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Quentin Boehler, Marc Vedrines, Salih Abdelaziz, Philippe Poignet, Pierre Renaud. Synthesis Method for the Design of Variable Stiffness Components Using Prestressed Singular Elastic Systems. *Mechanism and Machine Theory*, 2018, 121, pp.598-612. 10.1016/j.mechmachtheory.2017.11.013 . lirmm-01643079

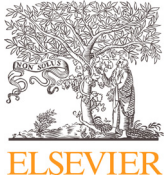
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Research paper

Synthesis method for the design of variable stiffness components using prestressed singular elastic systems

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ARTICLE INFO

Article history:

Received 8 April 2017

Revised 25 October 2017

Accepted 17 November 2017

Keywords:

Variable stiffness

Antagonistic stiffness

Prestressed elastic systems

Synthesis method

ABSTRACT

The design of variable stiffness components is of interest in several applicative contexts. One way to provide large stiffness variation is to use antagonistic stiffness in prestressed elastic systems composed of linear springs. Interestingly, such systems can be designed so that the stiffness in specific directions is only controlled by the prestress within the springs. In the absence of an adequate synthesis method, their exploitation for this purpose however relies today solely on the designer ability to find an arrangement of the springs that meet his requirements in terms of antagonistic stiffness variation. In this paper, a method is introduced for the synthesis of variable stiffness components using prestressed elastic systems. This method takes into account the antagonistic stiffness coming from the prestress and thus provides an efficient way to meet user-defined requirements. Several synthesis problems for usual variable stiffness components are assessed. It shows the effectiveness of the method to provide new arrangements suitable for implementations. The ability of the method to identify alternate arrangements for a given problem is also shown to ease the design, notably by introducing an exploration strategy using a predictor-corrector method.

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1. Introduction

The design of variable stiffness components is of importance in several contexts, ranging from semi-active vibration reduction [1] to compliant robotics, for which several design strategies of variable stiffness actuators are based on such components [2].

In absence of external load, the configuration of such component is in equilibrium and stable. An external load applied on the component will create a small displacement, and its stiffness is defined as the ratio of the external load to the infinitesimal displacement it creates [2]. A variable stiffness component is then defined as a mechanical system whose stiffness can be modulated in pre-defined directions while keeping unmodified its unloaded equilibrium configuration, reached in the absence of external load acting on the system, as considered in [1]. In the following, we consider only infinitesimal

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Nomenclature

\wedge	cross product operator
a	level of prestress
c	small scalar
m	number of independent inextensional mechanisms
n	number of springs
q	number of disturbance modes
s	number of independent states of self-stress
r	rank of the elastic stiffness matrix
\mathcal{B}	base of the elastic system
\mathcal{M}	m -dimensional manifold of the inextensional mechanisms
\mathcal{M}_i	i -th independent unitary inextensional mechanism
\mathcal{N}	nullspace of the non-linear system Jacobian matrix
\mathcal{P}	platform of the elastic system
\mathcal{R}	non-linear system
$\delta\mathcal{R}$	residual of the non-linear system
\mathcal{S}	s -dimensional manifold of the prestress sets
\mathbf{p}	parameter vector
$\delta\mathbf{p}$	disturbance of the parameter vector
\mathbf{r}_i	spring anchoring distance vector
\mathbf{s}_i	spring axis unit vector
$\boldsymbol{\tau}$	springs tension vector
\mathbf{S}_i	spring screw
τ_i	spring tension
l_i, l_{i0}	spring length, free-length
k_i	spring axial stiffness
\mathbf{J}	Jacobian matrix of the non-linear system
\mathbf{K}_e	elastic stiffness matrix (6×6)
\mathbf{K}_a	antagonistic stiffness matrix (6×6)
$\boldsymbol{\Omega}$	springs axial stiffness matrix ($n \times n$)
\mathbf{N}_S	states of self-stress matrix ($n \times s$)
\mathbf{N}_i	i -th disturbance mode
\mathbf{S}	unitary screws matrix ($6 \times n$)

displacement under external load, so that the behavior of the system can be linearized about this configuration to describe its behavior by a stiffness matrix.

There are several design strategies of variable stiffness components. One of them is the use of prestress elastic systems which has been reported to be particularly relevant to design variable stiffness component in this context [3]. Generally speaking, elastic systems are constituted by a set of springs that relates a rigid platform to a rigid base [4] (Fig. 1). Their lightweight nature and their proximity with bioinspired mechanisms [5,6] make them of interest. The stiffness of the coupling between the base and the platform partly depends on the *antagonistic stiffness* coming from the level of internal forces within the springs, namely the *prestress*. The use of the prestress modulation to generate variable stiffness can be very effective to obtain large stiffness variations without modifying the equilibrium configuration [1].

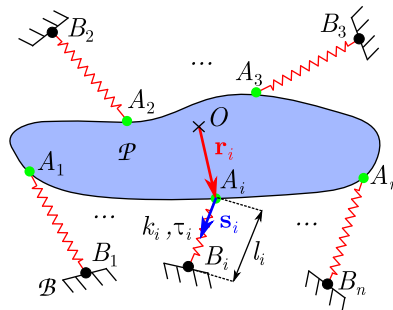


Fig. 1. General structure of elastic system. The platform \mathcal{P} is connected to the base \mathcal{B} by means of n springs.

An antagonistic stiffness optimization method of parallel robots has been proposed in [7,8] for manipulation tasks. In [3], an existing tensegrity prism is evaluated as a variable stiffness system for reduction of vibrations. The antagonistic stiffness is also exploited in parallel architectures, including cable-driven robots, to increase the system stability [9]. In all these cases, the works are limited to the analysis of known prestressed elastic systems, without any synthesis method. The goal here is therefore to provide a synthesis method to design variable stiffness components based on prestressed elastic systems from user requirements. Developing such a method constitutes a significant contribution when looking at existing synthesis methods for elastic systems. Approaches for the synthesis of elastic systems have been elaborated for the design of remote center of compliance (RCC) [10] and grasping devices [11]. The goal is then to find a set of springs to meet a desired stiffness matrix. Such a matrix relates the infinitesimal motion of the system end-effector, to the infinitesimal load to which it is submitted. Different methods have been derived in [4,12–14] to determine sets of linear/torsion springs iteratively, in order to provide a desired stiffness matrix. Neither of these previous synthesis methods take into account the contribution of the antagonistic stiffness or the effect of the prestress on the system behavior. Besides, these methods only aim at finding a single arrangement for a given problem. This can be insufficient when integration issues are encountered later in the design process. The dimension of the solution space has been evaluated in [4], but to our knowledge no work on the exploration of this space has been conducted yet.

This paper contains three contributions. A synthesis method of prestressed elastic systems is introduced that explicitly takes the antagonistic stiffness into account and its exploitation for the design of variable stiffness components. The synthesis problem is formulated from the designer initial requirements to ease the use of the method. As a second contribution, an exploration method of the solution space around an initial solution is proposed using a predictor-corrector method. This method is built to bring alternative arrangements for a given design problem, and thus several options for the designer. As a last contribution, the synthesis method is used for the design of variable stiffness components of common interest. New arrangements are then provided which constitute novel architectures for the engineering community.

In Section 2, the required tools for the synthesis of prestressed elastic systems are provided. The synthesis conditions are then formulated regarding the requirements for their exploitation in the design of variable stiffness components. The proposed synthesis method is presented in Section 3. In Section 4, the exploration method is developed. The use of the synthesis method is then illustrated in Section 5 with emphasis on the new arrangements that can be produced. Alternative arrangements for the design of a variable stiffness prismatic joint are provided in Section 6. Conclusions are finally given as well as perspectives of this work.

