Partitioning Sparse Graphs into an Independent Set and a Forest of Bounded Degree

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Abstract

An $(\mathcal{I}, \mathcal{F}_d)$-partition of a graph is a partition of the vertices of the graph into two sets I and F, such that I is an independent set and F induces a forest of maximum degree at most d. We show that for all $M < 3$ and $d \geqslant \frac{2}{3-M} - 2$, if a graph has maximum average degree less than M, then it has an $(\mathcal{I}, \mathcal{F}_d)$-partition. Additionally, we prove that for all $\frac{8}{3} \leqslant M < 3$ and $d \geqslant \frac{1}{3-M}$, if a graph has maximum average degree less than M then it has an $(\mathcal{I}, \mathcal{F}_d)$-partition. It follows that planar graphs with girth at least 7 (resp. 8, 10) admit an $(\mathcal{I}, \mathcal{F}_5)$-partition (resp. $(\mathcal{I}, \mathcal{F}_3)$-partition, $(\mathcal{I}, \mathcal{F}_2)$-partition).

1 Introduction

In this paper, unless we specify otherwise, all the graphs considered are simple graphs, without loops or multiple edges.

For i classes of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_i$, a $(\mathcal{G}_1, \ldots, \mathcal{G}_i)$-*partition* of a graph G is a partition of the vertices of G into i sets $V_1, \ldots, V_i$ such that, for all $1 \leqslant j \leqslant i$, the graph $G[V_j]$ induced by V_j belongs to $\mathcal{G}_j$.

In the following we will consider the following classes of graphs:

- $\mathcal{F}$ the class of forests,

- $\mathcal{F}_d$ the class of forests with maximum degree at most d,

- Δ_d the class of graphs with maximum degree at most d,

- $\mathcal{I}$ the class of empty graphs (i.e. graphs with no edges).

For example, an $(\mathcal{I}, \mathcal{F}, \Delta_2)$-partition of G is a vertex-partition into three sets V_1, V_2, V_3 such that $G[V_1]$ is an empty graph, $G[V_2]$ is a forest, and $G[V_3]$ is a graph with maximum degree at most 2. Note that $\Delta_0 = \mathcal{F}_0 = I$ and $\Delta_1 = \mathcal{F}_1$.

The *average degree* of a graph G with n vertices and m edges, denoted by $\text{ad}(G)$, is equal to $\frac{2m}{n}$. The *maximum average degree* of a graph G, denoted by $\text{mad}(G)$, is the maximum of $\text{ad}(H)$ over all subgraphs H of G. The *girth* of a graph G is the length of a smallest cycle in G, and infinity if G has no cycle.

Many results on partitions of sparse graphs appear in the literature, where a graph is said to be sparse if it has a low maximum average degree, or if it is planar and has a large girth. The study of partitions of sparse graphs started with the Four Colour Theorem [1, 2], which states that every planar graph admits an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition. Borodin [3] proved that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$-partition, and Borodin and Glebov [4] proved that every planar graph with girth at least 5 admits an $(\mathcal{I}, \mathcal{F})$-partition. Poh [12] proved that every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition.

More recently, Borodin and Kostochka [7] showed that for all $j \geqslant 0$ and $k \geqslant 2j + 2$, every graph G with $\text{mad}(G) < 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ admits a (Δ_j, Δ_k)-partition. In particular, every graph G with $\text{mad}(G) < \frac{8}{3}$ admits an $(\mathcal{I}, \Delta_2)$-partition, and every graph G with $\text{mad}(G) < \frac{14}{5}$ admits an $(\mathcal{I}, \Delta_4)$-partition. With Euler's formula, this yields that planar graphs with girth at least 7 admit $(\mathcal{I}, \Delta_4)$-partitions, and that planar graphs with girth at least 8 admit $(\mathcal{I}, \Delta_2)$-partitions. Borodin and Kostochka [6] proved that every graph G with $\text{mad}(G) < \frac{12}{5}$ admits an $(\mathcal{I}, \Delta_1)$-partition, which implies that that every planar graph with girth at least 12 admits an $(\mathcal{I}, \Delta_1)$-partition. This last result was improved by Kim, Kostochka and Zhu [10], who proved that every triangle-free graph with maximum average degree at most $\frac{11}{9}$ admits an $(\mathcal{I}, \Delta_1)$-partition, and thus that every planar graph with girth at least 11 admits an $(\mathcal{I}, \Delta_1)$-partition. In contrast with these results, Borodin, Ivanova, Montassier, Ochem and Raspaud [5] proved that for every d, there exists a planar graph of girth 6 that admits no $(\mathcal{I}, \Delta_d)$-partition. Montassier and Ochem [11] showed that this implies that deciding if a planar graph of girth 6 admits an $(\mathcal{I}, \Delta_d)$-partition in an NP-complete problem for all $d \geqslant 1$, and they proved that deciding if a planar graph of girth 7 has an $(\mathcal{I}, \Delta_2)$-partition is NP-complete. Esperet, Montassier, Ochem, and Pinlou [8] showed that deciding if a planar graph of girth 9 has an $(\mathcal{I}, \Delta_1)$-partition is NP-complete.

