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Noname manuscript No.
(will be inserted by the editor)

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Abstract This paper presents a constructive design of a continuous finite time controller for a class of mechanical systems known as underactuated systems that satisfy the symmetry properties . An adaptive radial basis function neural networks (RBFNN) finite time control scheme is proposed to stabilize the underactuated system at a given equilibrium, regardless of the various uncertainties and disturbances that the system contains. First, a coordinate transformation is introduced to decouple the control input so that an n-th order underactuated system can be represented into a special cascade form. Next an adaptive robust finite-time controller is derived from the adding a power integrator (API) technique and the RBFNN to approximate the nonlinear unknown dynamics in the new space, whose bounds are supposedly unknown. The stability and finite-time convergence of the closed-loop system are established by using Lyapunov theory. To show the effectiveness of the proposed method, simulations are carried out on the rotary inverted pendulum, a typical example of an underactuated mechanical system .

Keywords: Adding Power Integrator, Underactuated Mechanical Systems, Model Uncertainties, RBFNN, Lyapunov Method.

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1 Introduction

Underactuated systems [8,9,19,20,28,42] are mechanical systems with less control inputs than degrees of freedom, consequently they have at least one unactuated degree of freedom. The control synthesis of such mechanical systems represents a challenging control problem and can be considered much more difficult than the synthesis of fully-actuated systems in [4,31,40,43]. In this context, our main focus of this paper will be on a class of underactuated systems enjoying symmetry properties [32], which, based on an appropriate global transformation, will be reduced to normal form, so that it can easily controlled using standard techniques like reminiscent backstepping technique.

In practice, most underactuated mechanical systems are uncertain and multi-variable in character. It is important then to investigate effective robust and adaptive control techniques for such kind of systems . The authors in [5] proposed a robust controller based on a combination of a PD controller and a twisting-like algorithm to stabilize the damped cart pole system. In this approach however, uncertain model parameters have not been considered. To deal with model uncertainties, a time-scale approach along with the Lyapunov design have been proposed in [38]. The authors in [33] used a sliding mode technique for the cart pole system to stabilize the system in presence of disturbances. Model uncertainties however have not been tackled in this work. In [1,6], disturbances and model uncertainties have been fully considered in a technique that involves a backstepping procedure combined with sliding mode, applied to the inverted pendulum system after the system was converted into a normal form. Integral sliding mode control was also applied in [29,46] to deal with uncertainties in the two-wheeled mobile inverted pendulum. The control law being designed is based on the linearized system dynamics which resulted in a small region of attraction for the equilibrium. The problem becomes more challenging whenever non-parametric uncertainties and unknown time-varying disturbances are considered in the design of effective and robust control law for the underactuated system.

An important research issue which has attracted the attention of the control community is the finite-time convergence and finite time stabilization [2,3,10–12,17], that not only ensures faster convergence rate, but also guarantees better disturbance rejection and greater robustness to uncertainty. Rigorous analysis for finite-time stability has been established in [3,13,30]. Several interesting design schemes for finite stabilization of certain class of nonlinear systems using state-feedback and output-feedback can be found in [24,36], just to quote a few, and references therein. Although interesting broad results have been obtained by using finite-time stabilizing controllers in the literature, they may not be effective in dealing simultaneously with unknowns and time-variations in the systems. In the aforementioned works these unknowns are either constants or time-varying only, that is, not state-dependent. Different attempts using adaptive design schemes have been conducted in [14,15] to resolve the problem encountered in the presence of unknowns and time-variation in the system. However, the adaptive scheme has been applied to smooth nonlinear systems of the p-normal form [7], However, to the authors' best knowledge, there is no result on adaptive finite time control scheme for underactuated system with symmetry [32] embodying time varying uncertainties with unknown bounds. Therefore, it is of interest to develop a concise method that is capable of globally finite-time stabilizing the underactuated mechanical system of class I (i.e., enjoying the symmetry properties of mechanical systems) with more involved unknowns and time-variations.

In this paper, an adaptive finite-time control strategy is investigated for a class of underactuated mechanical systems with tree structures by using radial basis neural networks approximation functions. The purpose of this paper is two-fold: to generalise the Olfati-Saber's transformation [32] to $n-$th order underactuated systems into a special upper-triangular form with mismatched time-varying uncertainties, and to use the adding a power integrator (API) technique in combination of the radial basis neural networks approximation functions as a basic design tool for finite stabilization of the underactuated mechanical system.

The main contribution of this paper is twofold:

1. A finite time controller is constructed based on the adding a power integrator technique [27,26] and backstepping method. The key feature of this technique is that it only requires knowledge of the upper bounds of nonlinearities involved in the system dynamics, therefore simple recursive domination design can be obtained for an explicit construction of a smooth state feedback controller. The explicit form of the finite time controller appears to be more concise compared with the finite time controller proposed in [6,16,23,47] for underactuated mechanical systems. Furthermore, by virtue of the adding a power integrator, finite time stabilization of the underactuated system can be obtained as opposed to [6,16] where only global uniform ultimate boundedness is achieved. Moreover the inherent chattering problem due to discontinuity of the sliding mode approaches is straightforwardly circumvented.
2. The proposed scheme does not need any prior information about the bound of mismatched uncertainties. The tuning rule incorporated to estimate the mismatched uncertainty using the RBFNNN borrows much from the adaptive tracking [22] approach including the notion of tuning functions. The proposed scheme can guarantee that the state of the closed loop system converges to the origin in finite-time and the parameter estimations are bounded.

The reminder of this paper is organized as follows: Section 2 discusses the control problem formulation of a class of of mechanical underactuated systems along with the coordinate transformation that transforms this model into an upper-triangular form representation. Section 3 propose a finite-time stabilization control strategy for the n-th order underactuated system. In Section 4, simulation results with an application to the rotary inverted pendulum system are presented and discussed. Finally, Section 5 draws a conclusion for this paper.

2 Problem formulation and preliminaries

2.1 Problem formulation

Consider an underactuated mechanical system described by [39]

$$M_1(q)\ddot{q}_u + M_2(q)\ddot{q}_a + N_1(q,\dot{q}) = 0 \tag{1}$$
$$M_2^\top(q)\ddot{q}_u + M_3(q)\ddot{q}_a + N_2(q,\dot{q}) = B(q)u \tag{2}$$

Where the first equation (1) represents the unactuated dynamics while the second equation denotes the actuated dynamics, with $q = [q_u^\top, q_a^\top]^\top$ is the generalized coordinates such that $q_a \in \mathbb{R}^n$ is the unactuated degrees of freedom, and $q_u \in \mathbb{R}^m$ represents the actuated degrees of freedom. The inertia matrix of the underactuated system is defined by

$$M = \begin{bmatrix} M_1(q) & M_2(q) \\ M_2^\top(q) & M_3(q) \end{bmatrix} \tag{3}$$

which is positive symmetric and positive definite for all $q \in \mathbb{R}^{n+m}$. $B(q)$ is an invertible matrix and u is the generalized control forces produced by the m actuators. $N_i(q), i = 1,2$ are the Coriolis, centrifugal and gravitational force vectors.

Control of underactuated system such as the system presented in (1)-(2), most often resorts to the well-known collocated partial feedback linearization proposed by [44], where the objective is to transform the dynamics of the underactuated system into a reduced-order system with cascaded nonlinear system in strict feedback form with zero-dynamics. However, with this approach there is an inherent requirement that the model parameters of the mechanical system must be perfectly known to be able to design controller for the transformed normal form.