2. Formulation of the synthesis conditions

Our synthesis method is built upon the conditions to be fulfilled by the stiffness matrix of a variable stiffness component. These conditions, never gathered for synthesis purpose, are therefore first derived, starting from the definition of a stiffness matrix for prestressed elastic systems.

2.1. Stiffness matrix of a prestressed elastic system

For an elastic system, n linear springs connect a rigid platform \mathcal{P} to a rigid base \mathcal{B} as depicted in Fig. 1 in its unloaded equilibrium configuration. The i th spring is anchored on \mathcal{P} and \mathcal{B} in A_i and B_i respectively. The i th spring configuration is represented by a set of two vectors:

- $\mathbf{s}_i = \overrightarrow{A_i B_i} / \|\overrightarrow{A_i B_i}\|$, the unit vector in the direction of the spring axis,
- $\mathbf{r}_i = \overrightarrow{O A_i}$ which describes the anchoring point with respect to O , a point chosen on \mathcal{P} .

For the i th linear spring of axial stiffness k_i , the elastic characteristic is defined by $\tau_i = k_i(l_i - l_{i0})$ with τ_i , l_i and l_{i0} the tension, homogeneous to a force, length and free-length respectively.

In the following, the set of tensions in the springs is designated as $\boldsymbol{\tau} = [\tau_1 \dots \tau_n]^T$ and their lengths as $\mathbf{l} = [l_1 \dots l_n]^T$.

The stiffness matrix \mathbf{K} of the system in O relates the infinitesimal change in load applied on \mathcal{P} at point O , represented by the wrench $\delta \mathbf{w}$ expressed in ray coordinates [15] as suggested in [4], to the infinitesimal displacement of \mathcal{P} , represented by the twist $\delta \mathbf{p}$ expressed in axis coordinates¹. It is a 6×6 matrix defined as

$$\delta \mathbf{w} = \mathbf{K} \delta \mathbf{p} \quad (1)$$

The stiffness derives both from the elastic deformation of the spring and their tensions. The matrix \mathbf{K} is classically expressed as [9]

$$\mathbf{K} = \mathbf{K}_e + \mathbf{K}_a \quad (2)$$

with \mathbf{K}_e the *elastic stiffness* which comes from the elastic deformation of the springs, and \mathbf{K}_a the *antagonistic stiffness* which is related to the spring tensions, the so-called *antagonistic forces*. This paper will focus on diagonal stiffness matrices, since they ensure that there is no static coupling between the different directions in space: a single force/moment only generates a translation/rotation in the same direction. This decoupling is essential in practical designs.

¹ In the following, the dependency in O is not reminded in the notations for sake of simplicity.

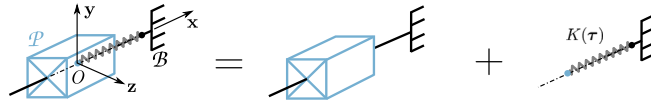


Fig. 2. Variable stiffness prismatic joint (left) decomposed as a prismatic kinematic pair (center) in parallel with a variable stiffness linear spring along its translation axis (right).

2.2. Exploitation of prestressed elastic systems for the design of variable stiffness components

Prestressed elastic systems are exploited in this paper for the design of variable stiffness component. Such exploitation of the elastic system here consists in using a variation of the antagonistic forces within the linear springs composing the system. The expected variation of the stiffness is then obtained by the modification of the antagonistic stiffness generated by these forces. The stiffness variation does not imply a modification of the geometry, or in other words, does not modify the unloaded equilibrium configuration of the system. The design issue is consequently decomposed in two tasks:

- the design of a mechanism that approximates satisfactorily the desired kinematic constraints in the coupling between B and P,
- the design of a variable stiffness elastic coupling between B and P in the unconstrained directions.

Such direction can be either a translation or a rotation [1]. Since only infinitesimal displacements along these directions are considered, the desired kinematics are considered locally about the unloaded equilibrium configuration.

As an example, the design of a variable stiffness prismatic joint can be advantageously considered to describe the two tasks. As illustrated in Fig. 2, the task can be decomposed in the synthesis of a prismatic kinematic pair associated in parallel with a variable stiffness elastic coupling in the x-axis translation direction. The only mobility of the component is here an infinitesimal displacement in this direction. It should be noticed that the spring depicted in Fig. 2 is used to conveniently represent the variable stiffness elastic coupling between B and P in the direction of the x-axis translation. In practice, the stiffness in this direction comes from the antagonistic stiffness of the elastic system used to implement such variable stiffness component. This antagonistic stiffness is generated by the prestress in the set of springs composing the elastic system. The stiffness matrix of a prestressed elastic system in its unloaded equilibrium configuration, expressed in the frame (O, x, y, z), is thus decomposed as:

$$\mathbf{K} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_t & 0 & 0 & 0 & 0 \\ 0 & 0 & K_t & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & K_\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & K_\theta \end{bmatrix}}_{\mathbf{K}_e} + \underbrace{\begin{bmatrix} K(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{K}_a} \tag{3}$$

Obviously, an elastic system is non-rigid as it is composed of linear springs. For this reason, the kinematic constraints related to the prismatic kinematic pair are approximated by a diagonal stiffness matrix with a null translational stiffness in the x-axis, and non-null values K_t and K_θ for other translational and rotational stiffnesses. Their values are to be chosen according to the desired kinematic behavior as they relate allowed parasitic motions to expected external loads. The kinematic behavior needs to be respected for any value of antagonistic forces, even null ones. For this reason, it is ensured by the elastic stiffness matrix \mathbf{K}_e of the prestressed elastic system, that is independent from antagonistic forces, as expressed in (3).

The variable elastic coupling is represented by a linear spring of variable stiffness, characterized by a rank-1 stiffness matrix with only one non-zero term $K(\tau)$. This element corresponds to the translational stiffness in the x-direction to be controlled by the antagonistic forces τ . This constitutes the antagonistic stiffness matrix \mathbf{K}_a of the prestressed elastic system to be synthesized, as expressed in (3).

With such design decomposition in two tasks, the stiffness in the desired direction is in the general case controlled by the antagonistic forces within the prestressed elastic system, and the proper kinematic behavior of the component is ensured by the elastic stiffness matrix \mathbf{K}_e , therefore for any set of antagonistic forces. The stiffness in the desired direction can thus be fully controlled through the springs tension that generates the antagonistic stiffness and set to zero if needed.

Two conditions on the elastic and antagonistic stiffness matrices stem from this design approach, which are now derived.

2.3. Condition on the elastic stiffness matrix

The expression of an elastic stiffness matrix has been derived analytically previously in [4]. Following this formulation, a spring axis is represented by the following screw

$$\mathbf{S}_i = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{r}_i \wedge \mathbf{s}_i \end{bmatrix} \tag{4}$$

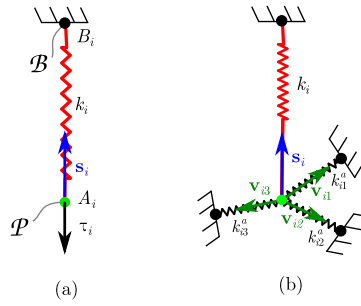


Fig. 3. Model of a loaded spring (a) using 4 unloaded springs (b).

so that the elastic stiffness matrix is expressed as

$$\mathbf{K}_e = \mathbf{S}\mathbf{\Omega}\mathbf{S}^T \tag{5}$$

where the i th column of \mathbf{S} is \mathbf{S}_i , and $\mathbf{\Omega}$ is a $n \times n$ diagonal matrix that includes the springs stiffnesses k_i on its diagonal. \mathbf{S} is a $6 \times n$ matrix called the *unitary screws matrix* (also sometimes referred as the *wrench matrix*) [16].

