}_D} = \{t \in S : V(t)$ intersects $\tilde{K}$, $\tilde{M}_D$ and $P'_{m,g}\}$, $S_{M_D} = \{t \in S : V(t)$ intersects K, M_D and $\tilde{K}\}$, $S_{\tilde{M}_G} = \{t \in S : V(t)$ intersects $\tilde{L}$, $\tilde{M}_G$ and $P_{n,g}\}$, $S_{M_G} = \{t \in S : V(t)$ intersects L, M_G and $\tilde{L}\}$, $S_D = \{t \in S : V(t) \subseteq V(D)\}$, $S_G = \{t \in S : V(t) \subseteq V(G)\}$, and $S_{GD} = \{t \in S : V(t)$ intersects $V(G)$ and $V(D)\}$. Notice that S_G, S_{GD}, S_D is a partition of S.

▶ Claim 13.1. If there exists a perfect packing S of $\mathcal{T}$, then $|S_{\tilde{M}_D}| = m$ and $|S_{M_D}| = (g-1)m$. This implies that $V(S_{\tilde{M}_D} \cup S_{M_D}) \cap V(\tilde{K}) = V(\tilde{K})$, meaning that the vertices of $\tilde{K}$ are entirely used by $S_{\tilde{M}_D} \cup S_{M_D}$.

Proof. We have $|S_{\tilde{M}_D}| \leq m$ since $|\tilde{M}_D| = m$. We obtain the equality since the vertices of $\tilde{M}_D$ only lie in the span of backward arcs from $P'_{m,g}$ to $\tilde{K}$, and they are not the head or the tail of a backward arc in $\mathcal{T}$. Thus, the only way to use vertices of $\tilde{M}_D$ is to create triangles in $S_{\tilde{M}_D}$, implying $|S_{\tilde{M}_D}| \geq m$. Using the same kind of arguments we also get $|S_{M_D}| = (g-1)m$. As $|V(\tilde{K})| = gm$ we get the last part of the claim. ◀

▶ Claim 13.2. If there exists a perfect packing S of $\mathcal{T}$, then there exists $q_0 \in [g]$ such that $\tilde{K}_S = \tilde{K}_{q_0}$, where $\tilde{K}_S = \tilde{K} \cap V(S_{\tilde{M}_D})$.

Proof. Let $S_{P'}$ be the triangles of S with at least one vertex in $P'_{m,g}$. As according to Claim 13.1 vertices of $\tilde{K}$ are entirely used by $S_{\tilde{M}_D} \cup S_{M_D}$, the only way to consume vertices of $P'_{m,g}$ is by creating local triangles in $P'_{m,g}$ or triangles in $S_{\tilde{M}_D}$. In particular, we cannot have a triangle (u, v, w) with $\{u, v\} \subseteq \tilde{K}$ and $w \in P'_{m,g}$, or with $u \in \tilde{K}$ and $\{v, w\} \subseteq P'_{m,g}$. More formally, we get the partition $S_{P'} = S_{\subseteq P'} \cup S_{\tilde{M}_D}$. As S is a perfect packing and uses in particular all vertices of $P'_{m,g}$ we get $|V(S_{P'})| = |V(P'_{m,g})|$, implying $|V(S_{\subseteq P'})| = |V(P'_{m,g})| - m$ by Claim 13.1. By Lemma 13, this implies that there exists $q_0 \in [g]$ such that $X' \subseteq X'^{q_0}$ where $X' = V(P'_{m,g}) \setminus V(S_{\subseteq P'})$. As X' are the only remaining vertices that can be used by triangles of $S_{\tilde{M}_D}$, we get that the m triangles of $S_{\tilde{M}_D}$ are of the form (u, v, w) with $u \in \tilde{K}_{q_0}$, $v \in \tilde{M}_D$, and $w \in X'$.