It can be interesting to find partitions of sparse graphs into an independent set and a forest of bounded degree, that is $(\mathcal{I}, \mathcal{F}_d)$-partitions. Note that if a graph admits an $(\mathcal{I}, \mathcal{F}_d)$-partition, then it admits an $(\mathcal{I}, \Delta_d)$-partition, and that an $(\mathcal{I}, \mathcal{F}_1)$-partition is the same as an $(\mathcal{I}, \Delta_1)$-partition. Therefore the previous results imply that:

- for every d, there exists a planar graph of girth 6 that admits no $(\mathcal{I}, \mathcal{F}_d)$-partition;

- there exists a planar graph of girth at least 7 that admits no $(\mathcal{I}, \mathcal{F}_2)$-partition;

- there exists a planar graph of girth at least 9 that admits no $(\mathcal{I}, \mathcal{F}_1)$-partition;

- every planar graph with girth at least 11 admits an $(\mathcal{I}, \mathcal{F}_1)$-partition.

Here are the main results of our paper:

Theorem 1. *Let M be a real number such that $M < 3$. Let $d \geqslant 0$ be an integer and let G be a graph with $\text{mad}(G) < M$. If $d \geqslant \frac{2}{3-M} - 2$, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$	The Four Color Theorem [1, 2]
	$(\mathcal{I},\mathcal{F},\mathcal{F})$	Borodin [3]
	$(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$	Poh [12]
Planar graphs with girth 5	$(\mathcal{I},\mathcal{F})$	Borodin and Glebov [4]
Planar graphs with girth 6	no $(\mathcal{I},\Delta_d)$	Borodin et al. [5]
Planar graphs with girth 7	no $(\mathcal{I},\Delta_2)$	Montassier and Ochem [11]
	$(\mathcal{I},\Delta_4)$	Borodin and Kostochka [7]
	$(\mathcal{I},\mathcal{F}_5)$	Present paper
Planar graphs with girth 8	$(\mathcal{I},\Delta_2)$	Borodin and Kostochka [7]
	$(\mathcal{I},\mathcal{F}_3)$	Present paper
Planar graphs with girth 9	no $(\mathcal{I},\Delta_1)$	Esperet et al.[8]
Planar graphs with girth 10	$(\mathcal{I},\mathcal{F}_2)$	Present paper
Planar graphs with girth 11	$(\mathcal{I},\Delta_1)$	Kim, Kostochka and Zhu [10]

Table 1: Known results on planar graphs.

Theorem 2. *Let M be a real number such that $\frac{8}{3} \leqslant M < 3$. Let $d \geqslant 0$ be an integer and let G be a graph with $\mathrm{mad}(G) < M$. If $d \geqslant \frac{1}{3-M}$, then G admits an $(\mathcal{I},\mathcal{F}_d)$-partition.*

By a direct application of Euler's formula, every planar graph with girth at least g has maximum average degree less than $\frac{2g}{g-2}$. That yields the following corollary:

Corollary 3. *Let G be a planar graph with girth at least g.*

1. *If $g \geqslant 7$, then G admits an $(\mathcal{I},\mathcal{F}_5)$-partition.*

2. *If $g \geqslant 8$, then G admits an $(\mathcal{I},\mathcal{F}_3)$-partition.*

3. *If $g \geqslant 10$, then G admits an $(\mathcal{I},\mathcal{F}_2)$-partition.*

Corollaries 3.1 and 3.2 are obtained from Theorem 2, whereas Corollary 3.3 is obtained from Theorem 1. See Table 1 for an overview of the results on vertex partitions of planar graphs presented above.

In term of tightness, it is not known whether every planar graph or girth 7, 8 or 10 admits an $(\mathcal{I},\mathcal{F}_d)$-partition for $d = 3$, $d = 2$ and $d = 1$ respectively. However, we note that Borodin, Ivanova, Montassier, Ochem and Raspaud [5] constructed, for all d, a planar graph of girth 6 with $16d + 14$ vertices that has no $(\mathcal{I},\Delta_d)$-partition (and thus in particular no $(\mathcal{I},\mathcal{F}_d)$-partition). Let us denote this graph by G_d. By Euler's formula, for every planar graph with girth at least g, n vertices, m edges, and at least one cycle, we have $\frac{m}{n-2} \leqslant \frac{g}{g-2}$. Moreover, if a graph with n vertices, m edges has no cycle, then $\frac{m}{n-2} \leqslant 1 \leqslant \frac{g}{g-2}$ for all $g \geqslant 3$. Therefore for every planar graph of girth at least 6 with n vertices and m edges, we have $\frac{2m}{n} \leqslant 3 \cdot \frac{n-2}{n}$. In particular, this is true for every subgraph

of G_d. Thus, as $\frac{n-2}{n}$ increases when $n \geqslant 2$ increases, if we denote the number of vertices of G_d by n, then $\mathrm{mad}(G_d) \leqslant 3 \cdot \frac{n-2}{n} = 3 - \frac{6}{n} = 3 - \frac{3}{8d+7} < 3 - \frac{3}{8d}$. That shows the following claim.

