Contraction-Bidimensionality of Geometric Intersection Graphs

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Abstract

Given a graph G, we define $\mathbf{bcg}(G)$ as the minimum k for which G can be contracted to the uniformly triangulated grid Γ_k. A graph class $\mathcal{G}$ has the SQGC property if every graph $G \in \mathcal{G}$ has treewidth $\mathcal{O}(\mathbf{bcg}(G)^c)$ for some $1 \leq c < 2$. The SQGC property is important for algorithm design as it defines the applicability horizon of a series of meta-algorithmic results, in the framework of bidimensionality theory, related to fast parameterized algorithms, kernelization, and approximation schemes. These results apply to a wide family of problems, namely problems that are *contraction-bidimensional*. Our main combinatorial result reveals a general family of graph classes that satisfy the SQGC property and includes bounded-degree string graphs. This considerably extends the applicability of bidimensionality theory for several intersection graph classes of 2-dimensional geometrical objects.

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1 Introduction

Treewidth is one of most well-studied parameters in graph algorithms. It serves as a measure of how close a graph is to the topological structure of a tree (see Section 2 for the formal definition). Gavril is the first to introduce the concept in [28] but it obtained its name in the second paper of the Graph Minors series of Robertson and Seymour in [36]. Treewidth has extensively used in graph algorithm design due to the fact that a wide class of intractable problems in graphs becomes tractable when restricted on graphs of bounded treewidth [1,4,5]. Before we present some key combinatorial properties of treewidth, we need some definitions.

1.1 Graph contractions and minors

Our first aim is the define some parameterized versions of the contraction relation on graphs. Given a non-negative integer c, two graphs H and G, and a surjection $\sigma : V(G) \to V(H)$ we write $H \leq_{\sigma}^{c} G$ if

- for every $x \in V(H)$, the graph $G[\sigma^{-1}(x)]$ is a non-empty graph of *diameter* at most c and
- for every $x, y \in V(H)$, $\{x, y\} \in E(H) \iff G[\sigma^{-1}(x) \cup \sigma^{-1}(y)]$ is connected.

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We say that H is a *c-diameter contraction* of G if there exists a surjection $\sigma : V(G) \to V(H)$ such that $H \leq_\sigma^c G$ and we write this $H \leq^c G$. Moreover, if σ is such that for every $x \in V(G)$, $|\sigma^{-1}(x)| \leq c+1$, then we say that H is a *c-size contraction* of G, and we write $H \leq^{(c)} G$.

1.2 Combinatorics of treewidth

One of the most celebrated structural results on treewidth is the following:

▶ **Proposition 1.** There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph excluding a $(k \times k)$-grid as a minor has treewidth at most $f(k)$.

A proof of Proposition 1 appeared for the first time by Robertson and Seymour in [37]. Other proofs, with better bounds to the function f, appeared in [38] and later in [17] (see also [31, 33]). Currently, the best bound for f is due to Chuzhoy, who proved in [3] that $f(k) = k^{19} \cdot \log^{\mathcal{O}(1)} k$. On the other side, it is possible to show that Proposition 1 is not correct when $f(k) = \mathcal{O}(k^2 \cdot \log k)$ (see [41]).

The potential of Proposition 1 on graph algorithms has been capitalized by the *theory of bidimensionality* that was introduced in [9] and has been further developed in [8, 12, 13, 15, 16, 21–23, 25, 27, 30]. This theory offered general techniques for designing efficient fixed-parameter algorithms and approximation schemes for NP-hard graph problems in broad classes of graphs (see [7, 10, 11, 14, 20]). In order to present the result of this paper we first give a brief presentation of this theory and of its applicability.

1.3 Optimization parameters and bidimensionality

A *graph parameter* is a function $\mathbf{p}$ mapping graphs to non-negative integers. We say that $\mathbf{p}$ is a *minimization graph parameter* if $\mathbf{p}(G) = \min\{k \mid \exists S \subseteq V(G) : |S| \leq k \text{ and } \phi(G, S) = \texttt{true}\}$, where ϕ is a some predicate on G and S. Similarly, we say that $\mathbf{p}$ is a *maximization graph parameter* if in the above definition we replace min by max and $\leq$ by $\geq$ respectivelly. Minimization or maximization parameters are briefly called *optimization parameters*.

Given two graphs G and H, if there exists an integer c such that $H \leq^c G$, then we say that H is a *contraction* of G, and we write $H \leq G$. Moreover, if there exists a subgraph G' of G such that $H \leq G'$, we say that H is a *minor* of G and we write this $H \preceq G$. A graph parameter $\mathbf{p}$ is *minor-closed* (resp. *contraction-closed*) when $H \preceq G \Rightarrow \mathbf{p}(H) \leq \mathbf{p}(G)$ (resp. $H \leq G \Rightarrow \mathbf{p}(H) \leq \mathbf{p}(G)$). We can now give the two following definitions:

$\mathbf{p}$ is *minor-bidimensional* if

- $\mathbf{p}$ is minor-closed, and
- $\exists k_0 \in \mathbb{N} : \forall k \geq k_0, \frac{\mathbf{p}(\boxplus_k)}{k^2} \geq \delta$

$\mathbf{p}$ is *contraction-bidimensional* if

- $\mathbf{p}$ is contraction-closed, and
- $\exists k_0 \in \mathbb{N} : \forall k \geq k_0, \frac{\mathbf{p}(\Gamma_k)}{k^2} \geq \delta$

for some $\delta > 0$. In the above definitions, we use $\boxplus_k$ for the $(k \times k)$-grid and Γ_k for the uniformly triangulated $(k \times k)$-grid (see Figure 1). If $\mathbf{p}$ is a minimization (resp. maximization) graph parameter, we denote by $\Pi_{\mathbf{p}}$ the problem that, given a graph G and a non-negative integer k, asks whether $\mathbf{p}(G) \leq k$ (resp. $\mathbf{p}(G) \geq k$). We say that a problem is *minor/contraction-bidimensional* if it is $\Pi_{\mathbf{p}}$ for some bidimensional optimization parameter $\mathbf{p}$.

A (non excaustive) list of minor-bidimensional problems is: Vertex Cover, Feedback Vertex Set, Longest Cycle, Longest Path, Cycle Packing, Path Packing, Diamond Hitting Set, Minimum Maximal Matching, Face Cover, and Max Bounded Degree Connected Subgraph. Some problems that are contraction-bidimensional (but not minor-bidimensional) are Connected Vertex Cover, Dominating Set, Connected Dominating Set, Connected Feedback Vertex Set, Induced Matching, Induced Cycle Packing, Cycle Domination, Connected Cycle Domination, d-Scattered Set, Induced Path Packing, r-Center, connected r-Center, Connected Diamond Hitting Set, Unweighted TSP Tour.

