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Structured Connectivity Augmentation*

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Abstract

We initiate the algorithmic study of the following “structured augmentation” question: is it possible to increase the connectivity of a given graph G by superposing it with another given graph H ? More precisely, graph F is the superposition of G and H with respect to injective mapping $\varphi: V(H) \rightarrow V(G)$ if every edge uv of F is either an edge of G , or $\varphi^{-1}(u)\varphi^{-1}(v)$ is an edge of H . Thus F contains both G and H as subgraphs, and the edge set of F is the union of the edge sets of G and $\varphi(H)$. We consider the following optimization problem. Given graphs G , H , and a weight function ω assigning non-negative weights to pairs of vertices of $V(G)$, the task is to find φ of minimum weight $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y))$ such that the edge connectivity of the superposition F of G and H with respect to φ is higher than the edge connectivity of G . Our main result is the following “dichotomy” complexity classification. We say that a class of graphs \mathcal{C} has *bounded vertex-cover number*, if there is a constant t depending on \mathcal{C} only such that the vertex-cover number of every graph from \mathcal{C} does not exceed t . We show that for every class of graphs \mathcal{C} with bounded vertex-cover number, the problems of superposing into a connected graph F and to 2-edge connected graph F , are solvable in polynomial time when $H \in \mathcal{C}$. On the other hand, for *any* hereditary class \mathcal{C} with unbounded vertex-cover number, both problems are NP-hard when $H \in \mathcal{C}$. For the unweighted variants of structured augmentation problems, i.e. the problems where the task is to identify whether there is a superposition of graphs of required connectivity, we provide necessary and sufficient combinatorial conditions on the existence of such superpositions. These conditions imply polynomial time algorithms solving the unweighted variants of the problems.

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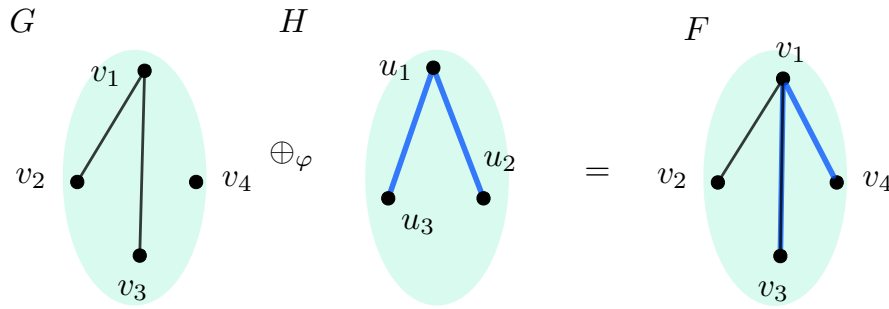
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1 Introduction

In connectivity augmentation problems, the input is a (multi) graph and the objective is to increase edge or vertex connectivity by adding the minimum number (weight) of additional edges, called links. This is a fundamental combinatorial problem with a number of important applications, we refer to the books of Nagamochi and Ibaraki [12] and Frank [6] for a detailed introduction to the topic. In this paper we initiate the study of a “structural” connectivity augmentation problem, where the set of additional edges should satisfy some additional

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■ **Figure 1** For injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $\varphi(u_1) = v_1$, $\varphi(u_2) = v_4$, and $\varphi(u_3) = v_3$, we have $F = G \oplus_{\varphi} H$.

constrains. For example, such constrains can be that all new edges should be visible from one vertex, i.e. the new set of edges forms a star, forms a cycle, or can be controlled from a small set of vertices, i.e. the graph formed by the additional edges has a small vertex cover.

It is convenient to model such an augmentation problem as a *graph superposition* problem. Let G and H be simple graphs (i.e. graphs without loops and multiple edges), $|V(G)| \geq |V(H)|$, and let $\varphi: V(H) \rightarrow V(G)$ be an injective mapping of the vertices of H to the set of vertices of $V(G)$. We say that a simple graph F is the *superposition of G and H with respect to φ* and write $F = G \oplus_{\varphi} H$ if $V(F) = V(G)$ and two distinct vertices $u, v \in V(F)$ are adjacent in F if and only if $uv \in E(G)$ or $u, v \in \varphi(V(H))$ and $\varphi^{-1}(u)\varphi^{-1}(v) \in E(H)$. See Fig. 1 for an example. Thus graph F contains G and H as subgraphs, and the edge set of F is the union of the edge sets of G and $\varphi(H)$.