Claim 4. *For all integer d, there exists a graph with maximum average degree less than M, with $d = \frac{3}{8(3-M)}$, that admits no $(\mathcal{I}, \Delta_d)$-partition (and thus no $(\mathcal{I}, \mathcal{F}_d)$-partition).*

That shows that Theorem 2 is tight up to a multiplicative factor of $\frac{3}{8}$.

2 Proof of Theorem 1

Let $M < 3$, and let d be an integer such that $d \geqslant \frac{2}{3-M} - 2$. Let us call a *good d-partition* of a graph G a partition (I, F) of the vertices of G such that I is an independent set of G, $G[F]$ is a graph with maximum degree at most d, and every cycle in $G[F]$ goes through a vertex with degree 2 in G. Note that for any graph G, if G admits a good d-partition, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition: while there is a vertex v with degree 2 in G that is in F and has two neighbours in F, move v from F to I. Theorem 1 is implied by the following lemma:

Lemma 5. *Every graph G with $\mathrm{mad}(G) < M$ has a good d-partition.*

Our proof uses the discharging method. For the sake of contradiction, assume that Lemma 5 is false. Let G be a counterexample to Lemma 5 with minimum order.

For all k, a vertex of degree k, at least k, or at most k in G is a *k-vertex*, a *k^+-vertex*, or a *k^--vertex* respectively. A $(d+1)^-$-vertex is a *small* vertex, and a $(d+2)^+$-vertex is a *big* vertex. Let v be a vertex of G and w be a neighbour of v. For all k, if w has degree k, at least k, or at most k in G, then w is a *k-neighbour*, a *k^+-neighbour*, or a *k^--neighbour* of v respectively. A neighbour of v that is a big vertex is a *big neighbour* of v, and a neighbour of v that is a small vertex is a *small neighbour* of v. We start by proving some lemmas on the structure of G. Specifically, we prove that some configurations are reducible, and thus cannot occur in G.

Lemma 6. *There are no 1^--vertices in G.*

Proof. Assume there is a 1^--vertex v in G. The graph $G - v$ has one fewer vertex than G, and thus, by minimality of G, admits a good d-partition (I, F). If v has no neighbours in I, then we can add it to I. Otherwise, it has no neighbours in F, and we can add it to F. In both cases, that leads to a good d-partition of G, a contradiction. $\square$

Lemma 7. *Every 2-vertex has at least one big neighbour.*

Proof. Assume v is a 2-vertex adjacent to two small vertices, u and w. The graph $G - v$ has one fewer vertex than G, and thus, by minimality of G, admits a good d-partition (I, F). If u and w are both in F, then we can put v in I, and if they are both in I, then we can put v in F. Therefore without loss of generality, we can assume that $u \in I$ and $w \in F$. If w has no neighbours in I, then we can put it in I, and put v in F. Therefore

we can assume that w has at least one neighbour in I and thus at most $d-1$ neighbours in F (since w is a small vertex in G). Then v has at most one neighbour in F, and this neighbour w has at most $d-1$ neighbours in $G[F]$, thus we can add v to F. In every case, this leads to a good d-partition of G, a contradiction. $\qquad\square$

A 2-vertex is a *light 2-vertex* if it is adjacent to a small vertex, and it is a *heavy 2-vertex* otherwise. Note that by Lemma 7, each 2-vertex has at most one small neighbour.

Lemma 8. *Let B be a set of small 3^+-vertices such that $G[B]$ is a tree. There exists a 3^+-vertex $v \notin B$ that is adjacent to a vertex of B.*

Proof. Assume that the lemma is false, that is every vertex that is not in B but has a neighbour in B is a 2-vertex. By minimality of G, $G - B$ admits a good d-partition (I, F). For every vertex v in B, successively, we put v in I if v has no neighbours in I and we put it in F otherwise. Note that this way a vertex that we add to F has at most d neighbours that are not in I, and we cannot make any cycle in $G[F]$ that does not go through a 2-vertex, since $G[B]$ is a tree. Thus we have a good d-partition of G, a contradiction. $\quad\square$

Let B be a set of small 3^+-vertices such that:

(a) $G[B]$ is a tree,

(b) there is only one edge that links a vertex of B to a 3^+-vertex u outside of B,

(c) u is a big vertex.