To address this observation, the control design in this paper avoids the collocated feedback linearization entirely. Before deriving the control scheme, the equations of motion of the underactuated system are converted to a reduced form through simple manipulation of (1)-(2) as in [44]

$$\ddot{q}_a = J_1(q)u + R_1(q,\dot{q}) \tag{4}$$
$$\ddot{q}_u = J_2(q)u + R_2(q,\dot{q}) \tag{5}$$

where

$$\begin{aligned}
R_1(q,\dot{q}) &= (M_3 - M_2^\top M_1^{-1}M_2)^{-1}[M_2^\top M_1^{-1}N_1(q,\dot{q}) \\
&\quad -N_2(q,\dot{q})] \\
J_1(q) &= (M_3 - M_2^\top M_1^{-1}M_2)^{-1}B(q) \\
R_2(q,\dot{q}) &= N_1 - M_1^{-1}M_2R_1(q,\dot{q}) \\
J_2(q) &= -M_1^{-1}M_2J_1(q)
\end{aligned}$$

The dynamic equations (4)-(5) define the coupled dynamics of the underactuated system (1)-(2), where it can be seen that the control input appears both in the actuated subsystem and the unactuated subsystem of the original system, which significantly increases complexity of the control design for underactuated systems. To resolve this issue, the author in [32], provided a suitable global change of coordinates that decouples the actuated from the unactuated dynamics with respect to the control input and allows to transform the system into a cascaded normal form.

Without loss of generality, an $n-$th order underactuated system can be described with a chain structure as follows

$$\begin{aligned}
\ddot{x}_i &= \phi_i(x_1,\dots,x_{i+1})u + \gamma_i(x_1,\dots,x_{i+1},\dot{x}_1,\dots,\dot{x}_i), \\
\ddot{x}_n &= \phi_n(\boldsymbol{x})u + \gamma_n(\boldsymbol{x},\dot{\boldsymbol{x}}) \quad i = 1,\dots,n-1
\end{aligned} \tag{6}$$

where $\boldsymbol{x} = [x_1,\dots,x_n]^\top \in \mathbb{R}^n$ are the state variables of the underactuated system and $u \in \mathbb{R}$ is the control input, $\gamma_i(\boldsymbol{x},\dot{\boldsymbol{x}}) \neq 0$, $\phi_i(\boldsymbol{x})$ are nonlinear functions for all $i = 1,\dots n$. We will use the technique presented in [32] to transform the system (6) to a system of upper-triangular form. As such, we introduce the following coordinate changes

$$\begin{aligned}
z_i &= x_i - \int_0^{x_{i+2}} \frac{\phi_{i+1}(s)}{\phi_{i+3}(s)}ds \quad & i = 1,3,5\dots,n-3 \\
z_i &= x_i - \frac{\phi_{i+1}(\boldsymbol{x})}{\phi_{i+2}(\boldsymbol{x})} & i = 2,4,6,\dots,n-2 \\
z_{n-1} &= x_{n-1} \\
z_n &= x_n
\end{aligned} \tag{7}$$

Remark 1 Note that the integral term in (7) has no real physical meaning except that the mathematical expression will help to globally transform the system (6) into normal form, where the actuated dynamics are decoupled from the unactuated ones. More about this transformation can be found in details in [32].

Hence, the system dynamics in a new Z-space coordinate can be obtained. Taking the time derivative of (7), we have

$$\begin{aligned}
\dot{z}_i &= z_{i+1} + d_i(\boldsymbol{z}), \quad 1 \le i \le n-1 \\
\dot{z}_n &= \phi_n(\boldsymbol{z})u + d_n(\boldsymbol{z}) \\
y &= h(\boldsymbol{z})
\end{aligned} \tag{8}$$

where $d_i(\boldsymbol{z})$ are defined as follows:

$$\begin{aligned}
d_1(\boldsymbol{z}) &= \frac{\phi_2(\boldsymbol{z})}{\phi_4(\boldsymbol{z})}z_4 - \frac{d}{dt}\int_0^{z_3}\frac{\phi_3(s)}{\phi_4(s)}ds \\
d_2(\boldsymbol{z}) &= -z_3 + \gamma_3(\boldsymbol{z}) - \frac{\phi_3(\boldsymbol{z})}{\phi_4(\boldsymbol{z})}\gamma_4 - \frac{d}{dt}\Big(\frac{\phi_3(\boldsymbol{z})}{\phi_4(\boldsymbol{z})}\Big)z_4 \\
&\vdots \\
d_{n-3} &= \frac{\phi_{n-2}(\boldsymbol{z})}{\phi_n(\boldsymbol{z})}z_n - \frac{d}{dt}\int_0^{z_{n-1}}\frac{\phi_{n-2}(s)}{\phi_n(s)}ds \\
d_{n-2} &= -z_{n-1} + \gamma_{n-2}(\boldsymbol{z}) - \frac{\phi_{n-2}(\boldsymbol{z})}{\phi_n(\boldsymbol{z})}\gamma_n - \frac{d}{dt}\Big(\frac{\phi_{n-2}(\boldsymbol{z})}{\phi_n(\boldsymbol{z})}\Big)z_n \\
d_{n-1} &= 0 \\
d_n &= \phi_n(\boldsymbol{z})
\end{aligned} \tag{9}$$

To control the normal form (8), we will consider the following assumptions :

Assumption 1 All modeling terms in (1)-(2), including M_1, M_2, M_3, N_1 and N_2 are considered unknown functions of time.

Assumption 2 The uncertain control coefficient $\phi_n(\boldsymbol{z})$ is bounded as $\phi_{\min} \le \phi_n(\boldsymbol{z}) \le \phi_{\max}$. Furthermore, it is convenient to assume that $\phi_n(\boldsymbol{z}) \neq 0$.

Remark 2 Note that the terms $d_i(\boldsymbol{z}) \in \mathbb{R}, i = 1,2,\dots,n$ in the transformed system (8) are regarded as matched and unmatched uncertainties. Also by construction, since $d_i(\boldsymbol{z}), i = 1,\dots n$ are functions of the states it will not be practical to assume that their variation bounds are known. In addition, it cannot be assumed to be linearly parameterizable into a multiplication of a known regressor and a vector of unknown constant parameters, this is mainly due to their complex expressions in (9).

Remark 3 In many robotic systems such as those classified in [32] as of Class-I underactuated systems, the sign of the control coefficient $\phi_n(.)$ is independent of the change of coordinates and is usually known and is nonzero.

In this paper, we consider a class of underactuated systems of the form (6), which can be transformed without collocated feedback linearization and through the global change of coordinates (8) into the cascaded form (8). Here, denote $\boldsymbol{z} = [z_1, z_2, \ldots, z_n]^\top \in \Upsilon$, where Υ is a compact set of $\mathbb{R}^n$, $u \in \mathbb{R}$ is the control input and $d_i(\boldsymbol{z}) \in \mathbb{R}, i = 1, \ldots, n-1$ are unmatched uncertainties while d_n is the matched uncertainty. Under Assumptions 2.1 and 2.2, the idea is to design the controller u such that the system state $\boldsymbol{z}$ tracks in finite time the desired trajectory $\boldsymbol{z}_d^* = [z_{1d}^*, z_{2d}^*, \ldots, z_{nd}^*]^\top \in \Upsilon_d$ where Υ_d is a compact set of Υ. This means that for any finite initial condition $z_i(0), i = 1, \ldots n$, there exists a finite time T, called the settling time of system (8) such that

$$\lim_{t\to T} e_i = \lim_{t\to T} \left| z_i - z_{id}^* \right| = 0, \quad i = 1, \ldots, n \tag{10}$$

2.2 Finite time stability

In what follows, we review some basic concepts of finite-time stability, its definition and some useful lemmas.

We denote the solution of the normal form system (8) by $\boldsymbol{z}(t, \boldsymbol{z}_0)$, where $\boldsymbol{z}_0$ is the initial state.

Definition 1 [3] The equilibrium point $\boldsymbol{z} = 0$ of (8) is finite-time convergent if there is an open neighborhood $\wp \subset \mathbb{R}^n$ of the origin and a function $T : \wp - \{0\} \to (t_0, +\infty)$, such that $\forall \boldsymbol{z}_0 \in \wp$, the solution $\boldsymbol{z}(t)$ of the system (8) exists and unique for all $t \in [0, T(\boldsymbol{z}_0, t_0))$, such that $\lim_{t\to T} \boldsymbol{z}(t) = 0$ and $\boldsymbol{z}(t) = 0$ $\forall t > T$. The instant time T is called the *settling time*. The equilibrium $\boldsymbol{z} = 0$ of the system (8) is finite-time stable if it is Lyapunov stable and finite convergent. In particular if $\wp = \mathbb{R}^n$, the equilibrium is said to be global finite-time stable (GFTS).