We study the algorithmic problem of increasing the edge-connectivity of graph G by superposing it with a graph H . We are interested in the weighted variant of the problem, where for every pair of vertices v and u of G , mapping the endpoints of an edge of H to u and v has a specified weight $\omega(uv)$. We consider the following problem.

STRUCTURED CONNECTIVITY AUGMENTATION

Input: Graphs G and H , a weight function $\omega: \binom{V(G)}{2} \rightarrow \mathbb{N}_0$, and a nonnegative integer W .

Task: Decide whether there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that graph $F = G \oplus_{\varphi} H$ is connected and the *weight* of the mapping $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y)) \leq W$.

We also study the problem of obtaining a 2-edge connected graph F by superposing graphs G and H . More precisely, we consider the following problem.

STRUCTURED 2-CONNECTIVITY AUGMENTATION

Input: Connected graph G and a graph H , a weight function $\omega: \binom{V(G)}{2} \rightarrow \mathbb{N}_0$ and a nonnegative integer W .

Task: Decide whether there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ of weight at most W such that $F = G \oplus_{\varphi} H$ is 2-edge connected.

Our results. Our main result is the following “dichotomy” complexity classification of structured augmentation problems. We say that a class of graphs \mathcal{C} has *bounded vertex-cover number*, if there is a constant t depending on \mathcal{C} only such that the vertex-cover number of every graph from \mathcal{C} does not exceed t . We show that for every class of graphs \mathcal{C} with bounded

vertex-cover number, STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION are solvable in polynomial time when $H \in \mathcal{C}$. We complement this result by showing that for *any* hereditary class \mathcal{C} with unbounded vertex-cover number, both problems are NP-complete when $H \in \mathcal{C}$. Thus for any hereditary class \mathcal{C} both problems with $H \in \mathcal{C}$ are NP-complete if and only if \mathcal{C} has unbounded vertex-cover number.

The running times of our algorithms solving STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION are of the form $|V(G)|^{\mathcal{O}(f(t))} \cdot \log W$, where f is some function and t is the vertex cover of H . Thus our algorithms are not fixed-parameter tractable when t is the parameter. We show that from the perspective of parameterized complexity, this situation is unavoidable. More precisely, we show that both problems are $W[1]$ -hard when parameterized by t . We refer to the book of Downey and Fellows [2] for an introduction to parameterized complexity.

We also consider the unweighted variants of STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION. In these cases, the weight is $\omega(uv) = 0$ for every pair of vertices of G and $W = 0$. The task is to identify whether there is a superposition of graphs G and H of edge connectivity 1 or 2, correspondingly. Here we obtain necessary and sufficient combinatorial conditions of the existence of an injective function φ such that $F = G \oplus_{\varphi} H$ is edge k -connected provided that G is edge $(k - 1)$ -connected, $k = 1, 2$. These conditions imply polynomial time algorithms solving the unweighted variants of the problems.

Due to space constraints some proof are either just sketched or omitted in this extended abstract. The full details are available in [4].

Related work. The problem of increasing graph connectivity by adding additional edges is the classic and well-studied problem. It was first studied by Eswaran and Tarjan [3] and Plesnik [13] who showed that increasing the edge connectivity of a given graph to 2 by adding minimum number of additional augmenting edges is polynomial time solvable. Subsequent work in [14, 5] showed that this problem is also polynomial time solvable for any given target value of edge connectivity to be achieved. However, if the set of augmenting edges is restricted, that is, there are pairs of vertices in the graph which do not constitute a new edge, or if the augmenting edges have (non-identical) weights on them, then the problem of computing the minimum size (or weight) augmenting set is NP-complete [3]. Augmentation problems with constraints like simplicity-preserving augmentations, augmentations with partition constraints, or planarity requirements can be found in the literature, see the book of Nagamochi and Ibaraki [12] for further references.