We call B a *bud* with father u. Note that every vertex that has a neighbour in a bud is a 2-vertex or a big vertex, or is in that bud. Therefore two different buds always have an empty intersection.

Let us build a structure by the following three steps. We note that some choices can be made in the construction, specifically the order in which the vertices are considered, and thus that the construction may not be unique. However, we do not care about that, and just build one such structure. We call this structure the *light forest L*.

We start with $L = (\emptyset, \emptyset)$, the graph with no vertices and no edges.

1. While there are light 2-vertices that are not in L, do the following. Pick a light 2-vertex v, and let u be the big neighbour of v (that exists by Lemma 7). Add to L the vertex v, the edge uv, and the vertex u (if it is not already in L). Also set that u is the *father* of v (and v is a *son* of u). See Figure 1, left. Note that by doing this, we obtain a star forest with only big vertices and light 2-vertices. Also note that the set of the big vertices and the set of the light 2-vertices are independent sets in L (but not necessarily in G).

2. While there are buds whose vertices are not all included in L, do the following. Pick a bud B. Let u be the father of B, and let v be the vertex of B adjacent to u. Add $G[B]$ to L, as well as the edge uv, and the vertex u (if it is not already in L). The vertex u is the father of v, and the father/son relationship in B is that of the
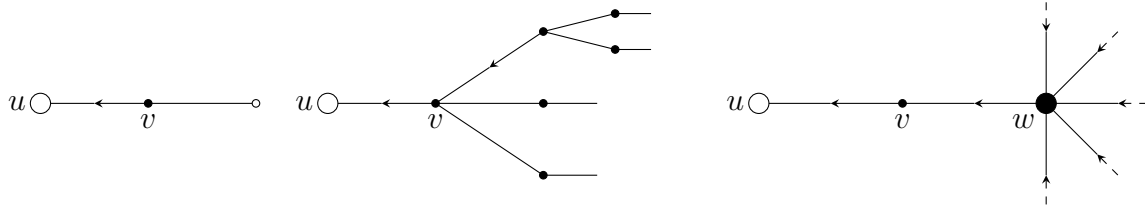
Figure 1: The construction of the light forest L. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented. The dashed lines are the continuation of the light forest. The arrows point from son to father in L.

> tree $G[B]$ rooted at v. See Figure 1, middle. We recall that the buds do not share vertices. Since we add the vertices of the buds to L all in one go, in the end all the vertices of the buds are in L. Note that each iteration of this step always adds to L a tree with at most one vertex (u) that was already in L. Hence, L is still a forest. Moreover, it is still a rooted forest since the tree we add has the orientation of a tree rooted at u.

3. While, for some k, there exists a big k-vertex $w \in L$ that has $k-1$ sons in L and a 2-neighbour v that is not in L, do the following. Let u be the neighbour of v distinct from w. Note that v is a heavy 2-vertex (since it was not added to L in Step 1), therefore u is a big vertex. Add to L the vertex v, the edges uv and vw, and the vertex u (if it is not already in L). We set that v is the father of w, and that u is the father of v. See Figure 1, right. Note that this operation just takes a root of L, adds a vertex as a father of this root and another vertex as a father of that new vertex. Therefore, L is still a rooted forest. Also note that each of the set of the big vertices and the set of the 2-vertices remains independent in L.

As noticed previously, L is a rooted forest. We say that a vertex v is a *descendant* of a vertex $u \neq v$ in L if there are vertices $v_0 = v$, v_1, ..., $v_k = u$ in L, such that for $i \in \{0, 1, \ldots, k-1\}$, v_{i+1} is the father of v_i in L. Consider a vertex v in L. If v is a heavy 2-vertex, then v was added in Step 3, and its two incident edges were added at the same time. If v is a big vertex and is not the root of its connected component in L, then the father of v was added in Step 3, thus all the neighbours of v distinct from its father are its sons in L. If v is a small 3^+-vertex in L, then v was added in Step 2 and thus is in a bud. Therefore, a vertex v in L is incident to an edge that is not in L only if either v is a big vertex and the root of its connected component in L, or v is a light 2-vertex, or v is in a bud. The *pending vertices* of L are the vertices that are not in L but are adjacent to a light 2-vertex. Note that the pending vertices are small.

Let B be a bud with father u. Let $S \subseteq V(G) \setminus (B \cup \{u\})$ and let (I, F) be a good d-partition of $S \cup \{u\}$ such that u either is in I or has at most $d-1$ neighbours in F. We show that we can extend the good d-partition to $S \cup \{u\} \cup B$. We proceed as follows: for every

vertex $v \in B$ successively, we add v to I if it has no neighbours in I or to F otherwise. The vertices in I clearly form an independent set. Moreover, $G[F]$ has maximum degree at most d and every cycle of $G[F]$ goes through a 2-vertex by construction of a bud. This leads to a good d-partition of $S \cup B \cup \{u\}$. We call that process *colouring the bud B*.