1.4 Subquadratic grid minor/contraction property

In order to present the meta-algorithmic potential of bidimensionality theory we need to define some *property on graph classes* that defines the horizon of its applicability. Let $\mathcal{G}$ be a graph class. We say that $\mathcal{G}$ has the *subquadratic grid minor property* (SQGM property for short) if there exist a constant $1 \leq c < 2$ such that every graph $G \in \mathcal{G}$ which excludes $\boxplus_t$ as a minor, for some integer t, has treewidth $\mathcal{O}(t^c)$. In other words, this property holds for $\mathcal{G}$ if Proposition 1 can be proven for a sub-quadratic f on the graphs of $\mathcal{G}$.

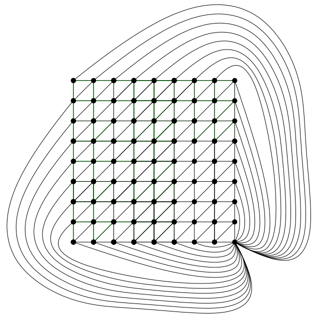

Figure 1 The graph Γ_9.

Similarly, we say that $\mathcal{G}$ has the *subquadratic grid contraction property* (SQGC property for short) if there exist a constant $1 \leq c < 2$ such that every graph $G \in \mathcal{G}$ which excludes Γ_t as a contraction, for some integer t, has treewidth $\mathcal{O}(t^c)$. For brevity we say that $\mathcal{G} \in \mathtt{SQGM}(c)$ (resp. $\mathcal{G} \in \mathtt{SQGC}(c)$) if $\mathcal{G}$ has the SQGM (resp SQGC) property for c. Notice that $\mathtt{SQGC}(c) \subseteq \mathtt{SQGM}(c)$ for every $1 \leq c < 2$.

1.5 Algorithmic implications

The meta-algorithmic consequences of bidimensionality theory are summarised as follows. Let $\mathcal{G} \in \mathtt{SQGM}(c)$, for $1 \leq c < 2$, and let $\mathbf{p}$ be a minor-bidimensional-optimization parameter.
[**A**] As it was observed in [9], the problem $\Pi_{\mathbf{p}}$ can be solved in $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ steps on $\mathcal{G}$, given that the computation of $\mathbf{p}$ can be done in $2^{\mathbf{tw}(G)} \cdot n^{\mathcal{O}(1)}$ steps (here $\mathbf{tw}(G)$ is the treewidth of the input graph G). This last condition can be implied by a purely meta-algorithmic condition that is based on some variant of *Modal Logic* [35]. There is a wealth of results that yield the last condition for various optimization problems either in classes satisfying the SQGM propety [18, 18, 19, 39, 40] or to general graphs [2, 6, 24].
[**B**] As it was shown in [25] (see also [26]), when the predicate ϕ can be expressed in Counting Monadic Second Order Logic (CMSOL) and $\mathbf{p}$ satisfies some additional combinatorial property called *separability*, then the problem $\Pi_{\mathbf{p}}$ admits a *linear kernel*, that is a polynomial-time algorithm that transforms (G, k) to an equivalent instance (G', k') of $\Pi_{\mathbf{p}}$ where G' has size $\mathcal{O}(k)$ and $k' \leq k$.
[**C**] It was proved in [22], that the problem of computing $\mathbf{p}(G)$ for $G \in \mathcal{G}$ admits a *Efficient Polynomial Approximation Scheme* (EPTAS) — that is an ϵ-approximation algorithm running in $f(\frac{1}{\epsilon}) \cdot n^{\mathcal{O}(1)}$ steps — given that $\mathcal{G}$ is hereditary and $\mathbf{p}$ satisfies the separability property and some reducibility property (related to CMSOL expresibility).

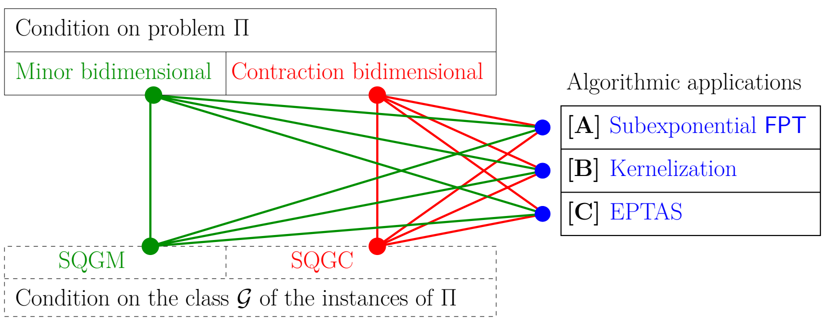

Figure 2 The applicability of bidimensionality theory.

All above results have their counterparts for *contraction-bidimensional* problems with the difference that one should instead demand that $\mathcal{G} \in \texttt{SQGC}(c)$. Clearly, the applicability of all above results is delimited by the SQGM/SQGC property. This is schematically depicted in Figure 2, where the green-triangles triangles indicate the applicability of minor-bidimensionality and the red triangle indicate the applicability of contraction-bidimensionality. The aforementioned $\Omega(k^2 \cdot \log k)$ lower bound to the function f of Proposition 1, indicates that $\texttt{SQGM}(c)$ does not contain all graphs (given that $c < 2$). The emerging direction of research is to detect the most general classes in $\texttt{SQGC}(c)$ and $\texttt{SQGC}(c)$. We denote by $\mathcal{G}_H$ the class of graphs that exclude H as a minor. Concerning the SQGM property, the following result was proven in [14].

▶ **Proposition 2.** For every graph H, $\mathcal{G}_H \in \texttt{SQGM}(1)$.

A graph H is an *apex graph* if it contains a vertex whose removal from H results to a planar graph. For for the SQGC property, the following counterpart of Proposition 2 was proven in [21].

▶ **Proposition 3.** For every apex graph H, $\mathcal{G}_H \in \texttt{SQGC}(1)$.

Notice that both above results concern graph classes that are defined by excluding some graph as a minor. For such graphs, Proposition 3 is indeed optimal. To see this, consider K_h-minor free graphs where $h \geq 6$ (these graphs are not apex graphs). Such classes do not satisfy the SQGC property: take Γ_k, add a new vertex, and make it adjacent, with all its vertices. The resulting graph excludes Γ_k as a contraction and has treewidth $> k$.

1.6 String graphs

An important step extending the applicability of bidimensionality theory further than H-minor free graphs, was done in [23]. *Unit disk* graphs are intersections graphs of unit disks in the plane and *map* graphs are intersection graphs of face boundaries of planar graph embeddings. We denote by $\mathcal{U}_d$ the set of unit disk graphs (resp. of $\mathcal{M}_d$ map graphs) of maximum degree d. The following was proved in [23].

▶ **Proposition 4.** For every positive integer d, $\mathcal{U}_d \in \texttt{SQGM}(1)$ and $\mathcal{M}_d \in \texttt{SQGM}(1)$.

Proposition 4 was further extended for intersection graphs of more general geometric objects (in 2 dimensions) in [30]. To explain the results of [30] we need to define a more general model of intersection graphs.