As it can be seen from Eq. (5), the elastic stiffness of the system depends on the springs arrangement described by \mathbf{S} . It was outlined in Section 2.2 that the components of the elastic stiffness matrix are set to zero if they correspond to the directions of variable stiffness. The elastic stiffness comes from the springs deformation. The springs should therefore be arranged so that an infinitesimal motion along these directions do not generate deformations in the springs. In this case, the elastic system is *singular* in the sense that \mathbf{K}_e and \mathbf{S}^T are rank deficient and the nullspace \mathcal{M} of \mathbf{S}^T includes all the motions of the platform \mathcal{P} that are possible without deformation in the springs. Such motions are called *inextensional mechanisms* as introduced in [17] and constitute the directions in which the elastic stiffness is null [1].

The manifold \mathcal{M} is spanned by m independent inextensional mechanisms so that $m = \dim(\mathcal{M})$. The rank r of the elastic stiffness matrix is then equal to $6 - m$. A singular elastic system is *kinematically indeterminate* [18], meaning that $m \geq 1$ and that \mathcal{M} is spanned by at least one inextensional mechanism. In this case, \mathbf{K}_e is thus rank deficient ($r < 6$). This constitutes a condition to be fulfilled by the elastic system under synthesis, which constrains the springs arrangement.

2.4. Condition on the antagonistic stiffness matrix

Derivation of the antagonistic stiffness matrix requires a proper modeling of spring pretension effect. The model introduced in [9] is here being used, based on an equivalent model composed of 4 unloaded springs.

A spring submitted to a tension τ_i is illustrated in Fig. 3(a). It can be modeled by 4 unloaded springs as represented in Fig. 3(b). The first one is aligned with the real spring axis, while the 3 others are in the plane orthogonal to this axis, along the axes \mathbf{v}_{i1} , \mathbf{v}_{i2} and \mathbf{v}_{i3} . Their stiffnesses k_{i1}^a , k_{i2}^a , k_{i3}^a are dependent on the tension τ_i and the length of the real spring. They represent the contribution of the spring tension to the stiffness matrix. The antagonistic stiffness matrix \mathbf{K}_a can then be computed with an expression similar to (5) for these three springs. A detailed explanation of this computation can be found in [9].

The variation of antagonistic stiffness within the elastic system is created by a variation of the antagonistic forces. Such forces can only be produced if the elastic system is prestressable, *i.e.* the spring tensions τ can be modified for a constant equilibrium configuration. Such tension sets, also called *prestress sets*, are included in the nullspace \mathcal{S} of \mathbf{S} [9]: these prestress sets ensure the static equilibrium of the system without modification of its configuration, and are thus compatible with the so-called *states of self-stress* of the system [18].

The manifold \mathcal{S} is spanned by s independent states of self-stress so that $s = \dim(\mathcal{S})$. The system is prestressable when it is *statically indeterminate* [18], meaning that $s \geq 1$ and that \mathcal{S} is spanned by at least one state of self-stress.

Given that $\text{rank}(\mathbf{S}) = \text{rank}(\mathbf{S}^T) = \text{rank}(\mathbf{K}_e) = r$, the following relationship between r , n , and s can be deduced from the rank-nullity theorem:

$$s = n - r \tag{6}$$

which is also known as the Maxwell's rule [18]. In the end, according to Eq. (6), the number n of springs needs to verify $n \geq r + 1$ since the system has to be prestressable to be exploited as a variable stiffness component. This constitutes a second condition to be fulfilled by the elastic system to be synthesized, which is here related to the number of springs.

3. Synthesis method

3.1. Problem formulation

During the synthesis of a variable stiffness component, the designer is in priority looking for a component with a required stiffness coming from the antagonistic forces in the springs. The synthesis method is thus first focused on the relation

between the desired variation of stiffness in pre-defined directions, and the desired variation of the prestress set to achieve it. The modification of the prestress set leads to the modification of the component stiffness about its unloaded equilibrium configuration, and not to platform motions. The designer is also looking for a suitable kinematic behavior, regardless of the antagonistic forces within the springs. As explained in Section 2.2, such behavior is characterized by the elastic stiffness matrix \mathbf{K}_e . The designer requirement has thus first to be expressed as a targeted singular elastic stiffness matrix, denoted in the following \mathbf{K}_e^* . A solution of interest for the designer corresponds to a diagonal elastic stiffness matrix, as outlined in Section 2.1. The inextensional mechanisms are then trivial to define, and this paper focuses on such situations. This matrix \mathbf{K}_e^* depends on the desired variable stiffness directions, and on the desired elastic stiffness in the other directions to ensure a proper kinematic behavior. This re-formulation from the designer requirements to a targeted stiffness matrix is indeed straightforward as shown in the different design problems assessed in Section 5.

The synthesis method proposed in this paper aims at finding arrangements of the springs that ensure:

- the existence of inextensional mechanisms along the desired direction of variable stiffness of the component,
- the desired variation of antagonistic stiffness in these directions for a predefined variation of the level of prestress within the springs.

In order to ease the synthesis formulation by the designer, the set of prestress in the springs is expressed as a product $a\boldsymbol{\tau}_0$, with a a scalar that conveniently represents the level of prestress within the springs and $\boldsymbol{\tau}_0$ a user-defined set of prestress compatible with the elastic system equilibrium ($\boldsymbol{\tau}_0 \in \mathcal{S}$). The designer requirements can thus be easily formulated as:

- A targeted matrix \mathbf{K}_e^* that corresponds to the desired kinematics of the component.
- And for each desired variable stiffness direction of the component:
 - a range of stiffness defined by an interval $(K_{\min}, K_{\max}) \in \mathbb{R}_+^2$,
 - a range of prestress level defined by an interval $(a_{\min}, a_{\max}) \in \mathbb{R}_+^2$.

The scalars K_{\min} and K_{\max} are the minimum and maximum stiffnesses that are respectively reached for the prestress sets $a_{\min}\boldsymbol{\tau}_0$ and $a_{\max}\boldsymbol{\tau}_0$. It can be noticed that $K_{\min} = 0$ if $a_{\min} = 0$ which constitutes the lowest achievable stiffness in the desired direction, reached in the absence of prestress within the springs. Variable stiffness is thus obtained, as the stiffness in the considered direction can be tuned between K_{\min} and K_{\max} by changing the prestress level between a_{\min} and a_{\max} .

From this formulation, two constraints have clearly to be integrated within the synthesis process. First, one has to ensure that the prestress set $\boldsymbol{\tau}_0$ is included in \mathcal{S} . It means that constraints on \mathcal{S} have to be formulated accordingly. Secondly, the antagonistic stiffness reached for the minimum and maximum level of prestress has to meet the user requirements.

3.2. Constraints on the states of self-stress

The manifold \mathcal{S} of the spring prestress sets is spanned by a basis of \mathcal{S} that constitutes the s independent states of self-stress. In this paper, the following constraints are formulated on this basis

$$\mathbf{N}_{\mathcal{S}} = \mathbf{0} \quad (7)$$

where $\mathbf{N}_{\mathcal{S}}$ is a full-rank $n \times s$ matrix whose columns impose the basis vectors of \mathcal{S} , i.e. the s independent states of self-stress. This constraint matrix thus leads to $6s$ independent scalar constraints. This matrix is designated as the *states of self-stress matrix*.