Let v be a 2-vertex of L, u its father and D_v the set of the descendants of v. Let (I, F) be a good d-partition of $S \subseteq V(G) \setminus (D_v \cup \{u, v\})$. We show that we can extend the good d-partition to $S \cup D_v \cup \{v\}$. We proceed as follows:

Step 1. We add every big vertex of D_v to I. We can do this, since big vertices form an independent set in L by construction, and as we noted previously, big vertices that are in L and are not the root of their component (in particular big vertices that are in D_v) have no incident edge outside of L.

Step 2. While there is a pending vertex $w \in S$ that has no neighbours in I, we add one such vertex to I.

Step 3. We add every 2-vertex of D_v and v to F. We can do this, since the 2-vertices of D_v form a stable set in L. Moreover, Step 2 ensures that the maximum degree of $G[F]$ is at most d.

Step 4. Finally, we colour every bud. Indeed, the father of every bud whose vertices are in D_v has been put in I in Step 1.

This leads to a good d-partition of $S \cup D_v \cup \{v\}$. We call that process *descending v*.

Lemma 9. *For all k, there are no big k-vertices in G that are in L and have k sons in L.*

Proof. Let u be a big k-vertex that has k sons in L. Note that this implies that u is the root of its connected component in L. Let C be the connected component of u in L. Let $H = G - V(C)$. The graph H has fewer vertices than G and thus, by minimality of G, H admits a good d-partition (I, F). Let N be the set of the 2-neighbours of u. We descend every vertex of N. Note that this implies that every vertex of N is put in F. We add u to I. Then we colour every bud of father u. This leads to a good d-partition of G, a contradiction. $\square$

Discharging procedure

Let $\epsilon = 3 - M$. Recall that $d \geqslant \frac{2}{3-M} - 2 = \frac{2}{\epsilon} - 2$, therefore $\epsilon \geqslant \frac{2}{d+2} > 0$. For all k, we assign to each k-vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G, the sum of the charges of the vertices is negative. The initial charge of each 3^+-vertex is at least ϵ, and thus is positive.

For every big vertex v, v gives charge $1 - \epsilon$ to each of its 2-neighbours that are its sons in L, does not give anything to its father in L (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other 2-neighbours.

Lemma 10. *Every vertex has non-negative charge at the end of the procedure.*

Proof. The small 3^+-vertices start with a non-negative charge, and do not give or receive charge throughout the procedure, thus they have non-negative charge at the end of the procedure.

Every 2-vertex is either in L, in which case it receives $1 - \epsilon$ from its father in L, or is not in L and is a heavy 2-vertex, in which case it receives $\frac{1-\epsilon}{2}$ from each of its neighbours. As 2-vertices have charge $\epsilon - 1$ at the start of the procedure, and as they receive $1 - \epsilon$, they have charge 0 at the end of the procedure.

Let v be a big k-vertex. By Lemma 9, v has at most $k - 1$ sons in L. Moreover, by construction of L, if v has $k - 1$ sons in L, then either its neighbour that is not its son in L is a 3^+-vertex, or it is the father of v in L (and in both cases v does not give charge to this vertex). Therefore v gives charge amounting to at most $(k - 1)(1 - \epsilon)$. Since its initial charge is $k - 3 + \epsilon$, in the end it has at least $k - 3 + \epsilon - (k - 1)(1 - \epsilon) = k\epsilon - 2$. Since every big vertex has degree at least $d + 2 \geqslant \frac{2}{\epsilon}$, the final charge of each big vertex is at least $\frac{2}{\epsilon}\epsilon - 2 = 2 - 2 = 0$. $\qquad\square$

By Lemma 10, every vertex has non-negative charge at the end of the procedure, thus the sum of the charges at the end of the procedure is non-negative. Since no charge was created nor removed, this is a contradiction with the fact that the initial sum of the charges is negative. That ends the proof of Lemma 5.

3   Proof of Theorem 2

This proof is similar to the proof of Theorem 1 above. Let $\frac{8}{3} \leqslant M < 3$, and let d be an integer such that $d \geqslant \frac{1}{3-M}$. We define good d-partitions as in Section 2. Theorem 2 is implied by the following lemma:

Lemma 11. *Every graph G with* $\mathrm{mad}(G) < M$ *has a good d-partition.*

For the sake of contradiction, assume that Lemma 11 is false. Let G be a counterexample to Lemma 11 with minimum order.