Let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be a collection of lines in the plane. We say that $\mathcal{L}$ is *normal* if there is no point belonging to more than two lines. The *intersection graph* $G_{\mathcal{L}}$ of $\mathcal{L}$, is the graph whose vertex set is $\mathcal{L}$ and where, for each i, j where $1 \leq i < j \leq k$, the edge $\{L_i, L_j\}$ has multiplicity $|L_1 \cap L_2|$. We denote by $\mathcal{S}_d$ the set containing every graph $G_{\mathcal{L}}$ where $\mathcal{L}$ is a normal collection of lines in the plane and where each vertex of $G_{\mathcal{L}}$ has edge-degree at most d. i.e., is incident to at most d edges. We call $\mathcal{S}_d$ *string graphs with edge-degree bounded by d*. It is easy to observe that $\mathcal{U}_d \cup \mathcal{M}_d \subseteq \mathcal{S}_{f(d)}$ for some quadratic function f. Moreover, apart from the classes considered in [23], $\mathcal{S}_d$ includes a much wider variety of classes of intersection graphs [30]. As an example, consider $\mathcal{C}_{d,\alpha}$ as the class of all graphs that are intersection graphs of α-convex 2-dimensional bodies[1] in the plane and have degree at most d. In [30], it was proven that $\mathcal{C}_{d,\alpha} \subseteq \mathcal{S}_c$ where c depends (polynomially) on d and c (see [34] for other examples of classes included in $\mathcal{S}_d$).

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[1] We call a set of points in the plane a *2-dimensional body* if it is homeomorphic to the closed disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$. A 2-dimensional body B is a *α-convex* if every two points can be the extremes of a line L consisting of α straight lines and where $L \subseteq B$.

Given a class of graph $\mathcal{G}$ and two integers c_1 and c_2, we define $\mathcal{G}^{(c_1,c_2)}$ as the set containing every graph H such that there exist a graph $G \in \mathcal{G}$ and a graph J that satisfy $G \leq^{(c_1)} J$ and $H \leq^{c_2} J$. Keep in mind that $\mathcal{G}^{(c_1,c_2)}$ and $\mathcal{G}^{(c_2,c_1)}$ are different graph classes. We also denote by $\mathcal{P}$ the class of all planar graphs. Using this notation, the two combinatorial results in [30] can be rewritten as follows:

▶ **Proposition 5.** Let c_1 and c_2 be two positive integers. If $\mathcal{G} \in \mathtt{SQGC}(c)$ for some $1 \leq c < 2$, then $\mathcal{G}^{(c_1,c_2)} \in \mathtt{SQGM}(c)$.

▶ **Proposition 6.** For every $d \in \mathbb{N}$, $\mathcal{S}_d \subseteq \mathcal{P}^{(1,d)}$.

Proposition 2, combined with Proposition 5, provided the wider, so far, framework on the applicability of minor-bidimensionality: $\mathtt{SQGM}(1)$ contains $\mathcal{G}_H^{(c_1,c_2)}$ for every apex graph H and positive integers c_1, c_2. As $\mathcal{P} \in \mathtt{SQGC}(1)$ (by, e.g., Proposition 3), Propositions 5 and 6 directly classifies in $\mathtt{SQGM}(1)$ the graph class $\mathcal{S}_d$, and therefore a large family of bounded degree intersection graphs (including $\mathcal{U}_d$ and $\mathcal{M}_d$). As a result of this, the applicability of bidimensionality theory for minor-bidimensional problems has been extended to much wider families (not necessarily minor-closed) of graph classes of geometric nature.

1.7 Our contrubution

Notice that Proposition 5 exhibits some apparent “lack of symmetry” as the assumption is “qualitatively stronger” than the conclusion. This does not permit the application of bidimensionality for *contraction*-bidimensional parameters on classes further than those of apex-minor free graphs. In other words, the results in [30] covered, for the case of $\mathcal{S}_d$, the green triangles in Figure 2 but left the red triangles open. The main result of this paper is to fill this gap by proving the following extension of Proposition 5:

▶ **Theorem 1.** *Let c_1 and c_2 be two positive integers. If $\mathcal{G} \in \mathtt{SQGC}(c)$ for some $1 \leq c < 2$, then $\mathcal{G}^{(c_1,c_2)} \in \mathtt{SQGC}(c)$.*

Combining Proposition 3 and Theorem 1 we extend the applicability horizon of contraction-bidimensionality further than apex-minor free graphs: $\mathtt{SQGC}(1)$ contains $\mathcal{G}_H^{(c_1,c_2)}$ for every apex graph H and positive integers c_1, c_2. As a special case of this, we have that $\mathcal{S}_d \in \mathtt{SQGC}(1)$. Therefore, on $\mathcal{S}_d$, the results described in Subsection 1.5 apply for contraction-bidimensional problems as well (such as those enumerated in the end of Subsection 1.3).

This paper is organized as follows. In Section 2, we give the necessary definitions and some preliminary results. Section 3 is dedicated to the proof of Theorem 1. We should stress that this proof is quite different than the one of Proposition 5 in [30]. Finally, Section 4 contains some discussion and open problems.

2 Definitions and preliminaries

All graphs in this paper are undirected, loop-less, and may have multiple edges. If a graph has no multiple edges, we call it *simple*. Given a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Let x be a vertex or an edge of a graph G and likewise for y; their *distance* in G, denoted by $\mathbf{dist}_G(x,y)$, is the smallest number of vertices of a path in G that contains them both. Moreover if G is a graph and $x \in V(G)$, we denote by $N_G^c(x)$ the set $\{y \mid y \in V(G),\ \mathbf{dist}_G(x,y) \leq c+1\}$. For any set of vertices $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by the vertices from S. If $G[S]$ is connected, then we say that S is a *connected vertex set* of G. We define the *diameter* of a connected subset S as the maximum pairwise distance between any two vertices of S. The *edge-degree* of a vertex $v \in V(G)$ is the number of edges that are incident to it (multi-edges contribute with their multiplicity to this number).

For our proofs, we also need the graph $\hat{\Gamma}_k$ that is the variant of Γ_k, depicted in Figure 3. Notice that Γ_k and $\hat{\Gamma}_k$ are both triangulated plane graphs, i.e., all their faces are triangles. In $\hat{\Gamma}_k$, we refer to the vertex a (as in Figure 3) as the *apex vertex* of $\hat{\Gamma}_k$. (We avoid the formal definitions of $\boxplus_k$, Γ_k, $\hat{\Gamma}_k$ in this extended abstract – see [21] for a more precise formalism.) In each of these graphs we denote the vertices of the underlying grid by their coordinates $(i, j) \in [0, k-1]^2$ agreeing that the upper-left corner is the vertex $(0, 0)$.

2.1 Treewidth

A *tree-decomposition* of a graph G, is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X} = \{X_t : t \in V(T)\}$ is a family of subsets of $V(G)$, called *bags*, such that the following three properties are satisfied:

- $\bigcup_{t \in V(T)} X_t = V(G)$,
- for every edge $e \in E(G)$ there exists $t \in V(T)$ such that $e \subseteq X_t$, and
- $\forall v \in V(G)$, the set $T_v = \{t \in V(T) \mid v \in X_t\}$ is a connected vertex set of T.

The *width* of a tree-decomposition is the cardinality of the maximum size bag minus 1 and the *treewidth* of a graph G is the minimum width over all the tree-decompositions of G. We denote the treewidth of G by $\mathbf{tw}(G)$.

