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► **To cite this version:**

Juanjo Rué, Dimitrios M. Thilikos, Vasiliki Velona. Structure and Enumeration of K_4 -minor-free links and link diagrams. *Electronic Notes in Discrete Mathematics*, 2018, 68, pp.119-124. 10.1016/j.endm.2018.06.021 . lirmm-01890505

HAL Id: lirmm-01890505

<https://hal-lirmm.ccsd.cnrs.fr/lirmm-01890505v1>

Submitted on 8 Oct 2018

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Structure and Enumeration of K_4 -minor-free links and link diagrams

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Abstract

We study the class \mathcal{L} of link types that admit a K_4 -minor-free diagram, i.e., they can be projected on the plane so that the resulting graph does not contain any subdivision of K_4 . We prove that \mathcal{L} is the closure of a subclass of torus links under the operation of connected sum. Using this structural result, we enumerate \mathcal{L} and subclasses of it, with respect to the minimal number of crossings or edges in a projection of $L \in \mathcal{L}$. Further, we enumerate (both exactly and asymptotically) all connected K_4 -minor-free link diagrams, all minimal connected K_4 -minor-free link diagrams, and all K_4 -minor-free diagrams of the unknot.

Keywords: series-parallel graphs, links, knots, generating functions, asymptotic enumeration, map enumeration

1 Introduction

The exhaustive generation of knots and links is a well-established problem in low dimensional geometry (see [7, Ch.5], for instance). However, there are very few enumerative results in the literature and they are relatively recent, (see [10] and [6]). Moreover, there seems to be no known results connecting graph theoretic classes with link classes.

Let us start with some formal definitions. A *knot* K is a smooth embedding of the 1-dimensional sphere \mathbb{S}^1 in \mathbb{R}^3 . A *link* is a finite disjoint union of knots $L = K_1 \cup \dots \cup K_n$. Two links L_1 and L_2 are equivalent if there is a continuous and injective function $h : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$, such that $h(L_1, 0) = L_1$ and $h(L_1, 1) = L_2$. A link equivalent to a set of non intertwined circles is called a *trivial link*. If it is a knot, we also call it the *unknot*.

Consider a link L and a sphere \mathbb{S}^2 embedded in such a way that it meets the link transversely in exactly two points P_1 and P_2 . Then we can discern two different links L_1, L_2 , after connecting P_1 and P_2 . The first corresponds to the part of L in the interior of the sphere and the second to the part in the exterior. We then call L a *connected sum* with *factors* L_1, L_2 , denoted $L_1 \# L_2$. A link that does not have non-trivial factors is called *prime*, otherwise *composite*. A link is *split* if there is a sphere embedded in the link complement that separates the link. Each of the components is called a *disjoint component* of the link and, conversely, the link is their *disjoint sum*.

Let $L \in \mathbb{R}^3$ be a link and let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a projection map. If for all $x \in L$, $|\pi^{-1}(x)| = 2$ or 1 , and all the double points are finite and transverse, then the projection is said to be *regular*. For all knots defined here, there exists a regular projection [2, Ch. 3]. This allows to work with *link-diagrams*, i.e. a triple (V, E, σ) , where (V, E) is a 4-regular plane graph and $\sigma : V(G) \rightarrow \binom{E(G)}{2}$, where, for every $v \in V(G)$, $\sigma(v)$ is a set of two opposite edges of the embedding of G and encodes which pair is overcrossing. Notice that a link-diagram may also have vertexless edges. We call these edges *trivial*. The *crossing number* of a link L is the minimum number of crossings that can have a diagram of it. Such a diagram is called *minimal*. We also say that a diagram (V, E, σ) is K_4 -minor-free if the graph (V, E) does not contain

¹The first and the third author have partially been supported by the MINECO projects MTM2014-54745-P, MTM2017-82166-P and the María de Maetzu research grant MDM-2014-0445 (BGSMath). The second author has been supported by the project “ESIGMA” (ANR-17-CE40-0028). The third author has been supported under an FPI grant from the MINECO research project MTM2015-67304-PI.

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any subgraph that is a subdivision of a K_4 .

A *torus link* is a link that can be embedded on the torus. They are denoted by $T(p, q)$, $p, q \in \mathbb{Z}$, where p and q are the number of times that the link crosses the meridian and the longitude cycle, respectively. Let \mathcal{T}_2 be the closure of torus links of type $T(2, q)$, $q \in \mathbb{Z} - \{0\}$, under the connected sum operation. Let \mathcal{L} be the class of links that have a K_4 -minor-free link-diagram and $\mathbf{mcl}(\mathcal{T}_2)$ be the closure of \mathcal{T}_2 under disjoint sum.