We take the same definitions as before. Lemmas 6–9 of the previous section still hold in this setting. Frank and Gyárfás [9] prove the following theorem:

Theorem 12 (Frank and Gyárfás [9])**.** *Let $H = (V, E)$ be a graph, and let $\omega : V \to \mathbb{N}$. There exists an orientation such that $\forall v \in V, d^+(v) \geqslant \omega(v)$ if and only if for all $X \subset V$, $\omega(X) \leqslant |\{uv \in E, u \in X\}|$.*

Given $H = (V, E)$ and $\omega : V \to \mathbb{N}$, a *good ω-orientation* of H is an orientation of H such that $\forall v \in V, d^+(v) \geqslant \omega(v)$. We prove some additional lemmas.

Lemma 13. *Let $H = (V, E)$ be a graph on $n \geqslant 1$ vertices and m edges. Let $\omega : V \to \mathbb{N}$ such that $\omega(V) \leqslant m$. There exists a subgraph S of H with at least one vertex such that S admits a good ω-orientation.*

Proof. For a graph I and a set $X \subset V(I)$, let $e_I(X) = |\{uv \in E(I), u \in X\}|$. If $\omega(X) \leqslant e_I(X)$, we say that X is *good* in I.

If every subset of V is good in H, then by Theorem 12, we have a good ω-orientation of H. Therefore we may assume that there is a subset of V that is not good in H. Let X be a maximum subset of V that is not good in H. Let $Y = V - X$, and let $H' = H[Y]$. Note that V is good in H, since $\omega(V) \leqslant m$, so $Y \neq \emptyset$.

If every subset of Y is good in H', then by Theorem 12, we have a good ω-orientation of H'. Therefore there is a $Z \subset Y$ such that Z is not good in H', i.e. $\omega(Z) > e_{H'}(Z)$. As X is not good in H, we also have $\omega(X) > e_H(X)$. Therefore we have $\omega(X \cup Z) = \omega(X) + \omega(Z) > e_H(X) + e_{H'}(Z) = |\{uv \in E(H), u \in X\}| + |\{uv \in E(H'), u \in Z\}| = |\{uv \in E(H), u \in (X \cup Z)\}| = e_H(X \cup Z)$. Therefore $X \cup Z$ is not good in H, which contradicts the maximality of X. $\qquad\square$

We recall that L is the light forest of G.

Lemma 14. *Let U be a non-empty subset of $V(L)$ with no small 3^+-vertices. Let $H = G[U]$ (i.e. the subgraph of G induced by the 2-vertices and the big vertices of $U \subseteq L$). Suppose:*

1. *There is an orientation of the edges of H such that every 2-vertex in H has at least one out-going edge, and for all $i \geqslant 1$, every big $(d + i + 1)$-vertex in G has at least i out-going edges.*

2. *There are no 1^--vertices in H.*

Then H contains an edge that is not in L and that is incident to a big vertex.

The graph H of Lemma 14 is as follows: it is composed by subtrees of L plus some additional edges (that do not belong to L). Such edges are edges between light 2-vertices, and maybe edges between roots of trees of L. The aim of Lemma 14 is to prove the existence of such latter edges.

The orientation in Lemma 14 does not correspond to the orientation defined by the father/son relation. This orientation will allow us to extend a partial partition (I, F): consider a big $(d + i + 1)$-vertex v being in F. Vertex v must have at least $i + 1$ neighbours in I. The orientation will point towards i sons of v that will be added to I. Moreover we will see that v will have one extra neighbour in I: either its father in L, or a neighbour outside L.

Proof of Lemma 14. Assume the lemma is false: every edge of H that is not in L is between two 2-vertices. Let R_0 be the set of the vertices of H that are not the descendants in L of a vertex of H. In particular, R_0 contains the roots of L that are in H, plus big vertices that have no ancestor in U. Note that R_0 contains only big vertices ; otherwise, H would contain 1-vertices. Moreover, $H - R_0$ has at least one vertex, otherwise U would contain only big vertices, there would be an edge between two big vertices, and this edge

could not be in L. Let S be the set of the vertices that are not in H, but are descendants of vertices of H.

By minimality of G, the graph $G - (V(H - R_0) \cup S)$ admits a good d-partition (I, F). While there is a vertex $v \in R_0$ that is in F and has no neighbours in I, we put v in I. Now we can assume that every vertex in $R_0 \cap F$ has a neighbour in I. Let $R = R_0$ (in the following we describe a procedure that modifies R but we need to refer to vertices of R_0). While there is a vertex in R, do the following:

- Suppose u is in I. We descend every 2-vertex with father u (by the procedure every 2-vertex is added to F) and colour every bud with father u. This leads to a good d-partition of u and all its descendants.