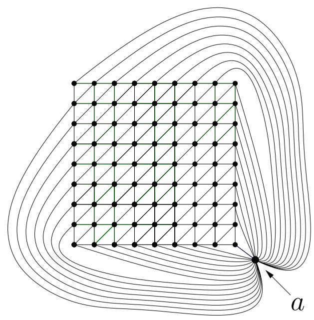

Figure 3 The uniformly triangulated grid $\hat{\Gamma}_9$.

▶ **Lemma 2.** *Let G be a graph and let H be a c-size contraction of G. Then $\mathbf{tw}(G) \leq (c+1) \cdot (\mathbf{tw}(H) + 1) - 1$.*

The proof of Lemma 2 is in Section A in the Appendix.

3 Proof of Theorem 1

Let H and G be graphs and c be a non-negative integer. If $H \leq^c_\sigma G$, then we say that H is a σ*-contraction* of G, and denote this by $H \leq_\sigma G$.

Before we proceed the the proof of Theorem 1 we make first the following three observations. (In all statements, we assume that G and H are two graphs and $\sigma : V(G) \to V(H)$ such that H is a σ-contraction of G.)

▶ Observation 1. Let S be a connected subset of $V(H)$. Then the set $\bigcup_{x \in S} \sigma^{-1}(x)$ is connected in G.

▶ Observation 2. Let $S_1 \subseteq S_2 \subseteq V(H)$. Then $\sigma^{-1}(S_1) \subseteq \sigma^{-1}(S_2) \subseteq V(G)$.

▶ Observation 3. Let S be a connected subset of $V(G)$. Then the diameter of $\sigma(S)$ in H is at most the diameter of S in G.

Given a graph G and $S_1, S_2 \subseteq V(G)$ we say that S_1 and S_2 *touch* if either $S_1 \cap S_2 \neq \emptyset$ or there is an edge of G with one endpoint in S_1 and the other in S_2.

We say that a collection $\mathcal{R}$ of paths of a graph is *internally disjoint* if none of the internal vertices, i.e., none of the vertex of degree 2, of some path in $\mathcal{R}$ is a vertex of some other path in $\mathcal{R}$. Let $\mathcal{A}$ be a collection of subsets of $V(G)$. We say that $\mathcal{A}$ is a *connected packing* of G if its elements are connected and pairwise disjoint. If additionally $\mathcal{A}$ is a partition of $V(G)$, then we say that $\mathcal{A}$ is a *connected partition* of G and if, additionally, all its elements have diameter bounded by some integer c, then we say that $\mathcal{A}$ is a c*-diameter partition* of G.

3.1 Λ-state configurations.

Let G be a graph. Let $\Lambda = (\mathcal{W}, \mathcal{E})$ be a graph whose vertex set is a connected packing of G, i.e., its vertices are connected subsets of $V(G)$. A Λ-*state configuration* of a graph G is a quadruple $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ where

1. $\mathcal{X}$ is a connected packing of G,
2. α is a bijection from $\mathcal{W}$ to $\mathcal{X}$ such that for every $W \in \mathcal{W}$, $W \subseteq \alpha(W)$,
3. $\mathcal{R}$ is a collection of internally disjoint paths of G, and
4. β is a bijection from $\mathcal{E}$ to $\mathcal{R}$ such that if $\{W_1, W_2\} \in \mathcal{E}$ then the endpoints of $\beta(\{W_1, W_1\})$ are in W_1 and W_2 and $V(\beta(\{W_1, W_2\})) \subseteq \alpha(W_1) \cup \alpha(W_2)$.

A Λ-state configuration $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ of G is *complete* if $\mathcal{X}$ is a partition of $V(G)$. We refer to the elements of $\mathcal{X}$ as the *states* of $\mathcal{S}$ and to the elements of $\mathcal{R}$ as the *freeways* of $\mathcal{S}$. We define $\mathsf{indep}(\mathcal{S}) = V(G) \setminus \bigcup_{X \in \mathcal{X}} X$. Note that if $\mathcal{S}$ is a Λ-state configuration of G, $\mathcal{S}$ is complete if and only if $\mathsf{indep}(\mathcal{S}) = \emptyset$.

Let $\mathcal{A}$ be a c-diameter partition of G. We refer to the sets of $\mathcal{A}$ as the $\mathcal{A}$-*clouds* of G. We define $\mathsf{front}_{\mathcal{A}}(\mathcal{S})$ as the set of all $\mathcal{A}$-clouds of G that are not subsets of some $X \in \mathcal{X}$. Given a $\mathcal{A}$-cloud C and a state X of $\mathcal{S}$ we say that C *shadows* X if $C \cap X \neq \emptyset$. The *coverage* $\mathsf{cov}_{\mathcal{S}}(C)$ of an $\mathcal{A}$-cloud C of G is the number of states of $\mathcal{S}$ that are shadowed by C. A Λ-state configuration $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ of G is $\mathcal{A}$-*normal* if its satisfies the following conditions:

(A) If a $\mathcal{A}$-cloud C intersects some $W \in \mathcal{W}$, then $C \subseteq \alpha(W)$.

(B) If a $\mathcal{A}$-cloud over $\mathcal{S}$ intersects the vertex set of at least two freeways of $\mathcal{S}$, then it shadows at most one state of $\mathcal{S}$.

We define $\mathsf{cost}_{\mathcal{A}}(\mathcal{S}) = \sum_{C \in \mathsf{front}_{\mathcal{A}}(\mathcal{S})} \mathsf{cov}_{\mathcal{S}}(C)$. Given $S_1 \subseteq S_2 \subseteq V(G)$ where S_1 is connected, we define $\mathsf{cc}_G(S_2, S_1)$ as the (unique) connected component of $G[S_2]$ that contains S_1.

3.2 Triangulated grids inside triangulated grids

▶ **Lemma 3.** *Let G and H be graphs and c, k be non-negative integers such that $H \leq^c G$ and $\Gamma_k \leq G$. Then $\Gamma_{k'} \leq H$ where $k' = \lfloor \frac{k-1}{2c+1} \rfloor - 1$.*

Proof. Let $k^* = 1 + (2c+1) \cdot (k'+1)$ and observe that $k^* \leq k$, therefore $\Gamma_{k^*} \leq \Gamma_k \leq G$. For simplicity we use $\Gamma = \Gamma_{k^*}$. Let $\phi : V(G) \to V(H)$ such that $H \leq^c_\phi G$ and let $\sigma : V(G) \to V(\Gamma)$ such that $\Gamma \leq_\sigma G$. We define $\mathcal{A} = \{\phi^{-1}(a) \mid a \in V(H)\}$. Notice that $\mathcal{A}$ is a c-diameter partition of G.