Lemma 1 *[3]: Consider the non-Lipschitz continuous autonomous system $\dot{z} = f(z), f(0) \in \mathbb{R}$. Assume there are C^1 function $V(z)$ defined on a neighborhood $\Im \subset \mathbb{R}$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$ such that*

1. *$V(z)$ is positive definite on $\Im$*
2. *$\dot{V}(z) + cV(z)^\alpha \leq 0, \quad \forall z \in \Im$*

Then the origin z is locally finite-time stable. If $\Im = \mathbb{R}$ and $V(z)$ is radially unbounded, then the origin $z = 0$ is globally finite time stable. Moreover, it can be verified that the settling time being dependent on the initial state $z(0) = z_0$ satisfies $T_z(z_0) \leq \frac{V(z_0)^{1-\alpha}}{c(1-\alpha)}$ for all z_0 in some open neighborhood of the origin.

Lemma 2 *[34] For any $x_i \in \mathbb{R}, i = 1, \ldots, n$ and $0 < p \leq 1$, the following inequality holds $(|x_1| + \ldots + |x_n|)^p \leq |x_1|^p + \ldots + |x_n|^p \leq n^{1-p}(|x_1| + \ldots + |x_n|)^p$. when $0 < p = \frac{p_1}{p_2} \leq 1$, where p_1 and p_2 are positive odd integers then the following holds $|x^p - y^p| \leq 2^{1-p}|x - y|^p$, for any $x, y \in \mathbb{R}$.*

Lemma 3 *[34] For any $x \in \mathbb{R}, y \in \mathbb{R}, c > 0, d > 0$ and $\gamma(x, y) > 0$ a real-valued function, the following holds: $|x|^c|y|^d \leq$
$(c\gamma(x, y)/(c + d))|x|^{c+d} + (d\gamma(x, y)^{-c/d}/(c + d))|y|^{c+d}$.*

2.3 RBF Neural Network Approximation

Consider a function $f(\boldsymbol{z}) : \mathbb{R}^m \to \mathbb{R}$. Suppose that $f(\boldsymbol{z})$ is unknown smooth nonlinear function and it can be approximated over a compact set $\Omega \subseteq \mathbb{R}^m$ with the following RBFNN:

$$f(\boldsymbol{z}) = {W^*}^\top \varpi(\boldsymbol{z}) + \delta_f(\boldsymbol{z}), \quad \forall \boldsymbol{z} \in \Omega \tag{11}$$

where the node number of the NN is l. More nodes mean more accurate approximation. $W^* \in \mathbb{R}^l$ represents the optimal weight vector, which is defined by

$$W^* = \arg\min_{\widehat{W}} \left\{ \sup_{\boldsymbol{z}\in\Omega} \left| f(\boldsymbol{z}) - \widehat{W}^\top \varpi(\boldsymbol{z}) \right| \right\} \tag{12}$$

where $\widehat{W}$ represents the estimate of W^*, $\varpi(\boldsymbol{z}) = [\varpi_1(\|\boldsymbol{z} - \theta_1\|), \varpi_2(\|\boldsymbol{z} - \theta_2\|), \ldots, \varpi_l(\|\boldsymbol{z} - \theta_l\|)]^\top : \Omega \to \mathbb{R}^l$ represents the regressor vector, with $\varpi_i(.)$ being an RBF and $\theta_i (i = 1, \ldots, l)$ are distinct points in the state space (also called centers). The elements of the regressor vector are chosen to be as the Gaussian function:

$$\varpi_i(\|\boldsymbol{z} - \theta_i\|) = \exp\left[\frac{-(\boldsymbol{z} - \theta_i)^\top (\boldsymbol{z} - \theta_i)}{\sigma^2} \right], \quad i = 1, \ldots, l \tag{13}$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_l]^\top$ is the center vector of the Gaussian basis function, and σ is the spread of the Gaussian basis function. $\delta_f(\boldsymbol{z})$ is the approximation error that is bounded over Ω, such that $|\delta_f(\boldsymbol{z})| \leq \bar{\delta}_f$, where $\bar{\delta}_f$ is an unknown constant.

Remark 4 Note that in RBF neural network approximation, the center parameters θ_i and the spread σ are generally chosen according to the scope of the input value. If the parameter values are badly chosen, the Gaussian function will not be effectively mapped and the RBF network will be invalid. In this paper, In order to reduce computational burden, we deliberately set θ_i and σ in the range of the input of the RBF and therefore we select the center θ_i and the spread σ values as constants.

3 Main results

The purpose of this section is to devise a control law that stabilizes the system (8) in finite time. Given the fact that the Z-system (8) contains unmatched uncertainty, traditional adaptive techniques like the backstepping technique [21] and the multiple-surface sliding [37] cannot be used, because the variation bounds of $d_i, i = 1, \dots, n$ are not assumed to be known. To solve the problem of finite-time stabilization with unmatched uncertainties, we propose a novel robust finite-time control scheme whereby a continuous recursive finite-time stabilizing control law is derived from the adding power integrator (API) technique [34] and the RBFNN technique.

In the following, to construct the finite-time controller we first define a set of virtual controllers $z^*_{1d}, \dots, z^*_{nd}$ and new variables $e_1, \dots, e_n$ as follows

$$\begin{aligned} z^*_{1d} &= 0, & e_1 &= z_1^{\frac{1}{q_1}} - z_{1d}^{*\frac{1}{q_1}} \\ z^*_{2d} &= -\beta_1 e_1^{q_2}, & e_2 &= z_2^{\frac{1}{q_2}} - z_{2d}^{*\frac{1}{q_2}} \\ &\vdots \\ z^*_{(n)d} &= -\beta_{n-1} e_n^{q_n}, & e_n &= z_n^{\frac{1}{q_n}} - z_{nd}^{*\frac{1}{q_n}} \end{aligned} \tag{14}$$

and choose the following controller and the adaptive law:

$$\begin{aligned} u &= -\mathrm{sign}(\phi_n(.))\xi(\bar{\boldsymbol{z}})e^{q_{n+1}} \\ \dot{\widehat{\Theta}} &= \Xi(\boldsymbol{z}, e_n) \end{aligned} \tag{15}$$

where the parameters $1 = q_1 > q_2 > \dots > q_m := \frac{4n+3-2m}{4n+1}$ with $m = 2, \dots, n$, the gain parameters β_i are positive constant to be determined during the control design. $\bar{\boldsymbol{z}} = [\boldsymbol{z}^\top, \widehat{\Theta}^\top]^\top$, $\widehat{\Theta}$ is the parameter estimation that will be defined later, $\xi(\bar{\boldsymbol{z}})$ and $\Xi(z_n, e_n)$ are smooth nonnegative functions to be also determined later.

The design procedure consists of n steps, throughout the $n-1$ steps, the virtual controllers $z^*_{(m-1)d}$ will be designed. Upon the completion of step n, the fast control term $\xi(\bar{\boldsymbol{z}})$ as well as the update law in (15) will be determined.