3.3. Constraints on the antagonistic stiffness

Given a maximum level of prestress a_{\max} , the maximum antagonistic stiffness along the inextensional mechanisms has to meet the user requirements for the prestress set $a_{\max}\boldsymbol{\tau}_0 \in \mathcal{S}$.

The manifold \mathcal{M} of the inextensional mechanisms is spanned by a basis of \mathcal{M} that constitutes the m independent inextensional mechanisms. Let \mathcal{M}_i be a unit 6-vector that describes the i th independent inextensional mechanism so that $[\mathcal{M}_1 \dots \mathcal{M}_m]$ is a normal basis of \mathcal{M} . The antagonistic stiffness in this direction can be easily computed as $\mathcal{M}_i^T \mathbf{K}_a \mathcal{M}_i$ that corresponds to the variation of elastic energy in the linearized system due to the unitary motion \mathcal{M}_i . If \mathcal{M}_i is a pure translation (resp. a pure rotation), this value corresponds to the translational (resp. rotational) stiffness in the direction of \mathcal{M}_i as $\|\mathcal{M}_i\| = 1$.

As a consequence, the following corresponding constraint can be formulated to ensure a maximum antagonistic stiffness K_{\max}^i in the direction of \mathcal{M}_i and for the springs prestress set $a_{\max}\boldsymbol{\tau}_0$

$$\mathcal{M}_i^T \mathbf{K}_a (a_{\max}\boldsymbol{\tau}_0) \mathcal{M}_i - K_{\max}^i = 0 \quad (8)$$

that leads to m independent scalar constraints. Same reasoning can be achieved for minimum level of prestress/stiffness to obtain the condition

$$\mathcal{M}_i^T \mathbf{K}_a (a_{\min}\boldsymbol{\tau}_0) \mathcal{M}_i - K_{\min}^i = 0 \quad (9)$$

3.4. Parameters set and synthesis method

Each spring is characterized by 8 parameters: 6 for \mathbf{s}_i and \mathbf{r}_i , 1 for its length l_i , and 1 for its stiffness k_i . A n -spring elastic system can thus be fully defined by an $8n \times 1$ parameter vector \mathbf{p} that gives the locations, lengths and the elastic characteristics of the springs.

The synthesis method introduced in this paper allows to generate systems for any stiffness matrix, from rank 1 to 6, even for an unlimited number of springs. The parameter vector \mathbf{p} is obtained in two steps:

Step 1. Express the design requirements as follows

- Define the minimum/maximum level of stiffness $(K_{\min}^i, K_{\max}^i) \forall i \in \{1, \dots, m\}$.
- Express the targeted matrix \mathbf{K}_e^* , that exhibits m independent inextensional mechanisms corresponding to the m directions of variable stiffness.
- Choose the number n of springs ensuring that the number s of states of self-stress is superior to 1 according to Eq. (6).
- Choose the prestress set $\boldsymbol{\tau}_0$ and the minimum/maximum level of prestress (a_{\min}, a_{\max}) .
- Choose the states of self-stress \mathbf{N}_S following Section 3.2 ensuring that $\boldsymbol{\tau}_0 \in \text{span}(\mathbf{N}_S)$.

Step 2.

- Formulate the synthesis problem as the following non-linear equations system

$$\begin{aligned} \mathbf{K}_e^* - \mathbf{S}(\mathbf{p})\boldsymbol{\Omega}(\mathbf{p})\mathbf{S}(\mathbf{p})^T &= \mathbf{0} \\ \mathbf{S}(\mathbf{p})\mathbf{N}_S &= \mathbf{0} \\ \mathcal{M}_i^T \mathbf{K}_a(a_{\min} \boldsymbol{\tau}_0) \mathcal{M}_i - K_{\min}^i &= 0 \quad \forall i \in \{1, \dots, m\} \\ \mathcal{M}_i^T \mathbf{K}_a(a_{\max} \boldsymbol{\tau}_0) \mathcal{M}_i - K_{\max}^i &= 0 \quad \forall i \in \{1, \dots, m\} \\ \mathbf{s}_i^T \mathbf{s}_i - 1 &= 0 \quad \forall i \in \{1, \dots, n\} \end{aligned} \quad (10)$$

under the constraints

$$\begin{cases} k_i > 0, \quad \forall i \in \{1, \dots, n\} \\ l_i > 0, \quad \forall i \in \{1, \dots, n\} \end{cases} \quad (11)$$

- Estimate the parameters vector \mathbf{p} that solves (10) subjected to (11) and the possible additional constraints.

Other constraints can be introduced in (11) to ensure the practical feasibility of the solution, with for instance equal length or stiffness for the springs set.

In the second step, one way to solve the constrained non-linear system (10) and (11) is to use well-known iterative numerical methods such as a Levenberg–Marquardt algorithm. In the following, its implementation in Matlab (The Mathworks Inc.) is being used. Several solutions may then be obtained by application of this method, depending on the initial parameter vector, the number of springs and the states of self-stress.

The method described here above is built for the synthesis of variable stiffness components using antagonistic stiffness. It is in that sense novel and can be a precious tool for the designer. It is however important to outline that the generation of alternative arrangements for a given design problem is often of interest for this designer since they constitute as many design options to be considered. Such arrangements can be obtained through several ways. First, as the solution depends on the initial parameter vector, this latter can be randomly initialized. The repetition of the method for a random initial parameter vector can possibly lead to different solutions. The efficiency of the approach is then however difficult to ensure. Second, the synthesis conditions, namely the number of springs, the number of states of self-stress and their basis, can be modified to provide alternative arrangements. Finally, the solutions space around an initial solution can be explored to determine a set of solutions for the same synthesis conditions. Such an exploration strategy is developed in Section 4, and the two latter approaches are applied in Section 6.

4. Exploration method

4.1. Predictor-corrector method

Let \mathcal{R} be the non-linear system (10) that describes the design problem. Let \mathbf{p}_0 be a solution to the design problem obtained after synthesis so that

$$\mathcal{R}(\mathbf{p}_0) = \mathbf{0} \quad (12)$$

Exploration of the solution space consists in determining solutions to the design problem in the neighborhood of \mathbf{p}_0 . A predictor-corrector method [19] is here proposed, that is based on the repetition of a 2-step process:

- **Prediction step:** the initial parameters set \mathbf{p}_0 is disturbed with a small variation $\delta\mathbf{p}$.
- **Correction step:** the system is solved with the disturbed parameters set $\mathbf{p}_0 + \delta\mathbf{p}$ as an initial condition in order to find a new solution.

Table 1
Description of the three case studies.