2 Structure of K_4 -links and enumeration

Our first result gives the type of links that admit a K_4 -minor-free diagram.

Theorem 2.1 *The links that admit a K_4 -minor-free diagram are exactly the ones in $\mathbf{mcl}(\mathcal{T}_2)$, i.e. $\mathcal{L} = \mathbf{mcl}(\mathcal{T}_2)$.*

Based on the above, we derive a precise description of links that admit a K_4 -minor-free diagram. Let $\bar{\mathcal{L}}$ be the class of non-split links in \mathcal{L} and $\hat{\mathcal{L}}$ the class of links in \mathcal{L} with no trivial disjoint components, with size defined as their crossing number. We will obtain the asymptotic growth of \mathcal{L} with respect to the number of edges in a minimal diagram (not the crossing number, so as to account also for trivial disjoint components). Let $L(z)$, $\bar{L}(z)$, and $\hat{L}(z)$ be the respective generating functions. Finally, let \mathcal{T} be the class of non plane, unrooted trees, where the vertices are labelled with multisets of the set $\{1\} \cup \{\pm 3, \pm 5, \dots\}$ and the edges are labelled with an element of the set $\{2, \pm 4, \pm 6, \dots\}$. These labels will encode crossing numbers, hence a label i corresponds to size $|i|$. The size of $T \in \mathcal{T}$ is the sum of its labels. See Figure 1 for an example of such a tree.

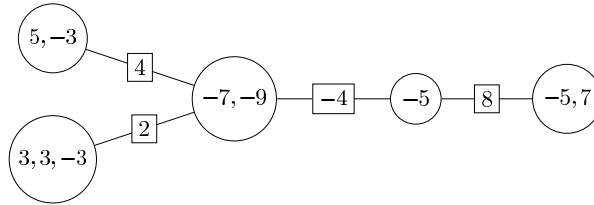


Figure 1. An object of \mathcal{T} .

Proposition 2.2 $\bar{\mathcal{L}} \cong \mathcal{T}$ and $\text{MSet}_{\geq 1}(\mathcal{T} \setminus T_1) \cong \hat{\mathcal{L}}$, where MSet is the multiset operator with at least one element and $T_1 \in \mathcal{T}$ is the tree with one vertex and label 1.

Using Proposition 2.2 and the framework of the *Symbolic Method* (see [4]), we derive combinatorial expressions for \mathcal{L} that translate to generating functions. Let $\bar{\mathcal{K}}$ and \mathcal{K} be the classes of prime knots and composite knots plus the trivial knot in \mathcal{L} , respectively. For prime knots in \mathcal{L} , i.e. prime torus knots $T(2, 2i + 1)$, $i \in \mathbb{Z} \setminus \{0, -1\}$, it holds that $\bar{K}(z) = 2 \sum_{i \geq 1} z^{2i+1} = \frac{2z^3}{1-z^2}$, counting according to their crossing number. For \mathcal{K} , it is enough to consider non-empty multisets of prime torus knots, therefore $\mathcal{K} = \text{MSet}_{\geq 1}(\bar{\mathcal{K}})$, which translates to the expression

$$K(z) = \exp \left(\sum_{k \geq 1} \frac{1}{k} \bar{K}(z^k) \right) - 1 = \exp \left(\sum_{k \geq 1} \frac{1}{k} \frac{2z^{3k}}{1-z^{2k}} \right) - 1.$$

Let \mathcal{G} be the class of unrooted and unlabelled non-plane trees. Notice that one cannot replace immediately z for \mathcal{K} in the ogf $G(z)$, since the vertices are not distinguishable. Hence, to continue, one needs to use cycle indices. Let \mathcal{G}^\bullet be the class of rooted and unlabelled non-plane trees. For the cycle index of \mathcal{G}^\bullet , it is known (see [1]) that

$$\mathcal{Z}_{\mathcal{G}^\bullet}(s_1, s_2, \dots) = s_1 \exp \left(\sum_{k \geq 1} \frac{1}{k} \mathcal{Z}_{\mathcal{G}^\bullet}(s_k, s_{2k}, \dots) \right).$$