Step 1: From (8) and the change of variables in (14), one has $\dot{e}_1 = z_2 + d_1(\boldsymbol{z})$. For the first step of the induction, we choose a Lyapunov function as

$$V_1(\boldsymbol{z}) = \frac{1}{1+S} e_1^{1+S} \tag{16}$$

where $S = (4n-1)/(4n+1)$. The time derivative of (16) gives

$$\begin{aligned} \dot{V}_1(\boldsymbol{z}) &= e_1^S \dot{e}_1 \\ &= e_1^S z_2^* + e_1^S (z_2 - z^*_{2d}) + e_1^S d_1(\boldsymbol{z}) \end{aligned}$$

By introducing the virtual control $z^*_{2d} = -\beta_1 e_1^S$, where β_1 is a design parameter, $\dot{V}_1(\boldsymbol{z})$ rewrites:

$$\dot{V}_1(\boldsymbol{z}) = -\beta_1 e_1^{2S} + e_1^S (z_2 - z^*_{2d}) + e_1^S d_1(\boldsymbol{z}) \tag{17}$$

In order to proceed with the second step of the design procedure, one needs to reveal the error variable e_2 into (17). This can be done by upper-bounding the second term of the righthand side of (17) by an expression that involves the error variable e_2. Using Lemma 2 and 3, the second term of (17) can be upper-bounded as follows

$$e_1^S(z_2 - z^*_{2d}) \le 2^{1-S}|e_2|^S|e_1|^S \le \frac{1}{2}|e_1|^{2S} + C_2|e_2|^{2S} \tag{18}$$

where C_2 is a positive constant. In light of (18), the time derivative of $V_1(\boldsymbol{z})$, can be written as follows:

$$\dot{V}_1(\boldsymbol{z}) \le -\beta_1 e_1^{2S} + \frac{1}{2}|e_1|^{2S} + C_2|e_2|^{2S} + e_1^S d_1(\boldsymbol{z}) \tag{19}$$

Step 2: By examining the dynamics of e_2, one has $\dot{e}_2 = \frac{1}{q_2}(z_3 + d_2(\boldsymbol{z}))z_2^{\frac{1}{q_2}-1} + \frac{S}{q_2}\beta_1 \dot{e}_1 e_1^{S-1} z_{2d}^{\frac{1}{q_2}-1}$. Clearly from this dynamic equation, it becomes hard to design a virtual controller z^*_{3d} to stabilize e_2 using standard backstepping technique (i.e, by constructing quadratic Lyapunov function). According to [18] and [35] , a $\mathcal{C}^1$, positive definite and proper Lyapunov candidate function is constructed as

$$V_2(\boldsymbol{z}) = V_1(\boldsymbol{z}) + W_2(z_1, z_2) \tag{20}$$

with

$$W_2(z_1, z_2) = \int_{z^*_{2d}}^{z_2} (\chi^{\frac{1}{q_2}} - z_{2d}^{*\frac{1}{q_2}})^{1+S-q_2} d\chi \tag{21}$$

being differential, positive definite and proper function, then similar to the proof in [26, 45], it is easy to show that the Lyapunov function candidate $V_2(\boldsymbol{z})$ is positive definite, proper and satisfies $V_2(\boldsymbol{z}) \le \max\{\frac{1}{1+S}, 2\}(e_1^{1+S} + e_2^{1+S})$.

Note also from [26], that the function $W_2(z_1, z_2)$ has the following property

$$\begin{aligned} \frac{\partial W_2(.)}{\partial z_2} &= e_2^{1+S-q_2} \\ \frac{\partial W_2(.)}{\partial z_1} &= -(1+S-q_2)\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1}\int_{z^*_{2d}}^{z_2}(\chi^{\frac{1}{q_2}} - z_{2d}^{*\frac{1}{q_2}})^{S-q_2}d\chi \\ &= -\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1}(z_2 - z^*_{2d}) \end{aligned}$$

Hence, the derivative of $W_2(.)$ can be computed as follows

$$\begin{aligned} \dot{W}_2(.) &= e_2 \dot{z}_2 - ((z_2 - z^*_{2d})\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1}\dot{z}_1) \\ &= e_2(z_3 - z^*_{3d}) - (z_2 - z^*_{2d})\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1}\dot{z}_1 + e_2 z_3^* \\ &\quad + e_2 d_2(z) \end{aligned} \tag{22}$$

In order to proceed further, the second and third terms on the right side of (22) should be estimated. Using the fact that

$$\Big|(z_2 - z_{2d}^*)\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1}\dot{z}_1\Big| \leq 2^{1-q_2} e_2^{q_2}\Big|\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1} z_2\Big| + 2^{1-q_2} e_2^{q_2}\Big|\frac{\partial z_{2d}^{*\frac{1}{q_2}}}{\partial z_1} d_1(z)\Big|$$

Let the first lumped unmatched uncertainty be $\tau_1 = e_2 d_2(\boldsymbol{z}) + 2^{1-q_2}\beta_1^{\frac{1}{q_2}} e_2^{q_2} d_1(z)$, it is then straightforward to obtain

$$\dot{W}_2 \leq e_2(z_3 - z_{3d}^*) + e_2 z_{3d}^* + 2^{1-q_2}\beta^{\frac{1}{q_2}} e_2^{q_2}|z_2| + \tau_1 \quad (23)$$

According to Lemma 2, the third term of the right hand side of (23) can be bounded such that

$$|z_2| \leq |e_2 + z_{2d}^{*\frac{1}{q_2}}|^{q_2} \leq |e_2|^{q_2} + \beta_1|e_1|^{q_2}$$

which implies that

$$\begin{aligned} |e|_2^{q_2} 2^{1-q_2}\beta^{\frac{1}{q_2}}|z_2| &\leq 2^{1-q_2}\beta^{\frac{1}{q_2}}|e_2|^{2q_2} + 2^{1-q_2}\beta^{1+\frac{1}{q_2}}|e_2|^{q_2} \\ &\quad \times |e_1|^{q_2} \\ &\leq 2^{2(1-q_2)-1}\beta^{2(1+\frac{1}{q_2})}|e_2|^{2q_2} + \frac{1}{2}|e_1|^{2q_2} \\ &\quad + 2^{1-q_2}\beta^{\frac{1}{q_2}}|e_2|^{2q_2} \\ &= \frac{1}{2}|e_1|^{2q_2} + \bar{C}_2|e_2|^{2q_2} \end{aligned} \quad (24)$$

where $\bar{C}_2 = 2^{2(1-q_2)}\beta^{2(1+\frac{1}{q_2})} + 2^{1-q_2}\beta^{\frac{1}{q_2}}$.

Combining (19), (23) and (24) yields the following

$$\begin{aligned} \dot{V}_2(.) &\leq -(\beta_1 - 1)|e_1|^{2S} + (C_1 + \bar{C}_2)|e_2|^{2S} \\ &\quad + e_2 z_{3d}^* + e_2(z_3 - z_{3d}^*) + \tau_1 \end{aligned} \quad (25)$$

At this stage, it is worth to note that $q_3 = q_2 - \frac{2}{4n+1}$, which implies that $q_3 = 1 + 2q_2 = 1 + 2S$. Therefore by selecting $\beta_1 > n - 1 - \kappa$, where $\kappa > 0$ is a design parameter, and the virtual control $z_{3d}^* = -\beta_2 e_2^{q_3}$ with $\beta_2 \geq n - 2 + \kappa + C_2 + \bar{C}_2$, we obtain

$$\dot{V}_2 \leq -(n-2+\kappa)|e_1|^{2S} - (n-2+\kappa)|e_2|^{2S} + e_2(z_3 - z_{3d}^*) + \tau_1 \quad (26)$$

From (26), it is clear that in the k-th step the time derivative of V_k should also be upper bounded by two negative terms, a crossing product and the lumped uncertainties. This observation will be shown by inductive steps as follows.

Step k ($k := 3 \ldots n-1$): We proceed to the derivation of the virtual control by using an inductive argument. Suppose at step $k-1$, there exists a C^1 Lyapunov candidate function $V_{k-1}(z_1, \ldots, z_{k-1})$, which is positive definite, proper that verifies

$$V_{k-1}(.) \leq \max\{\frac{1}{1+S}, 2\}\sum_{m=1}^{k-1} e_m^{1+S} \quad (27)$$

and a set of virtual controllers and coordinate transformations defined by

$$\begin{aligned} z_{1d}^* &= 0, & e_1 &= z_1^{\frac{1}{q_1}} - z_{1d}^{*\frac{1}{q_1}} \\ z_{2d}^* &= -\beta_1 e_1^{q_2}, & e_2 &= z_2^{\frac{1}{q_2}} - z_{2d}^{*\frac{1}{q_2}} \\ &\vdots \\ z_{(k)d}^* &= -\beta_{k-1} e_k^{q_k}, & e_k &= z_k^{\frac{1}{q_k}} - z_{kd}^{*\frac{1}{q_k}} \end{aligned} \quad (28)$$

where $\beta_1 > 0, \ldots, \beta_{k-1} > 0$ are positive design parameters such that,

$$\begin{aligned} \dot{V}_{k-1}(.) &\leq -(n-k+1+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + e_{k-1}^{1+S-q_{k-1}} \\ &\quad \times (z_k - z_{kd}^*) + \sum_{m=1}^{k-2}\tau_m \end{aligned} \quad (29)$$