Problem	Characteristics	Targeted elastic stiffness matrix
Spherical joint of center O	<ul style="list-style-type: none"> • $m = 3$ • $\mathcal{M}_1 = [0, 0, 0, 1, 0, 0]^T$ • $\mathcal{M}_2 = [0, 0, 0, 0, 1, 0]^T$ • $\mathcal{M}_3 = [0, 0, 0, 0, 0, 1]^T$ • $r = 3$ 	$\mathbf{K}_\xi^* = \text{diag}(K_t, K_t, K_t, 0, 0, 0)$
Revolute joint of axis (O, \mathbf{x})	<ul style="list-style-type: none"> • $m = 1$ • $\mathcal{M}_1 = [0, 0, 0, 1, 0, 0]^T$ • $r = 5$ 	$\mathbf{K}_\xi^* = \text{diag}(K_t, K_t, K_t, 0, K_\theta, K_\theta)$
Prismatic joint of axis (O, \mathbf{x})	<ul style="list-style-type: none"> • $m = 1$ • $\mathcal{M}_1 = [1, 0, 0, 0, 0, 0]^T$ • $r = 5$ 	$\mathbf{K}_\xi^* = \text{diag}(0, K_t, K_t, K_\theta, K_\theta, K_\theta)$

4.2. Tangent prediction

A key point in the method is the choice of an appropriate disturbance direction. For this reason, the Jacobian \mathbf{J} of the system \mathcal{R} is introduced as

$$\mathbf{J}\delta\mathbf{p} = \delta\mathcal{R} \tag{13}$$

with $\delta\mathcal{R}$ the residual introduced by the variation $\delta\mathbf{p}$ of the initial solution so that

$$\mathcal{R}(\mathbf{p}_0 + \delta\mathbf{p}) = \delta\mathcal{R} \tag{14}$$

A so-called *tangent predictor* is then considered, also known as Euler’s method [20]. To do so, the nullspace \mathcal{N} of \mathbf{J} is considered, since it spans the disturbances $\delta\mathbf{p}$ around \mathbf{p}_0 not modifying $\delta\mathcal{R}$ at the first order. For an infinitesimal variation $\delta\mathbf{p}$ it is possible to write

$$\forall \delta\mathbf{p} \in \mathcal{N}, \mathbf{J}\delta\mathbf{p} = \delta\mathcal{R} \simeq \mathbf{0} \tag{15}$$

Let $[\mathbf{N}_1, \dots, \mathbf{N}_q]$ be a basis of \mathcal{N} , \mathbf{N}_i is a $8n \times 1$ vector so that

$$\forall i \in \{1, \dots, q\} \mathcal{R}(\mathbf{p}_0 + c\mathbf{N}_i(\mathbf{p}_0)) = \mathbf{0} \tag{16}$$

with c a scalar. We denote \mathbf{N}_i as a *disturbance mode*. A disturbance $\delta\mathbf{p}$ computed as a linear combination of the basis vectors of \mathcal{N} , i.e. computed as

$$\delta\mathbf{p} = \sum_{i=1}^q c_i \mathbf{N}_i(\mathbf{p}_0) \tag{17}$$

does not modify the residual of \mathcal{R} at the first order if c is chosen small enough to ensure that the disturbance is infinitesimal. This iterative method thus provides alternative arrangements about an initial solution.

5. Application of the synthesis method

In this section, the interest of the synthesis method is outlined by the resolution of three design problems of variable stiffness components. Three relevant problems regarding applications in robotics are chosen, namely the design of spherical, revolute and prismatic joints of variable stiffnesses. The desired characteristics and the corresponding targeted stiffness matrix are summed up in Table 1. The revolute and prismatic joints both exhibit a single inextensional mechanism ($m = 1$) that corresponds to the rotation and the translation about which the stiffness variation is needed (here designated as the \mathbf{x} -axis). The spherical joint exhibits three inextensional mechanisms ($m = 3$) along the three rotations with respect to the joint center O . The corresponding elastic stiffness matrices \mathbf{K}_ξ^* are built from this data. These diagonal matrices, obviously rank deficient, clearly show the inextensional mechanisms. Translational stiffnesses K_t and angular stiffnesses K_θ about the non-singular directions are chosen equal.

The new arrangements presented in section 5.1, 5.2, 5.3 and 6.2 (corresponding to Fig. 4, 5, 6 and 8 respectively) are provided as interactive figures in the multimedia components provided as supplementary materials. Arbitrary numerical values are then chosen for the elastic stiffnesses namely $K_t = 1 \text{ N/m}$ and $K_\theta = 10 \text{ Nm/rad}$. Besides, as the minimum stiffness $K_{\min} = 0$ can always be reached for $a_{\min} = 0$, only one constraint on the maximum level of stiffness is considered. In order for the solution to be realistic from a designer point of view, additional constraints are added to the formulation given in (10). Namely, equal distances $\|\mathbf{r}_i\|$ are imposed for every spring, as well as equal spring stiffnesses and lengths.

5.1. Variable stiffness spherical joint

The synthesis of a spherical joint is first assessed. The targeted matrix \mathbf{K}_ξ^* as well as the corresponding directions of variable stiffness are provided in Table 1. The number of springs n is chosen equal to 6 so that $s = 3$. Minimum and maximum

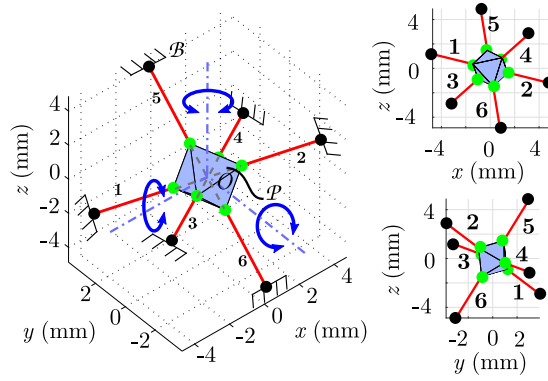


Fig. 4. Variable stiffness spherical joint of center O . Springs are in red with bold labels, anchor points on \mathcal{B} and \mathcal{P} in black and green respectively. \mathcal{P} is represented by the blue solid and O is in $(0, 0, 0)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

antagonistic stiffnesses to be reached are chosen equal to:

$$K_{\min}^1 = K_{\min}^2 = K_{\min}^3 = 0 \text{ Nm/rad} \quad (18)$$

$$K_{\max}^1 = K_{\max}^2 = K_{\max}^3 = 10 \text{ Nm/rad} \quad (19)$$

The prestress set is chosen as

$$\boldsymbol{\tau}_0 = [1 \dots 1]^T, \quad a_{\min} = 0, \quad a_{\max} = 1 \quad (20)$$

In order to ensure that $\boldsymbol{\tau}_0$ can be generated, the states of self-stress are chosen as

$$\mathbf{N}_S = \left[[1 \ 1 \ 0 \ 0 \ 0 \ 0]^T, [0 \ 0 \ 1 \ 1 \ 0 \ 0]^T, [0 \ 0 \ 0 \ 0 \ 1 \ 1]^T \right] \quad (21)$$

meaning that the pairs of springs $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ can be independently prestressed, which includes the prestress set $\boldsymbol{\tau}_0$.

A solution provided by the synthesis method for these conditions is depicted in Fig. 4 and the resulting parameters are given in Appendix A. The springs characteristics are defined by their axial stiffness and their length, and their pose are defined by the matrix \mathbf{S} . The last three columns are equal to zero which shows that all the directions of the springs are coincident in O . It is indeed a trivial solution that exhibits three pairs of colinear springs whose directions intersect in O thus creating three inextensional mechanisms in the direction of the three rotations represented by the blue arrows. This arrangement has been previously proposed in [21] for the design of a variable stiffness spherical joint. It is here interestingly automatically retrieved by this method.