For each $(i, j) \in [\![0, k'+1]\!]^2$, we define $b_{i,j}$ to be the vertex of Γ with coordinate $(i(2c+1), j(2c+1))$. We set $Q_{\text{in}} = \{b_{i,j} \mid (i,j) \in [\![1, k']\!]^2\}$ and $Q_{\text{out}} = \{b_{i,j} \mid (i,j) \in [\![0, k'+1]\!]^2\} \setminus Q_{\text{in}}$. Let also $Q = Q_{\text{in}} \cup \{b_{\text{out}}\}$ were b_{out} is a new element that does not belong in Q_{in}. Here b_{out} can be seen as a vertex that "represents" all vertices in Q_{out}.

Let q, p be two different elements of Q. We say that q and p are *linked* if they both belong in Q_{in} and their distance in Γ is $2c+1$ or one of them is b_{out} and the other is $b_{i,j}$ where $i \in \{1, k'\}$ or $j \in \{1, k'\}$.

For each $q \in Q_{\text{in}}$, we define $W_q = \sigma^{-1}(q)$. W_q is connected by the definition of σ. In case $q = b_{\text{out}}$ we define $W_q = \bigcup_{q' \in Q_{\text{out}}} \sigma^{-1}(q')$. Note that as Q_{out} is a connected set of Γ, then, by Observation 1, $W_{b_{\text{out}}}$ is connected in G. We also define $\mathcal{W} = \{W_q \mid q \in Q\}$. Given some $q \in Q$ we call W_q the q-*capital* of G and a subset S of $V(G)$ is a *capital* of G if it is the q-capital for some $q \in Q$. Notice that $\mathcal{W}$ is a connected packing of $V(G)$.

Let $q \in Q$. If $q \in Q_{\text{in}}$ then we set $N_q = N^c_\Gamma(q)$. If $q = b_{\text{out}}$, then we set $N_q = \bigcup_{q' \in Q_{\text{out}}} N^c_\Gamma(q')$. Note that for every $q \in Q$, $N_q \subseteq V(\Gamma)$. For every $q \in Q$, we define $X_q = \sigma^{-1}(N_q)$. Note that $X_q \subseteq V(G)$. We also set $\mathcal{X} = \{X_q \mid q \in Q\}$. Let q and p we two linked elements of Q. If both q and p belong to Q_{in}, and therefore are vertices of Γ, then we define $Z_{p,q}$ as the unique shortest path between them in Γ. If $p = b_{\text{out}}$ and $q \in Q_{\text{in}}$, then

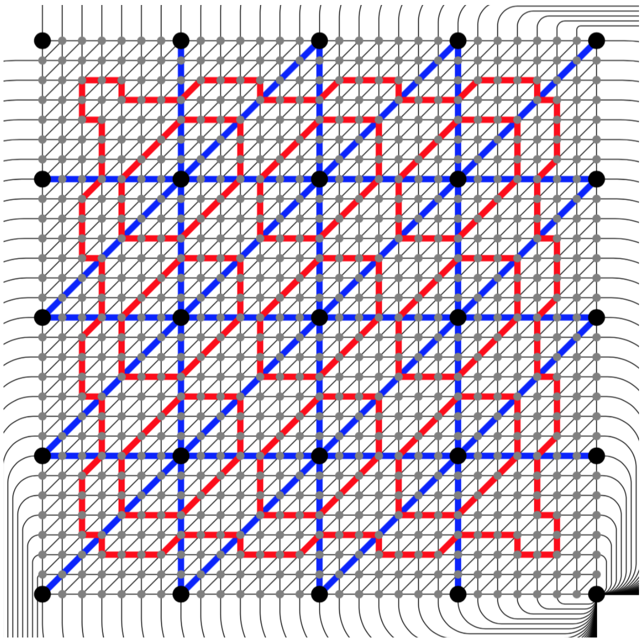

Figure 4 A visualization of the proof of Lemma 3. In this whole graph Γ_k, we initialize our reaserch of $\hat{\Gamma}_{k'}$ such that every internal red hexagon will become a vertex of $\hat{\Gamma}_{k'}$ and correspond to a state and the border, also circle by a red line will become the vertex b_{out}. The blue edges correspond to the freeways. Red cycles correspond to the boundaries of the starting countries. Blue paths between big-black vertices are the freeways. Big-black vertices are the capitals.

we know that $q = b_{i,j}$ where $i \in \{1, k'\}$ or $j \in \{1, k'\}$. In this case we define $Z_{p,q}$ as any shortest path in Γ between $b_{i,j}$ and the vertices in Q_{out}. In both cases, we define $P_{p,q}$ by picking some path between W_p and W_q in $G[\sigma^{-1}(V(Z_{p,q}))]$ such that $|V(P_{p,q}) \cap W_q| = 1$ and $|V(P_{p,q}) \cap W_p| = 1$.

Let $\mathcal{E} = \{\{W_p, W_q\} \mid \ p \text{ and } q \text{ are linked}\}$ and let $\Lambda = (\mathcal{W}, \mathcal{E})$. Notice that Λ is isomorphic to $\hat{\Gamma}_{k'}$ and consider the isomorphism that correspond each vertex $q = b_{i,j}$, $i, j \in [\![1, k']\!]^2$ to the vertex with coordinates (i, j). Moreover b_{out} corresponds to the apex vertex of $\hat{\Gamma}_{k'}$.

Let $\alpha : \mathcal{W} \to \mathcal{X}$ such that for every $q \in Q$, $\alpha(W_q) = X_q$. Let also $\mathcal{R} = \{P_{p,q} \mid p, q \in Q, \ p \text{ and } q \text{ are linked}\}$. We define $\beta : \mathcal{E} \to \mathcal{R}$ such that if q and p are linked, then $\beta(W_q, W_p) = P_{p,q}$. We use notation $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$.

▶ Claim 1. $\mathcal{S}$ is an $\mathcal{A}$-normal Λ-state configuration of G.

The proof of the Claim 1 is in Section B of the Appendix.

We define bellow three ways to transform a Λ-state configuration of G. In each of them, $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ is an $\mathcal{A}$-normal Λ-state configuration of G and C is an $\mathcal{A}$-cloud in $\mathsf{front}_{\mathcal{A}}(\mathcal{S})$.

1. The *expansion procedure* applies when C intersects at least two freeways of $\mathcal{S}$. Let X be the state of $\mathcal{S}$ shadowed by C (this state is unique because of property (B) of $\mathcal{A}$-normality). We define $(\mathcal{X}', \alpha', \mathcal{R}', \beta') = \mathsf{expand}(\mathcal{S}, C)$ such that
   - $\mathcal{X}' = \mathcal{X} \setminus \{X\} \cup \{X \cup C\}$,
   - for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where X' is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
   - $\mathcal{R}' = \mathcal{R}$, and $\beta' = \beta$.
2. The *clash procedure* applies when C intersects *exactly* one freeway P of $\mathcal{S}$. Let X_1, X_2 be the two states of $\mathcal{S}$ that intersect this freeway. Notice that $P = \beta(\alpha^{-1}(X_1), \alpha^{-1}(X_2))$, as

it is the only freeway with vertices in X_1 and X_2. Assume that $(C \cap V(P)) \cap X_1 \neq \emptyset$ (if, not, then swap the roles of X_1 and X_2). We define $(\mathcal{X}', \alpha', \mathcal{R}', \beta') = \mathsf{clash}(\mathcal{S}, C)$ as follows:

- $\mathcal{X}' = \{X_1 \cup C\} \cup \bigcup_{X \in \mathcal{X} \setminus \{X_1\}} \{\mathsf{cc}_G(X \setminus C, \alpha^{-1}(X))\}$ (notice that $\alpha^{-1}(X) \subseteq X \setminus C$, for every $X \in \mathcal{X}$, because of property (A) of $\mathcal{A}$-normality),
- for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where X' is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
- $\mathcal{R}' = \mathcal{R} \setminus \{P\} \cup \{P'\}$, where $P' = P_1 \cup P^* \cup P_2$ is defined as follows: let s_i be the first vertex of C that we meet while traversing P when starting from its endpoint that belongs in W_i and let P_i the subpath of P that we traversed that way, for $i \in \{1, 2\}$. We define P^* by taking any path between s_1 and s_2 inside $G[C]$, and
- $\beta' = \beta \setminus \{(\{W_1, W_2\}, P)\} \cup \{\{W_1, W_2\}, P'\}$.