◀

▶ Claim 13.3. If there exists a perfect packing S of $\mathcal{T}$, then there exists $q_0 \in [g]$ such that $V(S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) = V(D) \setminus K_{q_0}$.

Proof. By Claim 13.1 we know that $|S_{M_D}| = (g-1)m$. As by Claim 13.2 there exists $q_0 \in [g]$ such that $\tilde{K}_S = \tilde{K}_{q_0}$, we get that the $(g-1)m$ triangles of S_{M_D} are of the form (u, v, w) with $u \in K \setminus K_{q_0}$, $v \in M_D$, and $w \in \tilde{K} \setminus \tilde{K}_{q_0}$. ◀

▶ **Lemma 14.** *If there exists a perfect packing S of $\mathcal{T}$, then $V(S_{GD}) \cap V(G) \subseteq V(L)$. Informally, triangles of S_{GD} do not use any vertex of $M_G, \tilde{L}, \tilde{M}_T$ and $P_{n,g}$.*

Proof. By Claim 13.3, there exists $q_0 \in [g]$ such that $V(S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) = V(D) \setminus K_{q_0}$. By Theorem 7 we know that $K_{q_0} = K_{(q_0,1)} \dots K_{(q_0,m')}$ for some m' (we even have $m' = \frac{m}{2}$), where for each $j \in [m']$ we have $V(K_{(q_0,j)}) = (\theta_j, c_j)$. Moreover, for any $p \in [g]$, the last property of Theorem 7 ensures that for any $a \in \overleftarrow{A}(\mathcal{T}_{(p,q_0)})$, $V(a) \cap V(K_{q_0}) \neq \emptyset$ implies $a = vc_j$ for $v \in L_p$. So no arc of $\overleftarrow{A}(\mathcal{T}_{(p,q_0)})$, and thus no arc of $\mathcal{T}$ is entirely included in K_{q_0}. This implies that S cannot cover the vertices of K_{q_0} using triangles t with $V(t) \subseteq V(K_{q_0})$, and thus that all these vertices must be used by triangles of S_{GD}, implying that $V(S_{GD}) \cap V(D) = K_{q_0}$. The last property of Theorem 7 also implies that all the θ_j have a left degree equal to 0 in $\mathcal{T}$, or equivalently that there is no arc a of $\mathcal{T}$ such that $t(a) = \theta_j$ and $h(a) < \theta_j$. Thus, by induction for any j from m' to 1, we can prove that the only way for triangles of S_{GD} to use θ_j is to create a triangle $t_j = (v, \theta_j, c_j)$ with necessarily $v \in V(L)$. ◀

Lemma 14 will allow us to prove Claims 14.1, 14.2 and 14.3 using the same arguments as in the right part (D) of the tournament as all vertices of $M_G, \tilde{L}, \tilde{M}_T$ and $P_{n,g}$ must be used by triangles in S_G.

▶ Claim 14.1. If there exists a perfect packing S of $\mathcal{T}$, then $|S_{\tilde{M}_G}| = n$ and $|S_{M_G}| = (g-1)n$. This implies that $V(S_{\tilde{M}_G} \cup S_{M_G}) \cap V(\tilde{L}) = V(\tilde{L})$, meaning that vertices of $\tilde{L}$ are entirely used by $S_{\tilde{M}_G} \cup S_{M_G}$.

Proof. We have $|S_{\tilde{M}_G}| \leq n$ since $|\tilde{M}_G| = n$. Lemma 14 implies that all vertices of $\tilde{M}_G$ must be used by triangles of S_G, and thus using arcs whose both endpoints lie in $V(G)$. As vertices of $\tilde{M}_G$ are not the head or the tail of a backward arc in $\mathcal{T}$, we get that the only way for S_G to use vertices of $\tilde{M}_G$ is to create triangles in $S_{\tilde{M}_G}$, implying $|S_{\tilde{M}_G}| \geq n$. Using the same kind of arguments (and as all vertices of M_G must also be used by triangles of S_G) we also get $|S_{M_G}| = (g-1)n$. As $|V(\tilde{L})| = gn$ we get the last part of the claim. ◀

▶ Claim 14.2. If there exists a perfect packing S of $\mathcal{T}$, then there exists $p_0 \in [g]$ such that $\tilde{L}_S = \tilde{L}_{p_0}$, where $\tilde{L}_S = \tilde{L} \cap V(S_{\tilde{M}_G})$.