5.2. Variable stiffness revolute joint

The synthesis of a variable stiffness revolute joint is then considered. The corresponding stiffness matrix and directions of variable stiffness are provided in Table 1. A 6-spring elastic system is here synthesized so that $s = 1$. The minimum and maximum rotational stiffness to be reached around x -axis is chosen equal to:

$$K_{\min}^1 = 0 \text{ Nm/rad} \quad (22)$$

$$K_{\max}^1 = 10 \text{ Nm/rad} \quad (23)$$

Considering

$$\boldsymbol{\tau}_0 = [1 \dots 1]^T, \quad a_{\min} = 0, \quad a_{\max} = 1, \quad (24)$$

$$\mathbf{N}_S = [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \quad (25)$$

so that all the springs of the system can indeed be uniformly prestressed.

The geometry of the synthesized system is depicted in Fig. 5. The directions of the springs are coincident on the \mathbf{x} -axis thus creating the inextensional mechanism about the rotation represented by the blue arrow (see the resulting parameters in Appendix A). The arrangement of the springs $\{1, 2, 4\}$ with respect to $\{3, 5, 6\}$ ensures that the 6 springs can be uniformly prestressed. Besides, the system symmetry about O , as illustrated on the projected views, ensures that the translational stiffnesses in the directions \mathbf{x} , \mathbf{y} and \mathbf{z} are the same, as well as the rotational stiffness around \mathbf{y} and \mathbf{z} . This solution is a new arrangement to the knowledge of the authors that exhibits interesting properties to be exploited as a variable stiffness component.

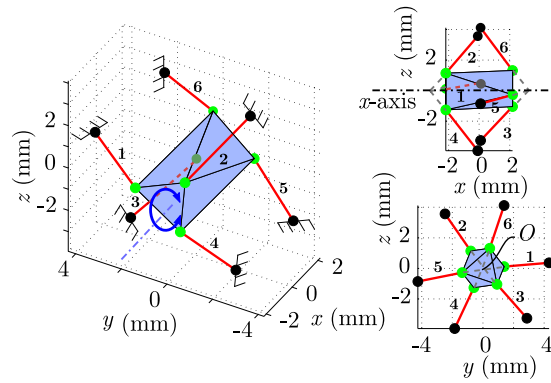


Fig. 5. Variable stiffness revolute joint of axis $(0, \mathbf{x})$. 3D-view on the left and two projected views on the right. Hidden springs are plotted in red dotted lines and intersecting spring directions in gray dotted lines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

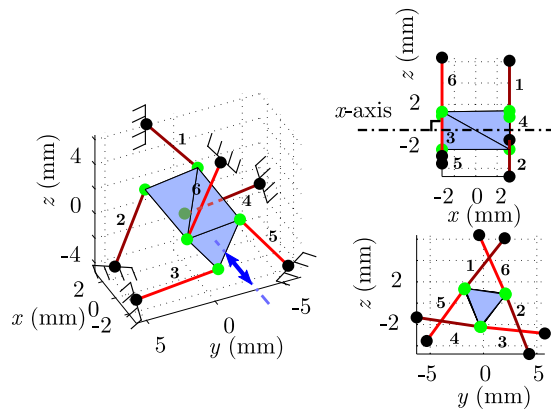


Fig. 6. Variable stiffness prismatic joint of axis $(0, \mathbf{x})$. 3D-view on the left and two projected views on the right. Springs 2, 4 and 5 have been darkened for a better reading of the figure.

5.3. Variable stiffness prismatic joint

Finally, a variable stiffness prismatic joint using a 6-spring elastic system is being synthesized considering $K_{\max}^1 = 1 \text{ N/m}$ as the translational stiffness to be reached in the x -axis. The states of self-stress \mathbf{N}_S and the prestress set $\boldsymbol{\tau}_0$ are the same as for the revolute joint.

The synthesized system is depicted in Fig. 6. The directions of the springs are orthogonal to the \mathbf{x} -axis thus creating the inextensional mechanism about the translation represented by the blue arrow (see the resulting parameters in Appendix A). Same remarks as before can be formulated regarding the arrangement of the springs to provide the uniform prestress and the homogeneous stiffness in the different directions. This also constitutes a new arrangement to the knowledge of the authors that can interestingly be exploited for the design of a variable stiffness prismatic joint.

6. Alternative arrangements

In this section, our ability to provide several arrangements for a design problem is demonstrated. These arrangements are obtained through two ways using either a modification of the number and the shape of states of self-stress in the synthesis method presented in Section 3, or the exploration method presented in Section 4. Both ways are performed to find alternative arrangements for the design of a variable stiffness prismatic joint and for which a first solution is provided in Section 5.3. This solution depicted in Fig. 6 is a 6-spring system with a single state of self-stress. A first way to provide alternative arrangements is to explore the solutions space around this initial solution and for the same synthesis conditions.

6.1. Exploration about an initial solution

The exploration method is first implemented for this purpose. The initial solution (see Fig. 6) is used as the starting point of the exploration. 20 iterations are performed. Arrangements corresponding to iterations 1, 10 and 20 are represented in

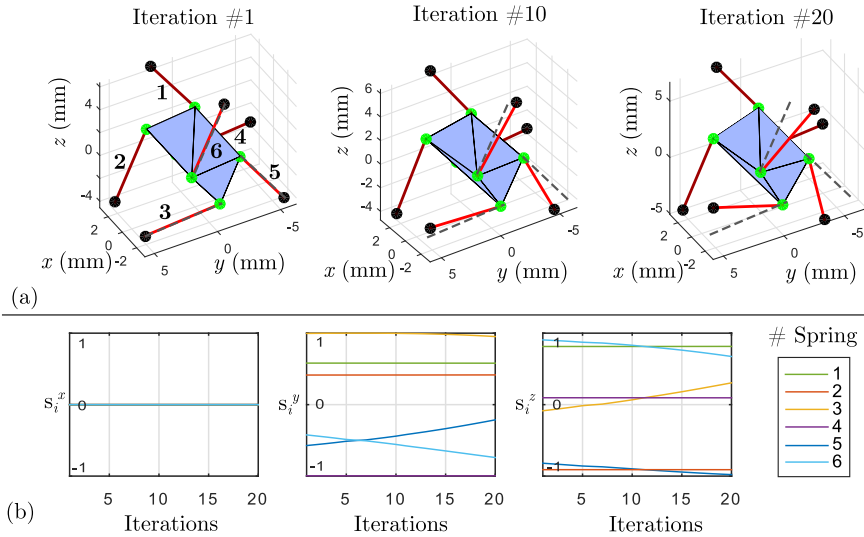


Fig. 7. Alternative arrangements for the variable stiffness prismatic joint using the exploration method. The initial axes of springs 3, 5 and 6 are represented with the dotted gray lines. (a) Solutions at iterations 1, 10 and 20, (b) evolution of the coordinates of \mathbf{s}_i along the iterations.

Fig. 7(a). The initial axes are represented with the dotted gray lines. The evolution of the coordinates s_i^x , s_i^y , s_i^z of the i th spring axis direction vector \mathbf{s}_i are represented in Fig. 7(b).

The reconfiguration is being performed while ensuring the presence of the inextensional mechanism along the \mathbf{x} -translation, since the springs remain orthogonal to the joint axis ($s_i^x = 0$). Their symmetric arrangement about this axis also ensures that the elastic stiffness matrix and the states of self stress fulfill the design requirements.

In this example, the use of the exploration method allows to find 20 alternative arrangements that are also solutions for the design of a variable stiffness prismatic joint. The figure clearly shows the axis reconfiguration of the springs 3, 5 and 6 during the exploration and the large modification of geometry along the iterations.

This thus constitutes a first way to provide alternative arrangements as solutions of a synthesis problem.