3: The *annex procedure* applies when C intersects no freeway of $\mathcal{S}$ and *touches* some country $X \in \mathcal{X}$. We define $(\mathcal{X}', \alpha', \mathcal{R}', \beta') = \mathsf{anex}(\mathcal{S}, C)$ such that

- $\mathcal{X}' = \{X_1 \cup C\} \cup \bigcup_{X \in \mathcal{X} \setminus \{X_1\}} \{\mathsf{cc}_G(X \setminus C, \alpha^{-1}(X))\}$ (notice that $\alpha^{-1}(X) \subseteq X \setminus C$, for every $X \in \mathcal{X}$, because of property (A) of $\mathcal{A}$-normality),
- for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where X' is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
- $\mathcal{R}' = \mathcal{R}$, and $\beta' = \beta$.

▶ Claim 2. Let $\mathcal{S} = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ be an $\mathcal{A}$-normal Λ-state configuration of G, and $C \in \mathsf{front}_{\mathcal{A}}(\mathcal{S})$. Let $\mathcal{S}' = \mathsf{action}(\mathcal{S}, C)$ where $\mathsf{action} \in \{\mathsf{expand}, \mathsf{clash}, \mathsf{anex}\}$. Then $\mathcal{S}'$ is an $\mathcal{A}$-normal Λ-state configuration of G where $\mathsf{cost}(\mathcal{S}', \mathcal{A}) \leq \mathsf{cost}(\mathcal{S}, \mathcal{A})$. Moreover, if $\mathsf{cov}_{\mathcal{S}}(C) \geq 1$, then $\mathsf{cost}(\mathcal{S}', \mathcal{A}) < \mathsf{cost}(\mathcal{S}, \mathcal{A})$ and if $\mathsf{cov}_{\mathcal{S}}(C) = 0$ (which may be the case only when $\mathsf{action} = \mathsf{anex}$), then $|\mathsf{indep}(\mathcal{S}')| < |\mathsf{indep}(\mathcal{S})|$.

The proof of the Claim 2 is in Section C of the Appendix.

To continue with the proof of Lemma 3 we explain how to transform the $\mathcal{A}$-normal Λ-state configuration $\mathcal{S}$ of G to a complete one. This is done in two phases. First, as long as there is an $\mathcal{A}$-cloud $C \in \mathsf{front}(\mathcal{S})$ where $\mathsf{cov}_{\mathcal{S}}(C) \geq 1$, we apply one of the above three procedures depending on the number of freeways intersected by C. We again use $\mathcal{S}$ to denote the $\mathcal{A}$-normal Λ-state configuration of G that is created in the end of this first phase. Notice that, as there is no $\mathcal{A}$-cloud with $\mathsf{cov}_{\mathcal{S}}(C) \geq 1$, then $\mathsf{cost}_{\mathcal{A}}(\mathcal{S}) = 0$. The second phase is the application of $\mathsf{anex}(\mathcal{S}, C)$, as long as some $C \in \mathsf{front}_{\mathcal{A}}(\mathcal{S})$ is touching some of the countries of $\mathcal{S}$. We claim that this procedure will be applied as long as there are vertices in $\mathsf{indep}(\mathcal{S})$. Indeed, if this is the case, the set $\mathsf{front}_{\mathcal{A}}(\mathcal{S})$ is non-empty and by the connectivity of G, there is always a $C \in \mathsf{front}_{\mathcal{A}}(\mathcal{S})$ that is touching some country of $\mathcal{S}$. Therefore, as $\mathsf{cost}_{\mathcal{A}}(\mathcal{S}) = 0$ (by Claim 2), procedure $\mathsf{anex}(\mathcal{S}, C)$ will be applied again.

By Claim 2, $|\mathsf{indep}(\mathcal{S})|$ is strictly decreasing during the second phase. We again use $\mathcal{S}$ for the final outcome of this second phase. We have that $\mathsf{indep}(\mathcal{S}) = \emptyset$ and we conclude that $\mathcal{S}$ is a complete $\mathcal{A}$-normal Λ-state configuration of G such that $|\mathsf{front}_{\mathcal{A}}(\mathcal{S})| = 0$.

We are now going to create a graph isomorphic to Λ only by doing contractions in G. For this we use $\mathcal{S}$, a complete $\mathcal{A}$-normal Λ-state configuration of G such that $|\mathsf{front}_{\mathcal{A}}(\mathcal{S})| = 0$, obtained as describe before. We contract in G every country of $\mathcal{S}$ into a unique vertex. This can be done because the countries of $\mathcal{S}$ are connected. Let G' be the resulting graph. By construction of $\mathcal{S}$, G' is a contraction of H. Because of Condition 4 of Λ-state configuration, every freeway of $\mathcal{S}$ becomes an edge in G'. This implies that there is a graph isomorphic to Λ that is a subgraph of G'. So $\hat{\Gamma}_{k'}$ is isomorphic to a subgraph of G' with the same number of vertices. Let see $\hat{\Gamma}_{k'}$ as a subgraph of G' and let e be an edge of G' that is not an edge of $\hat{\Gamma}_{k'}$. As e is an edge of G', this implies that in G, there is two states of $\mathcal{S}$ such that there is no freeway between them but still an edge. This is not possible by construction of $\mathcal{S}$. We deduce that G' is isomorphic to $\hat{\Gamma}_{k'}$. Moreover, as $|\mathsf{front}_{\mathcal{A}}(\mathcal{S})| = 0$, then every cloud is a subset of a country. This implies that G' is also a contraction of H. By contracting in G' the edge corresponding to $\{a, (k'-1, k'-1)\}$ in $\hat{\Gamma}_{k'}$, we obtain that $\Gamma_{k'}$ is a contraction of H. Lemma 3 follows. ◀

3.3 Proof of the main result

Lemmata 2 and 3 are the basic ingredients for the proof of Theorem 1. The proof can be found in Section D of the Appendix.