Let \mathcal{E} be the combinatorial class of the edge labels of \mathcal{T} , hence $E(z) = z^2 + \frac{2z^4}{1-z^2}$, and let $f(z) = E(z)K(z)$. We can now obtain the ordinary generating function of $\mathcal{F} = \mathcal{T}^\bullet \circ (\mathcal{E} \times \mathcal{L}_c)$. By Polya's Theorem, the latter satisfies the equation $F(z) = \mathcal{Z}_{\mathcal{G}^\bullet}(f(z), f(z^2), \dots)$. Then, it holds that

$$\bar{L}(z) = \frac{F(z)}{E(z)} + \frac{E(z)}{2} \left(-\frac{F(z)^2}{E(z)^2} + \frac{F(z^2)}{E(z^2)} \right),$$

by the Dissymmetry Theorem (see [1]). The first terms of $\bar{L}(z)$ are as follows:

$$\bar{L} = 1 + z^2 + 2z^3 + 3z^4 + 4z^5 + 9z^6 + 12z^7 + 26z^8 + 40z^9 + \dots$$

We can then get asymptotic estimates by means of complex analytic tools:

Theorem 2.3 $\bar{\mathcal{L}}$ has asymptotic growth of the form:

$$[z^n] \bar{L}(z) \sim \frac{cn^{-3/2}}{\Gamma(-1/2)} \rho^{-n}, \quad \rho \approx 0.44074, \quad c \approx 1.45557.$$

Additionally, $\hat{\mathcal{L}}$ has the same type of asymptotic growth with the same ρ and $c \approx 3.61691$.

The generating function $L(z)$ is equal to $\hat{L}(z^2) \frac{1}{1-z}$, since a link diagram of n vertices has $2n$ non-trivial edges and a number of trivial edges. We thus obtain the following corollary.

Corollary 2.4 \mathcal{L} has asymptotic growth of the form:

$$[z^n]L(z) \sim \frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n},$$

where $\rho \approx 0.44074$ and $c \approx 8.97779$ or $c \approx 3.95687$, depending on whether n is even or odd, respectively.

Finally, we also get asymptotic estimates for the coefficients of $K(z)$. The following result is a consequence of Meinardus' Theorem (see [4, VIII.23]).

Theorem 2.5 \mathcal{K} has asymptotic growth of the form:

$$[z^n]K(z) \sim cn^{-7/4} \exp(c'n^{1/2}),$$

where $c = \frac{e^{2 \ln(2)\zeta(0)}(\Gamma(2)\zeta(2))^{5/4}}{2\sqrt{\pi}} \approx 0.26275$, $c' = 2\sqrt{\Gamma(2)\zeta(2)} \approx 2.56509$, and $\zeta(z)$ is the Riemann zeta function.

3 Enumeration of families of link diagrams

Enumerating connected K_4 -free link-diagrams is equivalent to enumerating 4-regular unrooted planar maps which are K_4 -minor-free, that we call $\bar{\mathcal{M}}_1$. We first give a construction for the rooted ones, \mathcal{M}_1 , with respect to edges, adapting the construction for rooted 4-regular maps in [8]. With this we obtain a functional system of equations that can be analyzed by using the analytic machinery developed by Drmota in [3]. Finally, by adapting an argument by Richmond and Wormald in [9], we are able to show that we can unroot the maps under study, showing that the maps in our class with non-trivial automorphisms are exponentially few. This study gives the following result:

Theorem 3.1 The class of connected K_4 -free link diagrams $\bar{\mathcal{M}}_1$ satisfies that

$$[z^n]\bar{\mathcal{M}}_1(z) \sim \frac{1}{2n} \frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n}2^n, \quad \rho \approx 0.31184, \quad c \approx 1.52265$$

Refining the combinatoric analysis in the previous case (with more auxiliary classes) and using again an adaptation of the argument of Richmond and Wormald in [9], we are able to asymptotically enumerate also the class of minimal link diagrams $\bar{\mathcal{M}}_2$, as well as the class of K_4 -free link diagrams of the unknot, $\bar{\mathcal{M}}_3$:

Theorem 3.2 The class of connected K_4 -free minimal link diagrams $\bar{\mathcal{M}}_2$, and

the class of K_4 -free link diagrams of the unknot, \bar{M}_3 satisfy:

$$[z^n]\bar{M}_2(z) \sim \frac{1}{2n} \frac{c_2 n^{-3/2}}{\Gamma(-1/2)} \rho_2^{-n}, \quad \rho_2 \approx 0.41456, \quad c_2 \approx 0.81415,$$

$$[z^n]\bar{M}_3(z) \sim \frac{1}{2n} \frac{c_3 n^{-3/2}}{\Gamma(-1/2)} \rho_3^{-n}, \quad \rho_3 \approx 0.23188, \quad c_3 \approx 2.19020.$$

Acknowledgement. All authors like to thank Lefteris Kirousis that contributed to the inception of this research.

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