Now let us claim that (27) and (29) are true at step k. To prove this claim, consider the following Lyapunov candidate function

$$V_k(z_1, \ldots, z_k) = V_{k-1}(.) + W_k(z_1, \ldots, z_k) \quad (30)$$

with

$$W_k(.) = \int_{z_{dk}^*}^{z_k} (\chi^{\frac{1}{q_k}} - z_{dk}^{*\frac{1}{q_k}})^{1+S-q_k} d\chi \quad (31)$$

From the previous step and according to [26,45] , it can be observed that $W_k(.)$ has the following properties:

$$\frac{\partial W_k(.)}{\partial z_k} = e^{1+S-q_k} \quad (32)$$

$$\frac{\partial W_k(.)}{\partial z_m} = -(1+S-q_k)\frac{\partial z_{dk}^{*\frac{1}{q_k}}}{\partial z_m}\int_{z_{dk}^*}^{z_k}(\chi^{\frac{1}{q_k}} - z_{dk}^{\frac{1}{q_k}})^{S-q_k} d\chi \quad (33)$$

also, it is easy to show that $V_k(.)$ is C^1, proper and positive definite, which verifies

$$V_k(.) = \leq \max\{\frac{1}{1+S}, 2\}\sum_{m=1}^{k} e_m^{1+S}$$

The time derivative of $V_k(.)$ satisifies

$$\begin{aligned}\dot{V}_k(.) \le & -(n-k+1+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + e_{k-1}^{1+S-q_{k-1}}(z_k - z_{dk}^*) \\ & + e_k^{1+S-q_k} z_{k+1} + e_k^{1+S-q_k} d_k(\boldsymbol{z}) + \sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m}\dot{z}_m \\ & + \sum_{m=1}^{k-1}\tau_m \\ \le & -(n-k+1+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + e_{k-1}^{1+S-q_{k-1}}(z_k - z_{dk}^*) \\ & + e_k^{1+S-q_k} z_{k+1} + \sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} z_{m+1} \\ & + \Big[e_k^{1+S-q_k} d_k(\boldsymbol{z}) + \sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} d_m(\boldsymbol{z})\Big] + \sum_{m=1}^{k-1}\tau_m \end{aligned} \tag{34}$$

If the k-th lumped unmatched uncertainty is defined as $\tau_k = e_k^{1+S-q_k} d_k(\boldsymbol{z}) + \sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} d_m(\boldsymbol{z})$, then (35) rewrites:

$$\begin{aligned}\dot{V}_k(.) \le & -(n-k+1+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + e_{k-1}^{1+S-q_{k-1}}(z_k - z_{dk}^*) \\ & + e_k^{1+S-q_k} z_{k+1} + \sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} z_{m+1} + \sum_{m=1}^{k}\tau_m \end{aligned} \tag{35}$$

Next we bound the second and the fourth term of the right hand side of (35). First according to Lemma 3, it holds that

$$\begin{aligned}|e_{k-1}^{1+S-q_{k-1}}(z_k - z_{dk}^*)| & \le 2^{1+S-q_{k-1}}|e_{k-1}|^{1+S-q_{k-1}}|e_k|^{q_k} \\ & \le \frac{|e_{k-1}|^{2S}}{2} + C_k|e_k^{2S}| \end{aligned} \tag{36}$$

with C_k a positive constant. As for the fourth term, it is easy to obtain the following:

$$\begin{aligned}\Big|\sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} z_{m+1}\Big| & \le (1+S-q_k)2^{1-q_k}|e_k|^S \\ & \times \Big|\sum_{m=2}^{k-1} \frac{\partial z_{dk}^*}{\partial z_m} z_{m+1}\Big| \end{aligned} \tag{37}$$

To further bound the fourth term in (35), we need to conduct the analysis by inductive argument and assume that at step $k-1$, the following holds

$$\Big|\sum_{m=2}^{k-2} \frac{\partial z_{dk}^*}{\partial z_m} z_{m+1}\Big| \le \sum_{m=1}^{k-1} \gamma_{(k-1)m} e_m^S \tag{38}$$

where $\gamma_{(k-1)m} \ge 0$, then show the inequality also holds for k. Therefore, we have

$$\begin{aligned}\Big|\sum_{m=2}^{k-1} \frac{\partial z_{dk}^*}{\partial z_m} z_{m+1}\Big| & \le \Big| -\beta_{k-1}^{\frac{1}{q_k}} \sum_{m=2}^{k-1} \frac{\partial e_{k-1}}{\partial z_m} z_{m+1}\Big| \\ & \le \beta_{k-1}^{\frac{1}{q_k}} \Big| \frac{z_{k-1}^{\frac{1}{q_k}-1}}{q_{k-1}} z_k + \sum_{m=2}^{k-2} \frac{\partial {z_{k-1}^*}^{\frac{1}{q_{k-1}}}}{\partial z_m} z_{m+1}\Big| \end{aligned} \tag{39}$$

In light of the definition of the tracking error, it is worth recalling that $e_m = z_m^{\frac{1}{q_m}} - {z_{md}^*}^{\frac{1}{q_m}}$ and $z_m^* = -\beta_{m-1} e_{m-1}^{q_m}$, it can then be inferred that ${z_m^*}^{\frac{1}{q_m}} = -\beta_{m-1}^{\frac{1}{q_m}} e_{m-1}$ and therefore the following inequality holds

$$|z_m| \le |e_m + {z_m^*}^{\frac{1}{q_m}}|^{q_m} \le |e_m|^{q_m} + \beta_{m-1}|e_{m-1}|^{q_m} \tag{40}$$

Applying (40) to the inequality (40) and using the assumption of the inequality (38) yields:

$$\begin{aligned}\Big|\sum_{m=2}^{k-1} \frac{\partial z_{dk}^*}{\partial z_m} z_{m+1}\Big| \le & \beta_{k-1}^{\frac{1}{q_k}} \Big[\frac{1}{q_{k-1}}\big(|e_{k-1}|^{1-q_{k-1}} + \beta_{k-2}^{\frac{1}{q_{k-1}}-1} \\ & \times e_{k-2}^{1-q_{k-1}}\big)\big(|e_k|^{q_k} + \beta_{k-1}|e_{k-1}|^{q_k}\big) \\ & + \sum_{m=2}^{k-2} \frac{\partial z_{dk}^*}{\partial z_m} z_{m+1}\Big] \\ \le & \sum_{m=1}^{k} \gamma_{km}|e_m|^S \end{aligned} \tag{41}$$

with γ_{km} being a positive constant. Therefore,

$$\begin{aligned}\Big|\sum_{m=2}^{k-1} \frac{\partial W_k(.)}{\partial z_m} z_{m+1}\Big| & \le (1+S-q_k)2^{1-q_k}|e_k|^S \\ & \times \Big(\sum_{m=1}^{k} \gamma_{km}|e_m|^S\Big) \\ & \le \frac{1}{2}\sum_{m=1}^{k-1} e_m^{2S} + \bar{C}_k|e_k|^{2S} \end{aligned} \tag{42}$$

with $\bar{C}_k \ge 0$ being a positive constant.