4 Conclusions and open problems

The main combinatorial result of this paper is that, for every d, the class $\mathcal{S}_d$ of string graphs with multi-degree at most d has the SQGC property for $c = 1$. This means that, for fixed d, if a graph in $\mathcal{S}_d$ excludes as a contraction the uniformly triangulated grid Γ_k, then its treewidth is bounded by a linear function of k. Recall that string graphs are intersection graphs of lines in the plane. It is easy to extend our results for intersection graphs of lines in some orientable (or non-orientable) surface of genus γ. Let $\mathcal{S}_{d,\gamma}$ be the corresponding class. To prove that $\mathcal{S}_{d,\gamma} \in \mathtt{SQGC}(1)$ we need first to extend Proposition 6 for $\mathcal{S}_{d,\gamma}$ (which is not hard) and then use Theorem 1 and the fact that the class of graphs of bounded genus belongs in $\mathtt{SQGC}(1)$ (see e.g., [16]).

Of course, the main general question is to detect wide graph classes with the SQGM/SQGC property. In this direction, some insisting open issues are the following:

- Is the bound on the degree (or multi-degree) necessary? Are there classes of intersection graphs with unbounded or "almost bounded" maximum degree that have the SQGM/SQGC property?
- All so far known results classify graph classes in $\mathtt{SQGM}(1)$ or $\mathtt{SQGC}(1)$. Are there (interesting) graph classes in $\mathtt{SQGM}(c)$ or $\mathtt{SQGC}(c)$ for some $1 < c < 2$ that do not belong in $\mathtt{SQGM}(1)$ or $\mathtt{SQGC}(1)$ respectively? An easy (but trivial) example of such a class is the class $\mathcal{Q}_d$ of the q-dimensional grids, i.e., the cartesian products of $q \geq 2$ equal length paths. It is easy to see that the maximum k for which an n-vertex graph $G \in \mathcal{Q}_q$ contains a $(k \times k)$-grid as a minor is $k = \Theta(n^{\frac{1}{2}})$. On the other size, it can also be proven that $\mathbf{tw}(G) = \Theta(n^{\frac{q-1}{q}})$. These two facts together imply that $\mathcal{Q}_q \in \mathtt{SQGM}(2 - \frac{2}{q})$ while $\mathcal{Q}_q \notin \mathtt{SQGM}(2 - \frac{2}{q} - \epsilon)$ for every $\epsilon > 0$.
- Usually the graph classes in $\mathtt{SQGC}(1)$ are characterised by some "flatness" property. For instance, see the results in [29, 32, 32] for H-minor free graphs, where H is an apex graph. Can $\mathtt{SQGC}(1)$ be useful as an intuitive definition of the "flatness" concept? Does this have some geometric interpretation?

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A Proof of Lemma 2

Proof. By definition, since H is a c-size contraction of G, there is a mapping between each vertex of H and a connected set of at most c edges in G, so that by contracting these edge sets we obtain H from G. The endpoints of these edges form disjoint connected sets in G, implying a partition of the vertices of G into connected sets $\{V_x \mid x \in V(H)\}$, where $|V_x| \leq c+1$ for any vertex $x \in V(H)$.

Consider now a tree decomposition $(T, \mathcal{X})$ of H. We claim that the pair $(T, \mathcal{X}')$, where $X'_t := \bigcup_{x \in X_t} V_x$ for $t \in T$ is a tree decomposition of G. Clearly all vertices of G are included in some bag, since all vertices of H did. Every edge of G with both endpoints in the same part of the partition is in a bag, as each of these vertex sets is placed as a whole in the same bag. If e is an edge of G with endpoints in different parts of the partition, say V_x and V_y, then this implies that $\{x, y\} \in E(H)$. Thus, there is a node t of T for which $x, y \in X_t$ and therefore $e \subseteq X'_t$. Moreover, the continuity property remains unaffected, since for any vertex $x \in V(H)$ each vertex in V_x induces the same subtree in T that x did. ◀

B Proof of Claim 1

Proof of Claim 1. We first see that $\mathcal{S}$ is a Λ-state configuration of G. Condition 1 follows by the definition of X_q and Observation 1. Condition 2 follows directly by the definitions of W_q and X_q. For Condition 3, we first observe that, by the construction of Γ and the definition of $Z_{p,q}$, for any two pairs p, q and p', q' of pairwise linked elements of Q, the paths $Z_{p,q}$ and $Z_{p',q'}$ are internally vertex disjoined paths of Γ. It implies that $P_{p,q}$ and $P_{p',q'}$ can intersect each other only on the vertices of $W_p \cup W_q \cup W_{p'} \cup W_{q'}$. But $P_{p,q}$ (resp. $P_{p',q'}$), by construction contains only two vertices of $W_p \cup W_q \cup W_{p'} \cup W_{q'}$ that are the extremities of $P_{p,q}$, (resp. $P_{p',q'}$). So $P_{p,q}$ and $P_{p',q'}$ are internally vertex disjoined, as required. For Condition 4, assume that $\{W_p, W_q\} \in \mathcal{E}$. The fact that the endpoints of $\beta(\{W_p, W_q\})$ are in W_p and W_q follows directly by the definition of $\beta(\{W_p, W_q\}) = P_{p,q}$. It remains to prove that $V(\beta(\{W_p, W_q\})) \subseteq \alpha(W_p) \cup \alpha(W_q)$ or equivalently, that $V(P_{p,q}) \subseteq X_p \cup X_q$. Observe that, if both $p, q \in Q_{\text{in}}$, then every vertex in the shortest path $Z_{p,q}$ should be within distance c from either p or q. Similarly, if $p \in Q_{\text{in}}$ and $q = b_{\text{out}}$, then every vertex in the shortest path $Z_{p,q}$ should be within distance c from either p or some vertex in Q_{out}. So for every $p, q \in Q$, with $p \neq q$, $Z_{p,q} \subseteq N_p \cup N_q$. By Observation 2, every vertex in $\sigma^{-1}(V(Z_{p,q}))$ belongs to $X_p \cup X_q$ and the required follows as $V(P_{p,q}) \subseteq \sigma^{-1}(V(Z_{p,q}))$. This completes the proof that $\mathcal{S}$ is a Λ-state configuration of G.

We now prove that $\mathcal{S}$ is $\mathcal{A}$-normal. Recall that $\mathcal{A}$ be a c-diameter partition of G. Let C be a $\mathcal{A}$-cloud and let $C' = \sigma(C)$ be a subset of $V(\Gamma)$. As C is of diameter at most c, then, from Observation 3, C' is also of diameter at most c. Notice that if C intersects some member W of $\mathcal{W}$, then $C' = \sigma(C)$ also intersects $\sigma(W)$, therefore C' intersects some element of $Q_{\text{in}} \cup Q_{\text{out}}$. Assume C' contains $p \in Q_{\text{in}} \cup Q_{\text{out}}$, then $C' \subseteq N_p$. From Observation 1, $C \subseteq X_p = \alpha(W_p)$, therefore C satisfies Condition (A).

By construction, the distance in Γ between two elements of Q_{in} is either $2c+1$ or at least $4c+2$. The distance in Γ between on elements of Q_{in} and any element of Q_{out} is a multiple of $2c+1$. This implies that if $p, q \in Q$, $p \neq q$, $N_p \cap C' \neq \emptyset$, and $N_q \cap C' \neq \emptyset$, then p and q are linked.