Proof. Lemma 14 implies that all vertices of $\tilde{M}_G$ and $P_{(n,g)}$ must be used by triangles in S_G. Let S_P be the triangles of S_G with at least one vertex in $P_{n,g}$. As according to Claim 14.1 vertices of $\tilde{L}$ are entirely used by $S_{\tilde{M}_G} \cup S_{M_G}$, the only way for S_G to consume vertices of $P_{n,g}$ is by creating local triangles in $P_{n,g}$ or triangles in $S_{\tilde{M}_G}$. In particular, we cannot have a triangle (u, v, w) with $\{u, v\} \subseteq \tilde{L}$ and $w \in P_{n,g}$, or with $u \in \tilde{L}$ and $\{v, w\} \subseteq P_{n,g}$. More formally, we get the partition $S_P = S_{\subseteq P} \cup S_{\tilde{M}_G}$. As S_G uses in particular all vertices of $P_{n,g}$ we get $|V(S_P)| = |V(P_{n,g})|$, implying $|V(S_{\subseteq P})| = |V(P_{n,g})| - n$ by Claim 14.1. By Lemma 13, this implies that there exists $p_0 \in [g]$ such that $X \subseteq X^{p_0}$ where $X = V(P_{n,g}) \setminus V(S_{\subseteq P})$. As X are the only remaining vertices that can be used by triangles of $S_{\tilde{M}_G}$, we get that the n triangles of $S_{\tilde{M}_G}$ are of the form (u, v, w) with $u \in \tilde{L}_{p_0}$, $v \in \tilde{M}_G$, and $w \in X$. ◀

▶ Claim 14.3. If there exists a perfect packing S of $\mathcal{T}$, then there exists $p_0 \in [g]$ such that $V(S_P \cup S_{\tilde{M}_G} \cup S_{M_G}) = V(G) \setminus L_{p_0}$.

Proof. By Claim 13.1 we know that $|S_{M_G}| = (g-1)n$. As by Claim 14.2 there exists $p_0 \in [g]$ such that $\tilde{L}_S = \tilde{L}_{p_0}$, we get that the $(g-1)n$ triangles of S_{M_G} are of the form (u, v, w) with $u \in L \setminus L_{p_0}$, $v \in M_G$, and $w \in \tilde{L} \setminus \tilde{L}_{p_0}$. ◀

We are now ready to state our final claim is now straightforward as according Claim 13.3 and 14.3 we can define $S_{(p_0,q_0)} = S \setminus ((S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) \cup (S_P \cup S_{\tilde{M}_G} \cup S_{M_G}))$.

▶ Claim 14.4. If there exists a perfect packing S of $\mathcal{T}$, there exists $p_0, q_0 \in [g]$ and $S_{(p_0,q_0)} \subseteq S$ such that $V(S_{(p_0,q_0)}) = V(\mathcal{T}_{(p_0,q_0)})$ (or equivalently such that $S_{(p_0,q_0)}$ is a perfect packing of $\mathcal{T}_{(p_0,q_0)}$).

Proof of the weak composition

▶ **Theorem 15.** *For any $\epsilon > 0$, C_3-Perfect-Packing-T (parameterized by the total number of vertices N) does not admit a polynomial (generalized) kernelization with size bound $\mathcal{O}(N^{2-\epsilon})$ unless* NP $\subseteq$ coNP/Poly.