Strongly relevant to structural augmentation is the MINIMUM STAR AUGMENTATION problem, see e.g. [12, Section 3.3.3] and [10]. Here one wants to increase the edge-connectivity of a given graph by adding a new vertices and connecting it with a small number of edges to the remaining vertices of the graph. In our setting this corresponds to the case of graph G having an isolate vertex, and graph H being a star (a tree with vertex-cover number 1). Tibor and Szigeti [10] studied a generalization of this problem where one wants to make a graph edge r -connected by attaching p stars of specified degrees. In particular, they provided combinatorial conditions which are necessary and sufficient for such an augmentation. Again, this problem can be seen as a special case of structural augmentation, where graph G has p isolated vertices and graph H is the union of stars of specified degrees.

2 Preliminaries

We consider only finite undirected graphs. For a graph G , $\binom{V(G)}{2}$ denotes the set of unordered pairs of distinct vertices of G . For uniformity, we denote the elements of $\binom{V(G)}{2}$ in the same way as edges, i.e., we write $uv \in \binom{V(G)}{2}$. A subgraph H of G is *spanning* if $V(H) = V(G)$. For a graph G and a subset $U \subseteq V(G)$ of vertices, we write $G[U]$ to denote the subgraph of G induced by U . We write $G - U$ to denote the graph $G[V(G) \setminus U]$. Let $S \subseteq E(G)$ for a graph G . By $G - S$ we denote by $G - S$ the graph obtained by the deletion of the edges of S . We write $G - e$ instead of $G - \{e\}$ for an edge e . For a vertex v , we denote by $N_G(v)$ the (*open*) *neighborhood* of v , i.e., the set of vertices that are adjacent to v in G . Two nonadjacent vertices u and v are (*false*) *twins* if $N_G(u) = N_G(v)$. A set of edges with pairwise distinct end-vertices is called a *matching*. A matching M is *induced* if the end-vertices of M are pairwise nonadjacent. A vertex v is *saturated* in a matching M if v is incident to an edge of M . We say that the disjoint union of copies of K_2 is a *matching graph*. A graph class \mathcal{C} is said to be *hereditary* if for every $G \in \mathcal{C}$ and every induced subgraph H of G , $H \in \mathcal{C}$. A set of vertices $X \subseteq V(G)$ is a *vertex cover* of a graph G if every edge of G has at least one of its end-vertices in X . The minimum size of a vertex cover is called the *vertex-cover number* of G and is denoted by $\beta(G)$.

Let k be a positive integer. A graph G is (*edge*) k -*connected* if for every $S \subseteq E(G)$ with $|S| \leq k - 1$, $G - S$ is connected. Since we consider only edge connectivity, whenever we say that a graph G is k -connected, we mean that G is edge k -connected. We assume that every graph is 0-connected. A set of edges $S \subseteq E(G)$ of a connected graph G is an *edge separator* if $G - S$ is disconnected. An edge e of a connected graph G is a *bridge* if $\{e\}$ is a separator. Clearly, a connected graph is 2-connected if and only if it has no bridge. Let B be the set of bridges of a connected graph G . We call a component of $G - B$ a *biconnected component* of G . In other words, a biconnected component is an inclusion-wise maximal induced 2-connected subgraph of G . We say that a biconnected component L of a graph G is a *pendant biconnected component* (or simply a *pendant*) if a unique bridge of G is incident to $V(L)$. A biconnected component is *trivial* if it has a single vertex. For a graph G , we denote by $c(G)$ the number a components of G , and for a connected graph G , $p(G)$ is the number of pendants. We also denote by $i(G)$ the number of isolated vertices of G .

Let S be an inclusion-wise minimal edge separator of a connected graph G . Then $G - S$ has exactly two components C_1 and C_2 . Let G be a spanning subgraph of F . We say that an edge $e \in E(F) \setminus E(G)$ *covers* a minimal separator S of G if e has its end-vertices in C_1 and C_2 . The following observation about separators is useful.