By construction, if p and q are linked, then for every $r \in Q$ and every $u \in Z_{p,q}$, $\mathbf{dist}_\Gamma(r, u) \geq \min(\mathbf{dist}_\Gamma(r, p), \mathbf{dist}_G(r, q))$, where for every $x \in Q_{\text{in}}$, the quantity $\mathbf{dist}_\Gamma(x, b_{\text{out}})$ is interpreted as $\min\{\mathbf{dist}_\Gamma(x, q') \mid q' \in Q_{\text{out}}\}$. This implies that if C' intersects $Z_{p,q}$ for some $p, q \in Q$, then for every $r \in Q \setminus \{p, q\}$, then C' does not intersect N_r. We will use this fact in the next paragraph towards completing the proof of Condition (B).

We now claim that if C' intersects two distinct paths in $\{Z_{p,q} \mid (p, q) \in Q^2, p \neq q\}$, then C' intersects at most one of the sets in $\{N_{q'} \mid q' \in Q\}$. Let $Z_{p,q}$ and $Z_{p',q'}$ be two distinct paths intersected by C'. We argue first that p, q, p', q' cannot be all different. Indeed, if this is the case, as C' intersects $Z_{p,q}$ then C' cannot intersect $N_{p'}$ or $N_{q'}$ as $p', q' \notin \{p, q\}$. As $Z_{p',q'} \subseteq N_{q'} \cup N_{p'}$, we have a contradiction. Assume now that $p = p'$ and $q \neq q'$. As C' intersects $Z_{p,q}$, then it does not intersect N_r for any $r \in Q \setminus \{p, q\}$, and as it intersects $Z_{p,q'}$, then it does not intersect N_r for any $r \in Q \setminus \{p, q'\}$. We obtain that C' intersects at most one of the sets in $\{N_r \mid r \in Q\}$ that is N_p. By definition of the states, we obtain that C shadows at most one state that is X_p. That completes the proof of condition (B). ◇

C Proof of Claim 2

Proof of Claim 2. We first show that $\mathcal{S}'$ is an $\mathcal{A}$-normal Λ-state configuration of G. In each case, the construction of $\mathcal{S}'$ makes sure that $\mathcal{X}'$ is a connected packing of G and that the countries are updated in a way that their capitals remain inside them. Moreover, the highways are updated so to remain internally disjoint and inside the corresponding updated countries. We next prove that $\mathcal{S}'$ is $\mathcal{A}$-normal. Condition (A) is invariant as the cloud we take into consideration cannot intersect any $W \in \mathcal{W}$ and a cloud intersecting some capital $W \in \mathcal{W}$ cannot be disconnected from W. It now remains to prove condition (B). Because of Condition 4 of the definition of a Λ-state configuration, if a cloud C intersects a freeway, then it shadows at least one state. Now assume that a cloud C intersects two freeways in $\mathcal{S}'$, then by construction of $\mathcal{S}'$, it also intersects at least the two same freeways in $\mathcal{S}$. This along with the fact that $\mathcal{S}$ satisfies Condition (B), implies that $\mathcal{S}'$ satisfies condition (B) as well, as required.

Notice that, for any cloud $C^* \in \mathcal{A} \setminus \{C\}$, if C^* does not intersect a state X in $\mathcal{S}$, then the corresponding state X' in $\mathcal{S}'$, i.e., the state $X' = \alpha'(\alpha^{-1}(X))$, also does not intersect C^*. This means that $\mathsf{cost}(\mathcal{S}', \mathcal{A}) \leq \mathsf{cost}(\mathcal{S}, \mathcal{A})$.

Notice now that by the construction of $\mathcal{S}'$, C is not in $\mathsf{front}_{\mathcal{A}}(\mathcal{S}')$. In the case where $\mathsf{cov}_{\mathcal{S}}(C) \geq 1$ we have that $\mathsf{cost}(\mathcal{S}', \mathcal{A}) < \mathsf{cost}(\mathcal{S}, \mathcal{A})$.

Notice that the case where $\mathsf{cov}_{\mathcal{S}}(C) = 0$ happens only when $\mathsf{action} = \mathsf{anex}$ and there is an edge with one endpoint in C and one in some country X^* of $\mathcal{S}$ that does not intersect C. Moreover $\mathsf{cc}_G(X \setminus C, \alpha^{-1}(X)) = X$, for every state X of $\mathcal{S}$. This implies that $\mathsf{indep}(\mathcal{S}') \subseteq \mathsf{indep}(\mathcal{S})$. As $C \subseteq \mathsf{indep}(\mathcal{S})$ and $C \cap \mathsf{indep}(\mathcal{S}') = \emptyset$, we conclude that $|\mathsf{indep}(\mathcal{S}')| < |\mathsf{indep}(\mathcal{S})|$ as required. ◇

D Appendix

Proof of Theorem 1. Given a graph G, we define $\mathbf{bcg}(G)$ as the minimum k for which G can be contracted to the uniformly triangulated grid Γ_k. Let λ, c, c_1, and c_2 be integers. It is enough to prove that there exists an integer $\lambda' = \mathcal{O}(\lambda \cdot c_1 \cdot (c_2)^c)$ such that for every graph class $\mathcal{G} \in \mathtt{SQGC}(c)$,

$$\forall G \in \mathcal{G} \quad \mathbf{tw}(G) \leq \lambda \cdot (\mathbf{bcg}(G))^c \quad \Rightarrow$$
$$\forall G \in \mathcal{G}^{(c_1,c_2)} \quad \mathbf{tw}(G) \leq \lambda' \cdot (\mathbf{bcg}(G))^c.$$

Let $\mathcal{G} \in \mathtt{SQGC}(c)$ be a class of graph such that $\forall G \in \mathcal{G}$ $\mathbf{tw}(G) \leq \lambda \cdot (\mathbf{bcg}(G))^c$. Let $H \in \mathcal{G}^{(c_1,c_2)}$ and let G and J be two graphs such that $G \in \mathcal{G}$, $G \leq^{(c_1)} J$, and $H \leq^{c_2} J$. G

and J exist by definition of $\mathcal{G}^{(c_1,c_2)}$.

- By definition of H and J, $\mathbf{tw}(H) \leq \mathbf{tw}(J)$.
- By Lemma 2, $\mathbf{tw}(J) \leq (c_1 + 1)(\mathbf{tw}(G) + 1) - 1$.
- By definition of $\mathcal{G}$, $\mathbf{tw}(G) \leq \lambda \cdot \mathbf{bcg}(G)^c$.
- By Lemma 3, $\mathbf{bcg}(G) \leq (2c_2 + 1)(\mathbf{bcg}(H) + 2) + 1$.

If we combine these four statements, we obtain that

$$\mathbf{tw}(H) \leq (c_1 + 1)(\lambda \cdot [(2c_2 + 1)(\mathbf{bcg}(H) + 2) + 1]^c + 1) - 1.$$

As the formula is independent of the graph class, the Theorem 1 follows. ◀