Substituting (36) and (42) into (35) leads to

$$\begin{aligned}\dot{V}_k(.) \le & -(n-k+1+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + (C_k + \bar{C}_k)|e_k|^{2S} \\ & + \frac{1}{2}\sum_{m=1}^{k-1} e_m^{2S} + \frac{|e_{k-1}|}{2} + e_k^{1+S-q_k} z_{k+1}^* \\ & + \sum_{m=1}^{k}\tau_k + e_k^{1+S-q_k}(z_{k+1} - z_{d(k+1)})^*) \\ \le & -(n-k+\kappa)\sum_{m=1}^{k-1} e_m^{2S} + (C_k + \bar{C}_k)|e_k|^{2S} \\ & + e_k^{1+S-q_k}(z_{k+1} - z_{d(k+1)}^*) + e_k^{1+S-q_k} z_{k+1}^* \\ & + \sum_{m=1}^{k}\tau_k \end{aligned} \tag{43}$$

By introducing the virtual control $z^*_{(k+1)d} = -\beta_k e_k^{q_{k+1}}$ with β_k selected such that $\beta_k \geq n-k+\kappa+C_k+\bar{C}_k > 0$, we get

$$\dot{V}_k(.) \leq -(n-k+\kappa)\sum_{m=1}^{k} e_m^{2S} + e_k^{1+S-q_k}(z_{k+1} - z^*_{(k+1)d}) + \sum_{m=1}^{k}\tau_k \quad (44)$$

This completes the inductive proof.

step n: This is the final stage of the design where the real control input appears in the dynamics. For this step, according to the inductive previous steps, the n-th part of the Lyapunov candidate function can be constructed accordingly

$$V_n(z_1,\dots,z_n) = V_{n-1}(.) + W_n(z_1,\dots,z_n) \quad (45)$$

where

$$W_n(.) = \int_{z^*_{nd}}^{z_n} (\chi^{\frac{1}{q_n}} - z_n^{\frac{1}{q_n}})^{1+S-q_n} d\chi \quad (46)$$

Then it is obvious to conclude that $V_n(.)$ is C^1, positive definite and satisfies

$$V_n(.) \leq \max\{\frac{1}{1+S}, 2\}\sum_{m=1}^{n} e_m^{1+S}$$

From the above inductive argument, one can conclude that

$$\begin{aligned}\dot{V}_n &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + e_n^{1+S-q_n}\dot{z}_n + (C_n+\bar{C}_n)e_n^{2S} + \sum_{m=1}^{n-1}\tau_m \\ &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + e_n^{1+S-q_n}(\phi_n u + d_n(\boldsymbol{z})) \\ &\quad +(C_n+\bar{C}_n)e_n^{2S} + \sum_{m=1}^{n-1}\tau_m \\ &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + \phi_n e_n^{1+S-q_n}u + (C_n+\bar{C}_n)e_n^{2S} \\ &\quad + e_n^{q_{n+1}}\left(e_n^{1+S-q_n-q_{n+1}}d_n(\boldsymbol{z}) + e_n^{-q_{n+1}}\sum_{m=1}^{n-1}\tau_m\right) \quad (47)\end{aligned}$$

where $C_n > 0$ and $\bar{C}_n > 0$ are positive constants. Denote by G the total lumped matched and unmatched uncertainty defined as

$$G = e_n^{1+S-q_n-q_{n+1}}d_n(\boldsymbol{z}) + e_n^{-q_{n+1}}\sum_{m=1}^{n-1}\tau_m$$

Since G is unknown continuous function, it cannot be directly compensated by the design of the control input u. Based on the RBFNN approximation, the function G can be modeled by RBFNN on the compact set $\varUpsilon$ as

$$G = W^{\top}\varpi(\boldsymbol{z}) + \delta_G(\boldsymbol{z}), \quad \forall \boldsymbol{z} \in \varUpsilon \quad (48)$$

where $W \in \mathbb{R}^M$ is the weight vector, $M > 1$ is the NN node number and $\delta_G(\boldsymbol{z}) \in \mathbb{R}$ is the approximation error satisfying $|\delta_G(\boldsymbol{z})| \leq \bar{\delta}_G$. Substituting (48) into (47), we get

$$\begin{aligned}\dot{V}_n &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + \phi_n e_n^{1+S-q_n}u + (C_n+\bar{C}_n)e_n^{2S} \\ &\quad + e_n^{q_{n+1}}(W^{\top}\varpi(\boldsymbol{z}) + \delta_G(\boldsymbol{z})) \\ &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + \phi_n e_n^{1+S-q_n}u + (C_n+\bar{C}_n)e_n^{2S} \\ &\quad + [W^{\top}, \bar{\delta}_G]\begin{bmatrix}\varpi(\boldsymbol{z}) \\ 1\end{bmatrix} e_n^{q_{n+1}} \quad (49)\end{aligned}$$

By denoting $\varTheta = [W^{\top}, \bar{\delta}_G]^{\top} \in \mathbb{R}^{M+1}$ and $\boldsymbol{\varpi}(\boldsymbol{z}) = [\varpi(\boldsymbol{z}), 1]^{\top} \in \mathbb{R}^{M+1}$, we have

$$\begin{aligned}\dot{V}_n &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + \phi_n e_n^{1+S-q_n}u + (C_n+\bar{C}_n)e_n^{2S} \\ &\quad + \varTheta^{\top}\boldsymbol{\varpi}(\boldsymbol{z})e_n^{q_{n+1}} \\ &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + \mathrm{sign}(\phi_n)\phi_{\min} e_n^{1+S-q_n}u \\ &\quad + (C_n+\bar{C}_n)e_n^{2S} + \varTheta^{\top}\boldsymbol{\varpi}(\boldsymbol{z})e_n^{q_{n+1}} \quad (50)\end{aligned}$$

Obviously, $\varTheta$ is an unknown parameter and $\boldsymbol{\varpi}(\boldsymbol{z})$ is a known vector function. The actual control law can therefore be designed as follows

$$\begin{aligned}u &= -\frac{\mathrm{sign}(\phi_n)}{\phi_{\min}} e^{q_{n+1}}(n\varrho + \widehat{\varTheta}^{\top}\boldsymbol{\varpi}(\boldsymbol{z})) \\ &= -\mathrm{sign}(\phi_n)\xi(\bar{z})e^{q_{n+1}} \quad (51)\end{aligned}$$

where $\xi(\bar{z}) = \phi_{\min}^{-1}(n\varrho + \widehat{\varTheta}^{\top}\boldsymbol{\varpi}(\boldsymbol{z}))$, $\varrho > 0$ is a designed parameter such that $\xi(\bar{z}) > 0$ and $\widehat{\varTheta}$ is the estimation of the unknown parameter $\varTheta$. Substituting (51) into (50), it gives

$$\begin{aligned}\dot{V}_n &\leq -\kappa\sum_{m=1}^{n-1} e_m^{2S} + (C_n+\bar{C}_n)e_n^{2S} - n\varrho e_n^{2S} \\ &\quad + e_n^{q_{n+1}}\widetilde{\varTheta}^{\top}\boldsymbol{\varpi}(\boldsymbol{z}) \quad (52)\end{aligned}$$

where $\widetilde{\varTheta} = \varTheta - \widehat{\varTheta}$ is the estimate error of the matched and unmatched uncertainty. It is easy to see that $\dot{\widetilde{\varTheta}} = -\dot{\widehat{\varTheta}}$. Then, to be able to design the parameter update law for $\widehat{\varTheta}$, we will consider the Lyapunov function candidate

$$V_T = V_n(.) + \frac{1}{2}\widetilde{\varTheta}^{\top}\varGamma^{-1}\widetilde{\varTheta} \quad (53)$$

where Γ is a positive definite diagonal matrix. Taking the time derivative of (53) along the solutions of (52), results in

$$\begin{aligned}\dot{V}_T &\le -\kappa\sum_{m=1}^{n-1} e_m^{2S} + (C_n+\bar{C}_n)e_n^{2S} - n\varrho e_n^{2S}\\ &\quad + e_n^{q_{n+1}}\widetilde{\Theta}^\top \boldsymbol{\varpi}(\boldsymbol{z}) - \widetilde{\Theta}^\top \Gamma^{-1}\dot{\widehat{\Theta}}\\ &\le -\kappa\sum_{m=1}^{n-1} e_m^{2S} + (C_n+\bar{C}_n)e_n^{2S} - n\varrho e_n^{2S}\\ &\quad -\widetilde{\Theta}^\top(\Gamma^{-1}\dot{\widehat{\Theta}} - e_n^{q_{n+1}}\boldsymbol{\varpi}(\boldsymbol{z}))\end{aligned} \tag{54}$$

From (54), one can choose the adaptive law as follows:

$$\dot{\widehat{\Theta}} = \Gamma\boldsymbol{\varpi}(\boldsymbol{z})e_n^{q_{n+1}} := \Xi(\boldsymbol{z}, e_n) \tag{55}$$

which results in

$$\dot{V}_T \le -\kappa\sum_{m=1}^{n-1} e_m^{2S} - (n\varrho - C_n - \bar{C}_n)e_n^{2S} \tag{56}$$

We are now ready to announce the main result of this paper.