► **Observation 1.** *Let $k \geq 2$ be an integer and let a $(k - 1)$ -connected graph G be a spanning subgraph of F . Then F is k -connected if and only if for each edge separator S of G with $|S| = k - 1$, F has an edge that covers it.*

We also need some additional terminology and folklore observations for the augmentation of a connected graph to a 2-connected graph. Let G be a connected graph and let x and y be distinct vertices of G . We say that a bridge uv of G *belongs* to an (x, y) -path P if $uv \in E(P)$. Similarly, a biconnected component Q is *crossed* by P if $V(Q) \cap V(P) \neq \emptyset$. The following observation show that the choice of an (x, y) -path is irrelevant if the biconnected components containing the end-vertices are given.

► **Observation 2.** *Let distinct $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be pairs of distinct vertices of a connected graph G such that x_1, x_2 are in the same biconnected component of G and, similarly, y_1, y_2*

are in the same biconnected component of G . Let also P_1 and P_2 be (x_1, y_1) and (x_2, y_2) -paths respectively. Then the following holds:

- a bridge uv of G belongs to P_1 if and only if uv belongs to P_2 ,
- a biconnected component Q is crossed by P_1 if and only if Q is crossed by P_2 .

► **Observation 3.** Let u and v be distinct nonadjacent vertices of a connected graph G and let F be a graph obtained from G by the addition of the edge uv . Then uv covers all bridges that belongs to a (u, v) -path P in G , and for the biconnected components Q_1, \dots, Q_s that are crossed by P , $F[V(Q_1) \cup \dots \cup V(Q_s)]$ is a biconnected component of F .

In the remaining part of the paper, we will be always assuming that in the instance of the structured augmentation problem, we have

- (i) $|V(H)| \leq |V(G)|$;
- (ii) Graph H has no isolated vertices.

Indeed, if $|V(H)| > |V(G)|$, then there is no superposition of G and H , and thus such an instance is a no-instance. For (ii), it is sufficient to observe that mapping of isolated vertices of H to vertices of G does not influence the connectivity of the superposition. Another technical detail should be mentioned here. In Theorems 5 and 7, we evaluate the running times of algorithms as a function of $|V(G)|$ and the vertex cover number of H . In order to do this, we should be able to recognize within this time the (trivial) no-instances, where $|V(H)| > |V(G)|$. We can verify this condition in time $|V(G)|^{\mathcal{O}(1)}$ just by refuting the instances of size more than $|V(G)|^{\mathcal{O}(1)}$ after reading the first $|V(G)|^{\mathcal{O}(1)}$ bits.

3 Augmenting by graphs with small vertex cover

In this section we consider the situation when graph H is from a graph class \mathcal{C} with bounded vertex-cover number. In Subsection 3.1 we show that in this case STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION are solvable in polynomial time. In Subsection 3.2 we show that this condition is tight by proving that for any hereditary graph class \mathcal{C} with unbounded vertex-cover number, both problems are NP-hard. Due to space restrictions, we only sketch our results.

3.1 Algorithms

We start with a solution for STRUCTURED CONNECTIVITY AUGMENTATION, which is simpler than the solution for STRUCTURED 2-CONNECTIVITY AUGMENTATION.

Structured Connectivity Augmentation. We need the following lemma.

► **Lemma 4.** Let G and H be graphs and let $\varphi: V(H) \rightarrow V(G)$ be an injection such that $F = G \oplus_{\varphi} H$ is connected. Let also X be a vertex cover of H of size t . Then there is a set $Y \subseteq V(H) \setminus X$ of size at most $2(t-1)$ such that for graph $H' = H[X \cup Y]$ and mapping $\psi = \varphi|_{X \cup Y}$, the vertices of $\psi(X \cup Y)$ are in the same connected component of $F' = G \oplus_{\psi} H'$.

Let us remind, that, given a positive integer t , a graph class \mathcal{C} has vertex-cover number at most t if every graph $H \in \mathcal{C}$ has a vertex cover of size at most t . We are ready to prove the main theorem about STRUCTURED CONNECTIVITY AUGMENTATION.

► **Theorem 5.** Let t be a positive integer and \mathcal{C} be a graph class of vertex-cover number at most t . Then for any $H \in \mathcal{C}$, STRUCTURED CONNECTIVITY AUGMENTATION is solvable in time $|V(G)|^{\mathcal{O}(t)} \cdot \log W$.