Proof. Given t instances $\{\mathcal{I}_l\}$ of C_3-Perfect-Packing-T restricted to instances of Theorem 7, we define an instance $\mathcal{T}$ of C_3-Perfect-Packing-T as defined in Section 4. We recall that $g = \sqrt{t}$, and that for any $l \in [t]$, $|V(L_l)| = n$ and $|V(K_l)| = m$. Let $N = |V(\mathcal{T})|$. As $N = |V(P'_{(m,g)})| + m + (g-1)m + 2mg + |V(P_{(n,g)})| + n + (g-1)n + 2ng$ and $|V(P_{(\omega,\gamma)})| = O(\omega\gamma)$ by Lemma 13, we get $N = \mathcal{O}(g(n+m)) = \mathcal{O}(t^{\frac{1}{2+o(1)}} \max(|\mathcal{I}_l|))$. Let us now verify that there exists $l \in [t]$ such that $\mathcal{I}_l$ admits a perfect packing iff $\mathcal{T}$ admits a perfect packing. First assume that there exist $p_0, q_0 \in [g]$ such that $\mathcal{I}_{(p_0,q_0)}$ admits a perfect packing. By Lemma 14.4, there is a packing $S_{P'}$ of $P'_{(m,g)}$ such that $V(S_{p'}) = V(P'_{(m,g)}) \setminus X'^{q_0}$. We define a set $S_{\tilde{M}_D}$ of m vertex disjoint triangles of the form (u, v, w) with $u \in \tilde{L}_{q_0}, v \in \tilde{M}_D, w \in X'^{q_0}$. Then, we define a set S_{M_D} of $(g-1)m$ vertex disjoint triangles of the form (u, v, w) with $u \in L \setminus L_{q_0}, v \in M_D, w \in \tilde{L} \setminus \tilde{L}_{q_0}$. In the same way we define S_P, $S_{\tilde{M}_G}$ and S_{M_G}. Observe that $V(\mathcal{T}) \setminus ((S_{P'} \cup S_{\tilde{M}_D} \cup S_{M_D}) \cup (S_P \cup S_{\tilde{M}_G} \cup S_{M_G})) = K_{q_0} \cup L_{p_0}$, and thus we can complete our packing into a perfect packing of $\mathcal{T}$ as $\mathcal{I}_{(p_0,q_0)}$ admits a perfect packing. Conversely if there exists a perfect packing S of $\mathcal{T}$, then by Claim 14.4 there exists $p_0, q_0 \in [g]$ and $S_{(p_0,q_0)} \subseteq S$ such that $V(S_{(p_0,q_0)}) = V(\mathcal{T}_{(p_0,q_0)})$, implying that $\mathcal{I}_{(p_0,q_0)}$ admits a perfect packing. ◀

▶ **Corollary 16.** *For any $\epsilon > 0$, C_3-Packing-T (parameterized by the size k of the solution) does not admit a polynomial kernel with size $\mathcal{O}(k^{2-\epsilon})$ unless* NP $\subseteq$ coNP/Poly.

5 Conclusion and open questions

Concerning approximation algorithms for C_3-Packing-T restricted to sparse instances, we have provided a $(1 + \frac{6}{c+5})$-approximation algorithm where c is a lower bound of the $\mathtt{m}inspan$ of the instance. On the other hand, it is not hard to solve by dynamic programming C_3-Packing-T for instances where $\mathtt{m}axspan$ is bounded above. Using these two opposite approaches it could be interesting to derive an approximation algorithm for C_3-Packing-T with factor better than 4/3 even for sparse tournaments.

Concerning FPT algorithms, the approach we used for sparse tournament (reducing to the case where $m = \mathcal{O}(k)$ and apply the $\mathcal{O}(m)$ vertices kernel) cannot work the general case. Indeed, if we were able to sparsify the initial input such that $m' = \mathcal{O}(k^{2-\epsilon})$, applying the kernel in $\mathcal{O}(m')$ would lead to a tournament of total bit size (by encoding the two endpoint of each arc) $\mathcal{O}(m' log(m')) = \mathcal{O}(k^{2-\epsilon})$, contradicting Corollary 16. Thus the situation for C_3-Packing-T could be as in vertex cover where there exists a kernel in $\mathcal{O}(k)$ vertices, derived from [16], but the resulting instance cannot have $\mathcal{O}(k^{2-\epsilon})$ edges [8]. So it is challenging question to provide a kernel in $\mathcal{O}(k)$ vertices for the general C_3-Packing-T problem.