Theorem 1 *Consider the n-th order underactuated system in the X-space represented by (6), through the coordinate transformation (8), with Assumptions 2.1-2.2 and under the finite time controller (51) and the adaptive law (55), the closed-loop of the underactuated system is finite time stable in the sense of Definition 2.1 and the parameter estimation $\widehat{\Theta}$ is bounded.*

Proof The proof is conducted similarly to the analysis presented in [14]. First to ensure (56) is negative definite, one can select ϱ such that such that $n\varrho - C_n - \bar{C}_n > \kappa$, then (56) rewrites

$$\dot{V}_T \le -\kappa\sum_{m=1}^{n} e_m^{2S} \tag{57}$$

Obviously, from (57) it can be inferred that $e_i, i = 1,\ldots,n$ and $\widetilde{\Theta}$ are bounded, so does $\widehat{\Theta}$ because Θ is a constant vector. Moreover, it can be seen from (55) that $\widehat{\Theta}$ does not change its sign because $\dot{\widehat{\Theta}}$ is nonnegative. Therefore, without loss of generality, we assume $\|\widehat{\Theta}\|_1 \in [0, C]$, where C is a constant depending on initial values of $e_i(0)$ and $\widehat{\Theta}(0)$.

Take the Lyapunov function

$$\tilde{V}(\boldsymbol{e}, \widehat{\Theta}) = V_T - \frac{1}{2}\widetilde{\Theta}^\top\Gamma^{-1}\widetilde{\Theta} := V_n(.)$$

which is positive definite for all $e_1, e_2, \ldots, e_n$ and for any fixed $\widehat{\Theta}$.

Noting that $V_n(.) \le \max\{\frac{1}{1+S}, 2\}\sum_{m=1}^{n} e_m^{1+S}$ and using Lemma 2.2, we have

$$\tilde{V}^{\frac{2S}{1+S}} \le \max\{\frac{1}{1+S}, 2\}\sum_{m=1}^{n} e_m^{2S} \tag{58}$$

Define $\zeta_0(e_n, \boldsymbol{z}) = e_n^{S+q_n-1}\|\varpi(\boldsymbol{z})\|_1$ which is continuous with $\zeta_(0, \widehat{\Theta}) = 0$.

By (57) and (58)

$$\begin{aligned}\dot{\tilde{V}} &\le \dot{V}_T + \widehat{\Theta}^\top\Gamma^{-1}\dot{\widehat{\Theta}}\\ &\le -\kappa\sum_{m=1}^{n} e_m^{2S} + e^{q_{n+1}}\widehat{\Theta}^\top\boldsymbol{\varpi}(\boldsymbol{z})\\ &\le -\frac{\kappa}{2}\sum_{m=1}^{n} e_m^{2S} - \frac{\kappa}{2}e_n^{2S} + e^{2S}\|\Theta\|_1\zeta_0\\ &\le -\frac{\kappa}{2}\sum_{m=1}^{n} e_m^{2S} - \frac{\kappa}{2}e_n^{2S}\Big(1 - \frac{2(C+\bar{\Theta})}{\kappa}\zeta_0\Big)\\ &\le -\frac{\kappa}{2\max\{\frac{1}{1+S}, 2\}}\tilde{V}^{\frac{2S}{1+S}} - \frac{\kappa}{2}e_n^{2S}\Big(1 - \frac{2(C+\bar{\Theta})}{\kappa}\zeta_0\Big)\end{aligned} \tag{59}$$

where $\|\Theta\|_1 \le \bar{\Theta}$, with $\bar{\Theta}$ is an unknown bound. Define a continuous function $\bar{V}(e_n, \boldsymbol{z}) = \frac{2(C+\bar{\Theta})}{\kappa}\zeta_0$ which satisfies $\bar{V}(0, \boldsymbol{z}) = 0$. It is easy to show then that for a given $\|\widehat{\Theta}\|_1 \in [0, C]$, there exists a constant $\epsilon > 0$ such that for any $\boldsymbol{e} \in \Upsilon = \Big\{(\boldsymbol{e}, \widehat{\Theta}) : \tilde{V}(\boldsymbol{e}, \widehat{\Theta}) \le \epsilon\Big\}, \bar{V} < 1$ and therefore $e_n^{2S}(1 - \bar{V}) > 0$ because e_n^{2S} is positive. Therefore once $(\boldsymbol{e}, \widehat{\Theta}) \in \Upsilon$, it will never escape.

From the above analysis two cases for finite time convergence are considered.

If the initial conditions are $(\boldsymbol{e}(0), \widehat{\Theta}(0)) \in \Upsilon$, it is straightforward to show that $\dot{\tilde{V}} \le -\frac{\kappa}{2\max\{\frac{1}{1+S},2\}}\tilde{V}^{\frac{2S}{1+S}}$, since $\alpha := \frac{2S}{1+S} < 1$ then $\tilde{V}$ is finite time convergent according Lemma 2.1. Therefore $\boldsymbol{e} = [e_1, e_2, \ldots, e_n]^\top$ becomes 0 within time T such that

$$T \le 2\max\{\frac{1}{1+S}, 2\}\frac{\tilde{V}(\boldsymbol{e}(0), \widehat{\Theta}(0))^{1-\alpha}}{\kappa(1-\alpha)}$$

If in the seconde case, the initial conditions $(\boldsymbol{e}(0), \widehat{\Theta}(0)) \notin \Upsilon$, then the first thing to do is to estimate the maximum reaching time of $(\boldsymbol{e}(t), \widehat{\Theta}(t))$ to Υ. Before the state enters Υ, we have $\tilde{V}(\boldsymbol{e}, \widehat{\Theta}) > \epsilon$, therefore

$$\begin{aligned}V_T(\boldsymbol{e}(0), \widehat{\Theta}(0)) &\ge V_T(\boldsymbol{e}(0), \widehat{\Theta}(0)) - V_T(\boldsymbol{e}(T_2), \widehat{\Theta_2}(T_2))\\ &\ge \int_0^{T_2}\kappa\sum_{m=1}^{n} e_m^{2S}ds\\ &\ge \int_0^{T_2}\frac{\kappa}{2\max\{\frac{1}{1+S}, 2\}}\tilde{V}^{\frac{2S}{1+S}}ds\\ &\ge \frac{\kappa}{2\max\{\frac{1}{1+S}, 2\}}\epsilon^{\frac{2S}{1+S}}T_2\end{aligned}$$

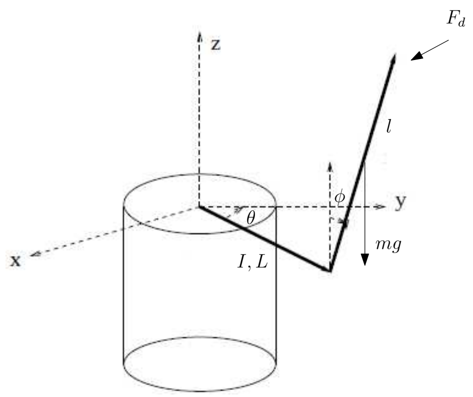

Fig. 1 Schematic view of the rotary inverted pendulum [32]
.

Therefore, $(\boldsymbol{e}, \widehat{\Theta})$ will enter Υ within T_2:

$$T_2 \leq \frac{2\max\{\frac{1}{1+S}, 2\}V_T(\boldsymbol{e}(0), \widehat{\Theta}(0)) + \widehat{\Theta}(0)^\top \Gamma \widehat{\Theta}(0)}{\kappa * \epsilon^{\frac{2S}{1+S}}}$$

after the reaching time T_2, the state will be in Υ. It will however take

$$T_1 \leq 2\max\{\frac{1}{1+S}, 2\}\frac{\epsilon^{1-\alpha}}{\kappa(1-\alpha)}$$

to attain the origin, thus $\boldsymbol{e} = [e_1, \ldots, e_n]^\top = 0$ within $T \leq T_1 + T_2$, therefore the closed-loop of the underactuated system is finite time convergent. This completes the proof.