Sketch of the proof. Let G and $H \in \mathcal{C}$ be graphs and let $\omega: \binom{V(G)}{2} \rightarrow \mathbb{N}_0$ be a weight function. We show that we can find in time $|V(G)|^{\mathcal{O}(t)} \cdot \log W$ an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is connected and $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y))$ is minimum if φ exists.

Let us remind that without loss of generality, we can assume that $|V(H)| \leq |V(G)|$ and H has no isolated vertices.

We start from finding a vertex cover X of size at most t in H . Since we aim for an algorithm with running time $|V(G)|^{\mathcal{O}(t)} \cdot \log W$, vertex cover X can be found by brute-force checking of all subsets of $V(H)$ of size at most t . If we fail to find X of size at most t , it means that $H \notin \mathcal{C}$, in this case we return the answer NO and stop. Assume that X exists.

Suppose that there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is connected and assume that for φ , $\omega(\varphi)$ is minimum. By Lemma 4, there is a set $Y \subseteq V(H) \setminus X$ of size at most $2(t-1)$ such that for $H' = H[X \cup Y]$ and $\psi = \varphi|_{X \cup Y}$, the vertices of $\psi(X \cup Y)$ are in the same component of $F' = G \oplus_{\psi} H'$. Considering all possibilities, we guess Y in time $|V(H)|^{\mathcal{O}(t)}$.

Now we consider all possible injective mapping $\psi: X \cup Y \rightarrow V(G)$ such that the vertices of $\psi(X \cup Y)$ are in the same connected component of $F' = G \oplus_{\psi} H'$, where $H' = H[X \cup Y]$. Notice that there are at most $|V(G)|^{3t-2}$ such mappings that can be generated in time $|V(G)|^{\mathcal{O}(t)}$. If we fail to find ψ , we reject the current choice of Y . Otherwise, for every ψ , we try to extend it to an injection $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is connected, and among all extensions we choose one that provides the minimum weight $\omega(\varphi)$.

Let $Z = V(H) \setminus (X \cup Y)$. The vertices of $\psi(X \cup Y)$ are in the same component of F' . Denote this component by F_0 and denote by F_1, \dots, F_r the other components of this graphs. Recall that Z is an independent set of H and each vertex of Z has an incident edge with one endpoint in X . It follows that for an injection $\varphi: V(H) \rightarrow V(G)$ such that $\psi = \varphi|_{X \cup Y}$, $F = G \oplus_{\varphi} H$ is connected if and only if for every $i \in \{1, \dots, r\}$, there is $v \in V(F_i)$ such that $v \in \varphi(Z)$. Hence, if $r > |Z|$, we cannot extend ψ . In this case we discard the current choice of ψ .

Assume from now that Y and ψ are fixed, $F' = G \oplus_{\psi} H'$ is connected and $r \leq |Z|$. For $z \in Z$ and $v \in V(G) \setminus \psi(X \cup Y)$, we define the weight of mapping z to v as

$$w(z, v) = \sum_{u \in N_G(v) \cap \psi(N_H(z))} \omega(uv),$$

that is, $w(z, v)$ is the weight of edges that is added to the weight of mapping if we decide to extend ψ by mapping z to v . Let $W = \max\{w(z, v) \mid z \in Z, v \in V(G) \setminus \psi(X \cup Y)\} + 1$. We construct the weighted auxiliary bipartite graph \mathcal{G} with the bipartition (A, B) of its vertex set and the weight function $f: E(\mathcal{G}) \rightarrow \mathbb{N}_0$ as follows.

- Set $A = (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_r) = V(G) \setminus \psi(X \cup Y)$.
- Construct a set of vertices S_0 of size $|V(F_0)| - |X \cup Y|$ and sets S_i of size $|V(F_i)| - 1$ for $i \in \{1, \dots, r\}$.
- Set $B = Z \cup S_0 \cup \dots \cup S_r$.
- For each $z \in Z$ and $v \in A$, construct an edge zv and set $f(zv) = w(z, v)$.
- For each $u \in S_0$ and $v \in V(F_0) \setminus \psi(X \cup Y)$, construct an edge uv and set $f(uv) = W$.
- For each $i \in \{1, \dots, r\}$, do the following: for each $u \in S_i$ and $v \in V(F_i)$, construct an edge uv and set $f(uv) = W$.