- Suppose u is in F. We remove u from R. For every 2-vertex v in H with father u such that the edge uv is oriented from u to v (according to the orientation defined in the statement of the lemma), we first add v to I, and then add the son of v to F and R. By hypothesis, if u is a $(d+i+1)$-vertex, then it has at least i outgoing edges. These edges lead to sons of u:

  - either $u \in R_0$, thus u has no ancestors in H by construction and all its neighbours in H are its sons ; moreover it has a neighbour outside H that is in I (by construction).

  - or, $u \in R \setminus R_0$, this means that u was added to R during the procedure, this implies that his father, say w, is a 2-vertex added to I and the edge wv is oriented from w to v (as every 2-vertex has an out-going edge by hypothesis). It follows that all the out-going neighbours of u are sons of u.

  It follows that u has at least $i + 1$ neighbours in I, and so all other neighbours can be added to F without violating the degree condition on F. Now we descend every 2-vertex $v \notin H$ with father u, and every 2-vertex $v \in H$ with father u such that the edge uv is oriented from v to u, and colour every bud with father u. The only problem that could occur is when two adjacent light 2-vertices ℓ and ℓ' are added to I: in that case, since ℓ and ℓ' were added to I, the edge that links ℓ (resp. ℓ') to its father is towards ℓ (resp. ℓ'); it follows that one of ℓ, ℓ' has no out-going edges, contradicting the hypothesis.

In all cases, that leads to a good d-partition of G, a contradiction. $\qquad\square$

Lemma 15. *Let U be a non-empty subset of $V(L)$ with no small 3^+-vertices. Let $H = G[U]$ (i.e. the subgraph of G induced by the 2-vertices and the big vertices of $U \subseteq L$). Suppose that H has no edge linking two roots of two connected components of L. Let us denote by $n_2^G(H)$ the number of vertices of H that are 2-vertices in G. Then,*

$$|E(H)| < n_2^G(H) + \sum_{big\ v \in H} (d_G(v) - d - 1).$$

Proof. By contradiction, suppose there exists such a subgraph H with $|E(H)| \geqslant n_2^G(H) + \sum_{big\ v \in H} (d_G(v) - d - 1)$. Let us define a weight function $\omega : V(H) \to \mathbb{N}$ such that, for every 2-vertex u, $\omega(u) = 1$ and, for every $(d+i+1)$-vertex v with $i \geqslant 1$ in $\mathbb{N}$, $\omega(v) = i$. By hypothesis, $|E(H)| \geqslant \sum_{v \in V(H)} \omega(v)$. By Lemma 13, H contains a non empty subgraph I that has a good ω-orientation.

Suppose that I has a vertex of degree 1, say v, and let u be the neighbour of v in I. As $\omega(v) \geqslant 1$, the only edge incident to v goes from v to u. It follows that, for all $w \neq v$, w has the same number of outgoing edges in I and in $I - \{v\}$. Hence $I - \{v\}$ is a subgraph of H with at least one vertex (it contains u) and has a good ω-orientation. By successively removing vertices of degree 1 from I, we can assume that I has no 1-vertices.

By Lemma 14, I has an edge e that is not in L and is incident to a big vertex. As no light 2-vertex is adjacent to a big vertex besides its father, edge e has to link the roots of two connected components of L, contradicting the hypothesis. $\square$

Let $\widehat{L}$ be the graph induced by $V(L)$, where we remove every edge that links the roots of two connected components of L and we remove every bud. An *internal* 2-vertex is a 2-vertex in $\widehat{L}$ that has its two neighbours in $\widehat{L}$. By applying Lemma 15 to $\widehat{L}$, we can bound the number of internal 2-vertices in $\widehat{L}$. We obtain the following lemma:

Lemma 16. *The number of internal 2-vertices is at most $2 \sum_{big\ v \in H}(d_G(v) - d - 1)$.*

Proof. Let $\widehat{L}'$ be the graph $\widehat{L}$ where every 2-vertex is removed in the following way: if v is an internal 2-vertex with neighbours u and w, then we remove v and add an edge from u to w, and we iterate. For the 2-vertices that are not internal, we just remove them. Note that $\widehat{L}'$ may have multiple edges and even loops. As for each 2-vertex that was removed, exactly one edge was removed, the number of edges in $\widehat{L}'$ is at most $\sum_{big\ v \in H} (d_G(v) - d - 1)$. By Lemma 7, every edge of $\widehat{L}'$ corresponds to at most two internal 2-vertices. Therefore there are at most $2 \sum_{big\ v \in H}(d_G(v) - d - 1)$ internal 2-vertices. $\square$

Discharging procedure

Let $\epsilon = 3 - M$ (recall that $\frac{8}{3} \leqslant M \leqslant 3$). Recall that $d \geqslant \frac{1}{3-M} = \frac{1}{\epsilon}$, therefore $\epsilon \geqslant \frac{1}{d} > 0$.

We assign to each k-vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G, the sum of the charges of the vertices is negative.