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A Definitions

Approximation

▶ **Definition 17** ([17]). Let Π and Π' be two optimization (maximization or minimization) problems. We say that Π L-reduces to Π' if there are two polynomial-time algorithms f, g, and constants $\alpha, \beta > 0$ such that for each instance I of Π

(a) Algorithm f produces an instance $I' = f(I)$ of Π' such that the optima of I and I', $OPT(I)$ and $OPT(I')$, respectively, satisfy $OPT(I') \leq \alpha OPT(I)$

(b) Given any solution of I' with cost c, algorithm g produces a solution of I with cost c such that $|c - OPT(I)| \leq \beta|c' - OPT(I')|$.

▶ **Definition 18.** Let A be an algorithm of a maximization (resp. minimization) problem Π. For $\rho \geq 1$, we say that A is a ρ-approximation of Π iff for any instance I of Π, $A_I \geq OPT(I)/\rho$ (resp. $A_I \leq \rho OPT(I)$) where A_I is the value of the solution $A(I)$ and $OPT(I)$ the value of a optimal solution of I.

▶ **Definition 19.** Let Π be a NP-optimization problem. The problem Π is in APX if there exists a constant $\rho > 1$ such that Π admits a ρ-approximation algorithm.

▶ **Definition 20.** Let Π be a NP-optimization problem. The problem Π admits a PTAS if for any $\epsilon > 0$, there exists a polynomial $(1 + \epsilon)$-approximation of Π.

Parameterized complexity

We refer the reader to [9] for more details on parameterized complexity and kernelization, and we recall here only some basic definitions. A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. For an instance $I = (x, k) \in \Sigma^* \times \mathbb{N}$, the integer k is called the *parameter*.

A parameterized problem is *fixed-parameter tractable* (FPT) if there exists an algorithm A, a computable function f, and a constant c such that given an instance $I = (x, k)$, A (called an FPT algorithm) correctly decides whether $I \in L$ in time bounded by $f(k) \cdot |I|^c$, where $|I|$ denotes the size of I. Given a computable function g, a *kernelization algorithm* (or simply a *kernel*) for a parameterized problem L of *size* g is an algorithm A that given any instance $I = (x, k)$ of L, runs in polynomial time and returns an equivalent instance $I' = (x', k')$ with $|I'| + k' \leq g(k)$. It is well-known that the existence of an FPT algorithm is equivalent to the existence of a kernel (whose size may be exponential), implying that problems admitting a polynomial kernel form a natural subclass of FPT. Among the wide literature on polynomial kernelization, we only recall in the notion of weak composition used to lower bound the size of a kernel.

▶ **Definition 21** (Definition as written in [12]). Let $L \subseteq \Sigma^*$ be a language, R be a polynomial equivalence relation on Σ^*, let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem, and let $f : \mathbb{N} \to \mathbb{N}$ be a function. An or-cross-composition of L into Q (with respect to R) of cost $f(t)$ is an algorithm that, given t instances $x_i \in \Sigma^*$ of L belonging to the same equivalence class of R, takes time polynomial in $\sum_{i\in[t]} |x_i|$ and outputs an instance $(y, k) \in \Sigma^* \times \mathbb{N}$ such that:

1. the parameter k is bounded by $\mathcal{O}(f(t) \max_i |x_i|^c)$, where c is some constant independent of t, and
2. $(y, k) \in Q$ if and only if there is an $i \in [t]$ such that $x_i \in L$.