We find a matching M in \mathcal{G} that saturates every vertex of A and has the minimum weight using the Hungarian algorithm [7, 11] in time $\mathcal{O}(|V(G)|^3 \cdot \log W)$.

Observe that a matching that saturates every vertex of A exists, because $r \leq Z$. We can construct such a matching by selecting one vertex in $V(F_i)$ for each $i \in \{1, \dots, r\}$ and matching it with a vertex of Z . Then we complement this set of edges to a matching saturating A by adding edges incident to $S_0 \cup \dots \cup S_r$. For the matching M that has minimum weight, we can also observe the following.

First, note that

- every vertex of Z is saturated by M . (1)

Indeed, targeting towards a contradiction, assume that $z \in Z$ is not saturated. Since $|V(H)| \leq |V(G)|$, there is $uv \in M$ such that $u \in S_0 \cup \dots \cup S_r$ and $v \in A$. We replace uv by zv in M . Because $f(uv) = W > w(zv)$, we obtain a matching with a smaller weight. This contradicts the choice of M .

Next, we claim that

- there is $zv \in M$ such that $z \in Z$ and $v \in V(F_i)$. (2)

Indeed, this is because the vertices of $V(F_i)$ are adjacent to $|V(F_i)| - 1$ vertices of S_i and all other their neighbors are in Z .

Finally, we have that among all matching saturating A , M is a matching satisfying (1) and (2) such that for $M' = \{zv \in M \mid z \in Z\}$, $f(M')$ is minimum. To see it, observe that $f(uv) = W$ for $uv \in M \setminus M'$. Hence, $f(M \setminus M') = (|A| - |Z|)W$, because $|M \setminus M'| = |A| - |Z|$ by (1). Therefore, $f(M') = f(M) - f(M \setminus M') = f(M) - (|A| - |Z|)W$.

For every $z \in Z$, we define $\varphi(z) = v$, where $zv \in M'$ and $\varphi(x) = \psi(x)$ for $x \in X \cup Y$. Clearly, φ is an extension of ψ . By (1), φ is an injective mapping of $V(H)$ to $V(G)$. By (2) and the choice of X and Y , we obtain that $G \oplus_\varphi H$ is connected. We claim that φ is an extension of ψ such that $F = G \oplus_\varphi H$ is connected that has the minimum total weight $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y))$.

Recall that we try all possible choices of Y and for every choice of Y , we consider all possible choices of ψ . If we fail to find an injection $\varphi: V(H) \rightarrow V(G)$ such that φ is an extension of ψ and $F = G \oplus_\varphi H$ is connected we return the answer NO. Otherwise, we return φ that provides the minimum weight.

To complete the proof, observe that the total running time of the algorithm is $|V(G)|^{\mathcal{O}(t)} \cdot \log W$. ◀

Structured 2-Connectivity Augmentation. As it could be expected, the algorithm for STRUCTURED 2-CONNECTIVITY AUGMENTATION is more technical. We start with a lemma, which is similar to Lemma 4. We show it by making use of Observations 1 and 3.

► **Lemma 6.** *Let G and H be graphs such that G is connected, and let $\varphi: V(H) \rightarrow V(G)$ be an injection such that $F = G \oplus_\varphi H$ is connected. Suppose that X is a vertex cover of H and $t = |X|$. Then there is a set $Y \subseteq V(H) \setminus X$ of size at most $2(t - 1)$ such that for $H' = H[X \cup Y]$ and $\psi = \varphi|_{X \cup Y}$, the vertices of $\psi(X \cup Y)$ are in the same biconnected component of $F' = G \oplus_\psi H'$.*

► **Theorem 7.** *Let t be a positive integer and \mathcal{C} be a graph class of vertex-cover number at most t . Then for any $H \in \mathcal{C}$, STRUCTURED 2-CONNECTIVITY AUGMENTATION is solvable in time $|V(G)|^{\mathcal{O}(2^t)} \log W$.*

Sketch of the proof. Let G and H be graphs such that G is connected and $H \in \mathcal{C}$. Let $\omega: \binom{V(G)}{2} \rightarrow \mathbb{N}_0$ be a weight function. Similarly to the proof of Theorem 5 we show that we can find in time $|V(G)|^{\mathcal{O}(2^t)} \cdot \log W$ the minimum value of $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y))$

for an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is connected if such a mapping φ exists.