Every 3^+-vertex has a charge of at least $\epsilon > 0$. Therefore every vertex that has a negative charge is a 2-vertex and has charge $\epsilon - 1$. We will redistribute the weight from the 3^+-vertices to the 2-vertices, in order to obtain a non-negative weight on each vertex, by the following three steps:

1. Let S be a maximal set of small 3^+-vertices such that $G[S]$ is connected. Let S_2 be a set of 2-vertices that have exactly one (by Lemma 7) neighbour in S. Note that since $\epsilon \leqslant 1$, every k-vertex in S has charge at least $(k-2)\epsilon$. The vertices in S give ϵ to each of the vertices in S_2.

   Suppose that the total charge of S becomes negative. This implies that the number of vertices in S_2 is more than $\sum_{v \in S}(d(v) - 2)$. Therefore there are at most $|S| - 1$

edges in $G[S]$. Since $G[S]$ is connected, this implies that $G[S]$ is a tree. Now by Lemma 8, there is at least one big vertex outside of S that has a neighbour in S. Note that if there are at least two of these vertices, or if one of them has at least two neighbours in S, then one can observe that $G[S]$ has at most $|S| - 2$ edges, contradicting the connectivity of $G[S]$. Therefore S is a bud. In this case, S ends up with a charge of at least $-\epsilon$. We will, in Step 2, make sure that every son of a big vertex in L receives at least $1 - 2\epsilon \geqslant \epsilon$ (since $\epsilon \leqslant \frac{1}{3}$) from its father, and this will ensure that every bud ends up with a non-negative charge.

We do this step for maximal every set S of small 3^+-vertices such that $G[S]$ is connected successively. Note that such sets are distinct.

2. For every big vertex v, v gives $1 - 2\epsilon$ to each of its sons, does not give anything to its father (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other 2-neighbours. Additionally, every big k-vertex gives $2(k - d - 1)\epsilon$ to a common pot.

3. The common pot gives ϵ to every internal 2-vertex.

Lemma 17. *Every vertex has non-negative charge at the end of the procedure.*

Proof. Note that by what precedes every small 3^+-vertex v has non-negative charge.

Every 2-vertex that is not in L receives $\frac{1-\epsilon}{2}$ from each of its neighbours. Every light 2-vertex that is not adjacent to a 2-vertex (i.e. every 2-vertex of L that is not an internal 2-vertex) receives $1 - 2\epsilon$ from its father and ϵ from its other neighbour (which is a small 3^+-vertex). Every internal 2-vertex receives $1 - 2\epsilon$ from its father and ϵ from the common pot. Therefore every 2-vertex has charge 0 at the end of the procedure.

Let us prove that every big vertex has non-negative charge at the end of the procedure. Let v be a big k-vertex. Let $c(v)$ be the initial charge of v, and $c'(v)$ be the final charge of v. Suppose by contradiction that $c'(v) < 0$. By Lemma 9, vertex v has at most $k - 1$ sons. Moreover, if v has $k - 1$ sons, then its last neighbour is either the father of v, or a 3^+-vertex (by construction of L). Recall that $\epsilon \leqslant \frac{1}{3}$, therefore $1 - 2\epsilon \geqslant \frac{1-2\epsilon}{2}$. If v has $k - 1$ sons, then v gives $1 - 2\epsilon$ to each of its $k - 1$ sons, and $2(k - d - 1)\epsilon$ to the common pot, therefore $c'(v) = c(v) - (1 - 2\epsilon)(k - 1) + 2(k - d - 1)\epsilon$, and thus as $c'(v) < 0$, we have $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$. If v has $k - 2$ sons, then it gives $1 - 2\epsilon$ to each of its $k - 2$ sons, and may give at most $\frac{1-\epsilon}{2}$ to its two other neighbours and $2(k - d - 1)\epsilon$ to the common pot, therefore $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$. If we decrease the number of sons of v further than $k - 2$, we will still have $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$.

Thus if $c(v) \geqslant (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$, we get a contradiction. Recall that $c(v)$ is equal to $k - 3 + \epsilon$. Therefore we only need to prove that $k - 3 + \epsilon \geqslant (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$, which is equivalent to $d \geqslant \frac{1}{\epsilon}$. $\square$

Let us prove that the common pot also has non-negative charge at the end of the procedure. It receives charge $\sum_{v \text{ big}} 2(d(v) - d - 1)\epsilon$. By Lemma 16, this charge is at least ϵ times the number of internal 2-vertices. The common pot gives ϵ to each internal 2-vertex, therefore it has non-negative charge at the end of the procedure.

By Lemma 17, every vertex has non-negative charge at the end of the procedure, thus the sum of the charges at the end of the procedure is non-negative. Since no charge was created nor removed, and since the common pot also has non-negative charge, this is a contradiction with the fact that the initial sum of the charges is negative. That ends the proof of Theorem 2.

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