▶ **Theorem 22** ([4]). *Let $L \subseteq \Sigma^*$ be a language, let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem, and let d, ϵ be positive reals. If L is* NP*-hard under Karp reductions, has an or-cross-composition into Q with cost $f(t) = t^{1/d+o(1)}$, where t denotes the number of instances,* *and Q has a polynomial (generalized) kernelization with size bound $\mathcal{O}(k^{d-\epsilon})$, then* NP $\subseteq$ coNP/Poly*.*

B Problems

▶ Problem 1. (FVS)
Input: A directed graph $D = (V, A)$.
Output: A set of vertices $X \subseteq V$ such that $D[V \setminus X]$ is acyclic.
Optimisation: Minimise $|X|$.
The problem is called FVST if the input is a tournament.

▶ Problem 2. (d-Set Packing)
Input: An integer $d \geq 3$ and a d-uniform hypergraph $G = (V, H)$.
Output: A subset of hyperedges $X = \{X_i, i \in [k]$ with $X_i \in H\}$ such that for every $i \neq j$, $X_i \cap X_j = \emptyset$.
Optimisation: Maximise k.

▶ Problem 3. (Perfect d-Set Packing)
Input: An integer $d \geq 3$ and a d-uniform hypergraph $G = (V, H)$.
Question: Is there a subset of hyperedges $X = \{X_i, i \in [k]$ with $X_i \in H\}$ such that for every $i \neq j$, $X_i \cap X_j = \emptyset$ and $\bigcup_{i \in [k]} X_i = V$?

▶ Problem 4. (H-Packing)
Input: A graph $G = (V, E)$ and a subgraph H.
Output: A collection of subgraphs $X = \{H_i, i \in [k]\}$ such that for every i, H_i is isomorphic to H and for every $j \neq i$, $V(H_i) \cap V(H_j) = \emptyset$.
Optimisation: Maximise k.

▶ Problem 5. (Perfect H-Packing)
Input: A graph $G = (V, E)$ and a subgraph H.
Question: Is there a collection of subgraphs $X = \{H_i, i \in [k]\}$ such that for every i, H_i is isomorphic to H, for every $j \neq i$, $V(H_i) \cap V(H_j) = \emptyset$ and $\bigcup_{i \in [k]} H_i = V$?

C Polynomial detection of sparse tournaments

▶ **Lemma 23.** *In polynomial time, we can decide if a tournament is sparse or not, and if so, to give a linear representation whose FAS is a matching*

Proof. Indeed if a tournament $\mathcal{T}$ is sparse we can detect the first vertex (or vertices) of a linear representation $\sigma(\mathcal{T})$ of $\mathcal{T}$ where $\overleftarrow{A}(\mathcal{T})$ is a matching. If T has a vertex x of indegree 0 then x must be the first or the second vertex of $\sigma(\mathcal{T})$, and we can always suppose that x is the first vertex of $\sigma(\mathcal{T})$. Otherwise, we look at Z the set of vertices of $\mathcal{T}$ with indegree 1. As $\mathcal{T}$ is a tournament we have $|Z| \leq 3$ and if $Z = \emptyset$ then T is not a sparse tournament. If $|Z| = 1$, then the only element of Z must be the first vertex of $\sigma(\mathcal{T})$. If $|Z| = 2$ with $Z = \{x, y\}$ such that xy is an arc of $\mathcal{T}$, then x must be the first element of $\sigma(\mathcal{T})$ and y its second element. Finally, if $|Z| = 3$ with $Z = \{x, y, z\}$ then xyz must be a triangle of $\mathcal{T}$ and must be placed at the beginning of $\sigma(\mathcal{T})$. So repeating inductively these arguments we obtain in polynomial time in $|\mathcal{T}|$ either $\sigma(\mathcal{T})$ such that $\overleftarrow{A}(\mathcal{T})$ is a matching or a certificate that $\mathcal{T}$ is not sparse. ◀