4 Simulation results

In this section, the effectiveness and the robustness of the proposed adaptive finite time controller augmented by the RBFNN is evaluated through two simulation cases on a rotary inverted pendulum (see Fig. 1), whose system dynamics are described by the following state space representation

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_2(\mathbf{x}) + b_2(\mathbf{x})u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_4(\mathbf{x}) + b_4(\mathbf{x})u
\end{aligned} \tag{60}$$

where, we designated by $[\theta, \dot{\theta}, \phi, \dot{\phi}]^\top = \mathbf{x}^\top = [x_1, x_2, x_3, x_4]^\top$ with θ is the angular displacement of the arm and ϕ is the angular displacement of the pendulum and f_2, f_4, b_2 and b_4 are defined in (61):

with $\Delta = (J + ml^2)(I + ml^2\sin(x_3)^2) + mJL^2 + m^2l^2L^2\sin(x_3)^2$, m and J are respectively the mass and the moment of inertia of the pendulum, l is the distance of the center of mass of the pendulum to its end point, I is the moment of inertia of the arm and F_d is the external disturbance acting on the pendulum. For convenience, the numerical values of the main physical parameters of the rotary inverted pendulum are provided in Table 1. Interested readers can refer to the literature [16] for more details. The initial conditions are chosen as $\mathbf{x} = [-\frac{\pi}{4}, 0, \frac{\pi}{3}, 0]^\top$, the desired state is $\mathbf{x}_d = [0, 0, 0, 0]^\top$. The control gains are chosen to be $\kappa = 10, \beta_1 = 13, \beta_2 = 12, \beta_3 = 11$, $\beta_4 = 20$ and $\Gamma_{ii} = 500$. Different simulation scenarios have been considered. The first simulation scenario is the control of uncertain cart-pole system without external disturbances. The second one considers the case of an uncertain cart-pole system with the presence of external disturbances.

1. **Scenario 1**: *Uncertain rotary inverted pendulum system without external disturbances*
   In this scenario, we assume that the system dynamics is completely unknown. The obtained simulation results are shown in Figures 2 to 3. It is clear in Figure 2 how the arm and the pendulum angles converge in finite time to zero despite the uncertainties present in the system dynamics. The control effort being deployed is shown in Figure 3(a). Figure 3(b) shows the convergence to zero of the state variables in the Z-space. Clearly the algorithm performs well in the presence of model parameters' uncertainties.
2. **Scenario 2**: *Uncertain rotary inverted pendulum system with external disturbances*
   In this simulation scenario, on top of the uncertainties that the system dynamics contains, a periodic perturbation $d(t) = \sin(2t)$ is added as an external excitation to the rotary inverted pendulum in order to test the robustness of the proposed control approach. The obtained simulation results are depicted in Figure 4 and Figure 5. From Figure 4, it can be noticed that despite the existing uncertainties and the considered external disturbance, the controller is able to restore the system to its desired values. The control effort after the application of the disturbances is shown Figure 5(e). Clearly the control input remains with reasonable amplitude even with the external perturbation. The convergence of the state variables in Z-space are shown in Figure 5(a)-(d). From these figures it can be seen that our control design is robust to tolerate significant variation of the system parameters as well as the external disturbances.

$$
\begin{aligned}
f_2(\mathbf{x}) &= \frac{(J+ml^2)mlx_4[Lx_4\sin(x3) - lx_2\sin(2x_3)]}{\Delta} - \frac{m^2l^2L\cos(x_3)[g\sin(x_3)+0.5lx_2^2\sin(2x_3)]}{\Delta} - \frac{mlL\cos(x_3)}{\Delta}F_d \\
b_2(\mathbf{x}) &= \frac{J+ml^2}{\Delta} \\
f_4(\mathbf{x}) &= \frac{ml\sin(x_3)[(I+mL^2l\sin(x_3)^2)g - mlL^2x_3^2\cos(x_3)]}{\Delta} - \frac{ml^2x_2\sin(2x_3)x_2[mlLx_4\cos(x_3)+0.5x_2(I+mL^2+l^2\sin(x_3)^2)]}{\Delta} \\
&\quad + \frac{(I+mL^2+ml^2\sin(x_3)^2)}{\Delta}Fd \\
b_4(\mathbf{x}) &= -\frac{mlL\cos(x_3)}{\Delta}
\end{aligned} \tag{61}
$$

Table 1 Physical parameters of the rotary inverted pendulum

Parameter	Nominal Value (unit)	Added Uncertainty
g	$9.8ms^{-2}$	0
m	5.38×10^{-2}kg	$0.0263+0.01\sin 40t$
I	1.75×10^{-2}kg.m^2	0.01
J	1.98×10^{-4}kg.m^2	0.01
l	0.113m	$0.07+0.01\sin 2t$
L	0.215m	$0.07+0.01\sin 2t$

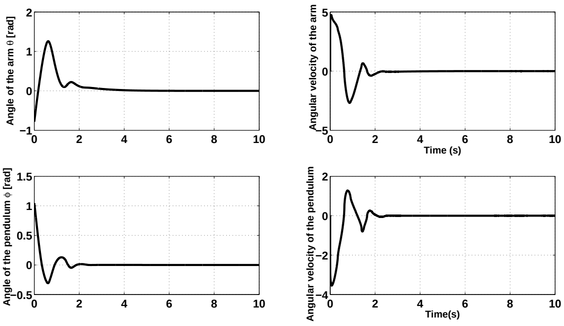

Fig. 2 System response arm and pendulum angle versus time.

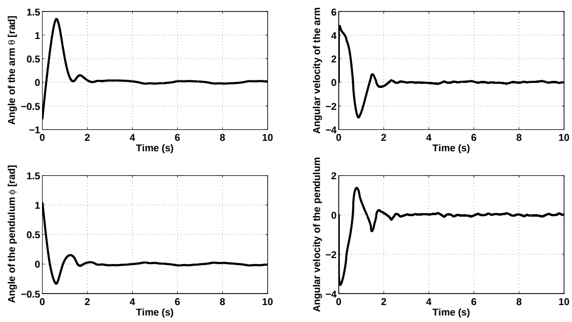

Fig. 4 System response arm and pendulum angle versus time.

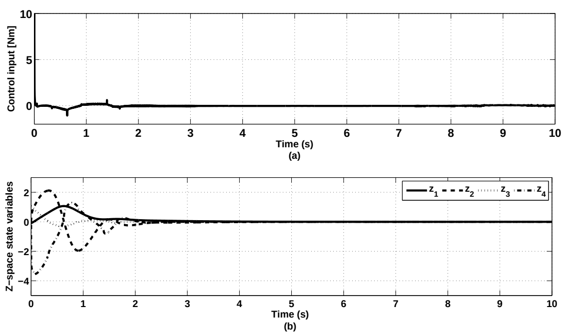

Fig. 3 Evolution of the control effort and Z-space state versus time.

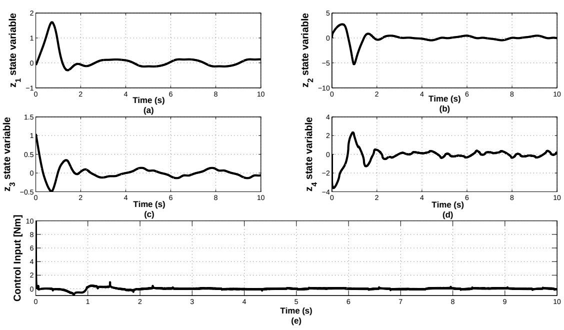

Fig. 5 Evolution of the control effort and Z-space state versus time.

5 Conclusion

In this paper, the problem of finite time stabilization was addressed for the control of a class of uncertain underactuated mechanical systems. By integrating a fractional power feedback control method with an adaptive RBFNNs scheme, the uncertainties in the system can be effectively handled and finite time stabilization is achieved. Future work is to extend the current design technique to tracking control problem of underactuated systems operating under modelling uncertainties and stochastic perturbations with fault-tolerant control. Effective strategies that could handle all these challenges can be found in [25] and [41]. However, the solutions presented so far in these works, can only ensure asymptotic stability. Our endeavor will be to introduce finite-time convergence to stochastic system model.

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