6.2. Multiple states of self-stress for alternative arrangements

Another way to provide new solutions is to increase the number of springs, and thus the number of states of self-stress. Alternative arrangements can then be obtained by a different choice of the states of self-stress that spans S and that are defined by the s basis vectors stacked in \mathbf{N}_S . As explained in Section 3.1, this basis can be freely chosen, under the condition that it can span the prestress set $\boldsymbol{\tau}_0$. The number of states of self-stress s linearly increases with the number of springs n (see Eq. (6)) so that adding a spring increases the basis \mathbf{N}_S dimension by adding an additional state of self-stress. Such option is thus interesting to provide alternative solutions to a given design problem.

In the case of the variable stiffness prismatic joint, the synthesis is now considered with a 10-spring system so that the number of states of self-stress is equal to five ($s = 5$). The prestress set $\boldsymbol{\tau}_0 = [1 \dots 1]^T$ is not modified. In this example, two different states of self-stress basis that both span $\boldsymbol{\tau}_0$ are considered, namely

$$\mathbf{N}_{S1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T, \quad \mathbf{N}_{S2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \quad \text{so that re-}$$

spectively 5 pairs and 5 sextuples of springs can be independently prestressed.

The results obtained with the synthesis method under these new conditions are depicted in Fig. 8 and the resulting parameters are shown in Appendix A. For the first solution with \mathbf{N}_{S1} , the antagonistic arrangements of the successive pairs of springs ensure the desired states of self-stress (springs 3 and 4 for example). It can be noticed that the method does not manage the springs overlap, such as springs 6 and 10. Similar overlaps occur for the second solution (springs 2 and 8, 1 and 7, etc). Even if less interesting for novel designs, these solutions show the efficiency of the method to provide numerous different arrangements to solve the same synthesis problem with different conditions.

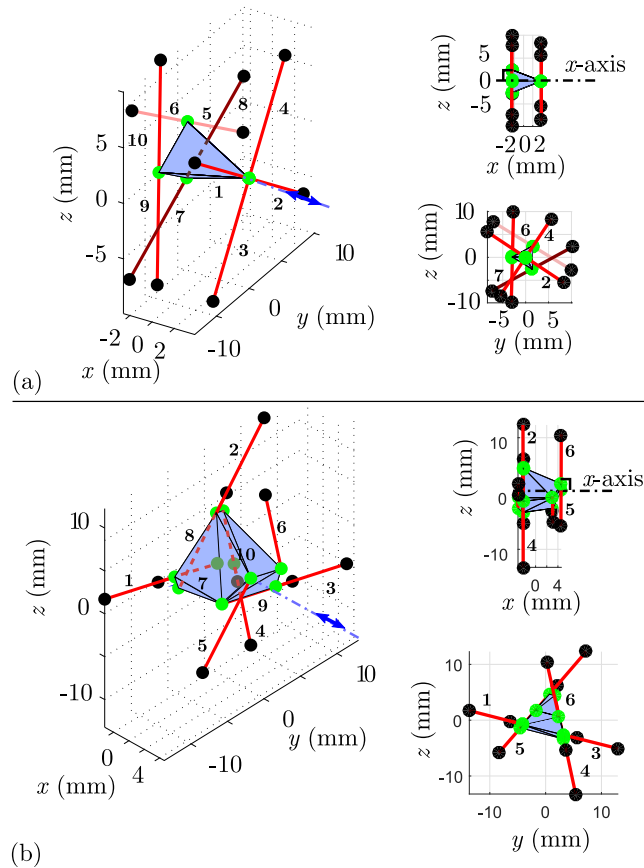


Fig. 8. Alternative arrangements for a variable stiffness prismatic joint using 10 springs. (a) and (b) with states of self-stress N_{S1} and N_{S2} respectively.

7. Conclusions and perspectives

In this paper, a new method is introduced for the synthesis of variable stiffness components using prestressed singular elastic systems. This work is motivated by the design of new variable stiffness components that can be obtained through the use of antagonistic stiffness variation in the singular directions of such systems. For that purpose, the states of self-stress as well as the level of antagonistic stiffness are taken into account during the process. This constitutes one main contribution of this paper. Additionally, an exploration method is proposed and enables the designer to explore the solution space starting from an initial solution generated by this synthesis method. This second contribution can notably be useful to provide alternative arrangements that solve a given design problem.

The effectiveness of these approaches is demonstrated through several applications. New arrangements for spherical, revolute and prismatic joints with variable stiffness have been provided. These arrangements are suitable for implementation and constitutes the third contribution of the paper. Alternative arrangements for the design of a variable stiffness prismatic joint were ultimately presented. Such arrangements are obtained using the proposed exploration method and also by increasing the number of springs and changing the shape of the states of self-stress.

Several extensions of the proposed synthesis and exploration methods can be foreseen. First, springs overlaps condition could be integrated as it may be required in some cases. This aspect has already been assessed for interference situations in parallel robots and cable-driven robots [22,23] from which corresponding synthesis constraints could stem to provide this overlapping avoidance feature. In practice, the designer can however already choose favorable synthesis conditions to produce viable solutions, as demonstrated in [21] for the design of a variable stiffness spherical joint using multimaterial additive manufacturing combined with pneumatic actuation of the prestress. Practical implementation of these arrangements can indeed be considered using different integration methods. Another extension would be to develop a classification of the singularities of the synthesized systems for a better understanding of the solution space to be explored. This could be performed using the Grassmann varieties as it was already applied for the study of parallel manipulators singularities in [24].

Acknowledgment

This work was supported by French state funds managed by the ANR within the Investissements d’Avenir programme (Robotex ANR-10-EQPX-44, Labex CAMI - ANR-11-LABX-0004) and by the Région Alsace and Aviesan France Life Imaging infrastructure.

Appendix A. Detailed results

A1. Parameters

Resulting parameters for the different solutions are shown in Table A.1.

A2. Coordinates of the springs anchoring points

The coordinates of the springs anchoring points for the different provided designs are shown in Table A.2. They are expressed in mm in the frame (O, **x**, **y**, **z**), the *i*th row corresponding to the coordinates of A_i or B_i in their respective column.

Table A1
Parameters of the solutions.

		k (N/mm)	l (mm)	S^T
6-spring systems	Spherical joint (see Section 5.1)	0.50	4	$\begin{bmatrix} 0.76 & -0.58 & 0.30 & 0 & 0 & 0 \\ -0.76 & 0.58 & -0.30 & 0 & 0 & 0 \\ 0.15 & 0.59 & 0.79 & 0 & 0 & 0 \\ -0.15 & -0.59 & -0.79 & 0 & 0 & 0 \\ -0.63 & -0.56 & 0.55 & 0 & 0 & 0 \\ 0.63 & 0.56 & -0.55 & 0 & 0 & 0 \end{bmatrix}$
	Revolute joint (see Section 5.2)	0.50	3.6	$\begin{bmatrix} 0.58 & 0.81 & 0.069 & 0 & 0.22 & -2.6 \\ 0.58 & -0.47 & 0.67 & 0 & 2.1 & 1.5 \\ -0.58 & 0.54 & -0.61 & 0 & 1.9 & 1.7 \\ 0.58 & -0.35 & -0.74 & 0 & -2.3 & 1.1 \\ -0.58 & -0.8 & -0.16 & 0 & 0.51 & -2.5 \\ -0.58 & 0.26 & 0.77 & 0 & -2.4 & 0.82 \end{bmatrix}$
	Prismatic joint (see Section 5.3)	0.33	6	$\begin{bmatrix} 0 & -0.59 & -0.81 & 2.2 & -2.6 & 1.9 \\ 0 & 0.37 & -0.93 & -2.2 & 2.9 & 1.2 \\ 0 & 0.99 & -0.1 & 2.2 & -0.33 & -3.1 \\ 0 & -0.99 & 0.15 & -2.2 & -0.47 & -3.1 \\ 0 & 0.62 & 0.78 & -2.2 & -2.5 & 2.0 \\ 0 & -0.41 & 0.91 & 2.2 & 2.9 & 1.3 \end{bmatrix}$
10-spring prismatic joints	Solution for N_{S1} (see Section 6.2)	0.20	10	$\begin{bmatrix} 0 & -0.83 & 0.55 & 0 & -2.1 & -3.2 \\ 0 & 0.83 & -0.55 & 0 & 2.1 & 3.2 \\ 0 & -0.55 & -0.83 & 0 & 3.2 & -2.1 \\ 0 & 0.55 & 0.83 & 0 & -3.2 & 2.1 \\ 0 & 0.85 & -0.52 & -2.9 & -1.3 & -2.2 \\ 0 & -0.85 & 0.52 & 2.9 & 1.3 & 2.2 \\ 0 & -0.88 & -0.48 & -2.9 & -1.2 & 2.3 \\ 0 & 0.88 & 0.48 & 2.9 & 1.2 & -2.3 \\ 0 & -0.025 & -1.0 & 2.9 & -2.6 & 0.065 \\ 0 & 0.025 & 1.0 & -2.9 & 2.6 & -0.065 \end{bmatrix}$
	Solution for N_{S2} (see Section 6.2)	0.20	10	$\begin{bmatrix} 0 & -0.96 & 0.24 & -1.7 & 0.77 & 3.0 \\ 0 & 0.66 & 0.76 & -2.5 & 1.7 & -1.5 \\ 0 & 0.97 & -0.24 & 1.8 & 0.74 & 2.9 \\ 0 & 0.21 & -0.99 & -2.5 & -2.2 & -0.47 \\ 0 & -0.66 & -0.76 & 2.5 & 3.4 & -3.0 \\ 0 & -0.22 & 0.98 & 2.5 & -4.5 & -1.0 \\ 0 & -0.96 & 0.24 & -1.7 & 0.77 & 3.0 \\ 0 & 0.66 & 0.76 & -2.5 & 1.7 & -1.5 \\ 0 & 0.97 & -0.24 & 1.8 & 0.74 & 2.9 \\ 0 & 0.21 & -0.98 & -2.5 & -2.2 & -0.47 \end{bmatrix}$

Table A2
Coordinates in mm of the anchoring points A_i and B_i in the frame $(O, \mathbf{x}, \mathbf{y}, \mathbf{z})$.

		A_i	B_i
6-spring systems	Spherical joint (see Section 5.1)	$\begin{bmatrix} 1.3 & -1.0 & 0.51 \\ -1.3 & 1.0 & -0.51 \\ 0.25 & 1.0 & 1.4 \\ -0.25 & -1.0 & -1.4 \\ -1.1 & -0.97 & 0.93 \\ 1.1 & 0.97 & -0.93 \end{bmatrix}$	$\begin{bmatrix} 4.4 & -3.3 & 1.7 \\ -4.4 & 3.3 & -1.7 \\ 0.83 & 3.4 & 4.5 \\ -0.83 & -3.4 & -4.5 \\ -3.6 & -3.2 & 3.1 \\ 3.6 & 3.2 & -3.1 \end{bmatrix}$
	Revolute joint (see Section 5.2)	$\begin{bmatrix} -2.2 & 1.4 & 0.12 \\ -2.2 & -0.79 & 1.1 \\ 2.2 & 0.92 & -1.0 \\ -2.2 & -0.59 & -1.3 \\ 2.2 & -1.4 & -0.27 \\ 2.2 & 0.44 & 1.3 \end{bmatrix}$	$\begin{bmatrix} -0.092 & 4.3 & 0.37 \\ -0.092 & -2.5 & 3.6 \\ 0.092 & 2.9 & -3.3 \\ -0.092 & -1.8 & -3.9 \\ 0.092 & -4.3 & -0.86 \\ 0.092 & 1.4 & 4.1 \end{bmatrix}$
	Prismatic joint (see Section 5.3)	$\begin{bmatrix} -3.2 & -1.8 & 1.3 \\ 3.2 & 2.1 & 0.82 \\ -3.2 & -0.23 & -2.2 \\ 3.2 & -0.33 & -2.2 \\ 3.2 & -1.8 & 1.4 \\ -3.2 & 2.0 & 0.91 \end{bmatrix}$	$\begin{bmatrix} -3.2 & -5.3 & -3.5 \\ 3.2 & 4.3 & -4.8 \\ -3.2 & 5.7 & -2.8 \\ 3.2 & -6.3 & -1.3 \\ 3.2 & 2.0 & 6.1 \\ -3.2 & -0.41 & 6.4 \end{bmatrix}$
10-spring prismatic joints	Solution for N_{S1} (see Section 6.2)	$\begin{bmatrix} 3.9 & 0 & 0 \\ 3.9 & 0 & 0 \\ 3.9 & 0 & 0 \\ 3.9 & 0 & 0 \\ -2.6 & 1.5 & 2.5 \\ -2.6 & 1.5 & 2.5 \\ -2.6 & 1.4 & -2.5 \\ -2.6 & 1.4 & -2.5 \\ -2.6 & -2.9 & 0.072 \\ -2.6 & -2.9 & 0.072 \end{bmatrix}$	$\begin{bmatrix} 3.9 & -8.3 & 5.5 \\ 3.9 & 8.3 & -5.5 \\ 3.9 & -5.5 & -8.3 \\ 3.9 & 5.5 & 8.3 \\ -2.6 & 10.0 & -2.8 \\ -2.6 & -7.0 & 7.7 \\ -2.6 & -7.4 & -7.3 \\ -2.6 & 10.0 & 2.2 \\ -2.6 & -3.1 & -9.9 \\ -2.6 & -2.6 & 10.0 \end{bmatrix}$
	Solution for N_{S2} (see Section 6.2)	$\begin{bmatrix} -3.1 & -4.1 & -0.74 \\ -2.2 & 0.68 & 4.6 \\ 3.1 & 3.3 & -2.7 \\ -2.2 & 3.3 & -3.4 \\ 4.6 & -1.8 & 1.7 \\ 4.6 & 2.4 & 0.66 \\ -3.1 & 3.2 & -2.6 \\ -2.2 & -4.5 & -1.4 \\ 3.1 & -4.1 & -0.8 \\ -2.2 & 1.6 & 4.4 \end{bmatrix}$	$\begin{bmatrix} -3.1 & -14.0 & 1.7 \\ -2.3 & 7.2 & 12.0 \\ 3.1 & 13.0 & -5.1 \\ -2.2 & 5.4 & -13.0 \\ 4.5 & -8.4 & -5.8 \\ 4.6 & 0.19 & 10.0 \\ -3.1 & -6.3 & -0.17 \\ -2.2 & 2.1 & 6.3 \\ 3.1 & 5.5 & -3.2 \\ -2.2 & 3.7 & -5.4 \end{bmatrix}$

Supplementary material

Supplementary material associated with this article can be found, in the online version, at [10.1016/j.mechmachtheory.2017.11.013](https://doi.org/10.1016/j.mechmachtheory.2017.11.013).

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