The first steps of our algorithm are the same as in the proof of Theorem 5. Again, we remind that $|V(H)| \leq |V(G)|$ and that H has no isolated vertices.

Next, we find a vertex cover X of minimum size in H of size at most t in time $|V(G)|^{\mathcal{O}(t)}$. If we fail to find X of size at most t , then $H \notin \mathcal{C}$. We return NO and stop. From now on we assume that X exists.

Suppose that there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is 2-connected and assume that for φ , $\omega(\varphi)$ is minimum. By Lemma 6, there is a set $Y \subseteq V(H) \setminus X$ of size at most $2(t-1)$ such that for $H' = H[X \cup Y]$ and $\psi = \varphi|_{X \cup Y}$, the vertices of $\psi(X \cup Y)$ are in the same biconnected component of $F' = G \oplus_{\psi} H'$. Considering all possibilities, we guess Y in time $|V(H)|^{\mathcal{O}(t)}$.

Now we consider all possible injective mapping $\psi: X \cup Y \rightarrow V(G)$ such that the vertices of $\psi(X \cup Y)$ are in the same biconnected component of $F' = G \oplus_{\psi} H'$ where $H' = H[X \cup Y]$. Notice that there at most $|V(G)|^{3t-2}$ such mappings that can be generated in time $|V(G)|^{\mathcal{O}(t)}$. If we fail to find ψ , we reject the current choice of Y . Otherwise, for every ψ , we try to extend it to an injection $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is 2-connected, and among all extensions we choose one that provides the minimum weight $\omega(\varphi)$.

Let $Z = V(H) \setminus (X \cup Y)$. The vertices of $\psi(X \cup Y)$ are in the same biconnected component of F' . Denote this biconnected component by F_0 and denote by F_1, \dots, F_r the pendant biconnected components of F' that are distinct from F_0 . Recall that Z is an independent set of H and each vertex of Z has an incident edge with one endpoint in X . By Observation 1, we obtain the following crucial property.

For an injection $\varphi: V(H) \rightarrow V(G)$ such that $\psi = \varphi|_{X \cup Y}$, $F = G \oplus_{\varphi} H$ is 2-connected if and only if

- (i) for every $i \in \{1, \dots, r\}$, there is $v \in V(F_i)$ such that $v \in \varphi(Z)$, and
- (ii) if v is the unique element of $V(F_i) \cap \varphi(Z)$ and v is incident to a bridge vu of G , then there is $x \in X$ such that $\varphi(x) \neq u$ and x is adjacent to $\varphi^{-1}(v)$ in H .

Similarly to the proof of Theorem 5, we solve auxiliary matching problems to find the minimum weight of φ but now, due the condition (ii), the algorithm becomes more complicated and we are using dynamic programming.

For $z \in Z$ and $v \in V(G) \setminus \psi(X \cup Y)$, we define the weight of mapping z to v as

$$w(z, v) = \sum_{u \in N_G(v) \cap \psi(N_H(z))} \omega(uv), \quad (3)$$

that is, $w(z, v)$ is the weight of edges that is added to the weight of mapping if we decide to extend ψ by mapping z to v . Our aim is to find the extension φ of ψ that satisfies (i) and (ii) such that the total weight of the mapping of the vertices of Z to vertices of $V(G) \setminus \psi(X \cup Y)$ by φ is minimum.

Since X is a vertex cover of H of size t , the set Z can be partitioned into $s \leq 2^t$ classes of false twins Z_1, \dots, Z_s . Let $p_i = |Z_i|$ for $i \in \{1, \dots, s\}$. We exploit the following property of false twins in Z : if $x, y \in Z_i$, then $w(x, v) = w(y, v)$ for $v \in V(G) \setminus \psi(X \cup Y)$.

For each s -tuple of integers (q_1, \dots, q_s) such that $0 \leq q_i \leq p_i$, for $i \in \{1, \dots, s\}$ and each $h \in \{0, \dots, r\}$, we define

$$\alpha_h(q_1, \dots, q_s) = \min_{\xi} \sum_{z \in Z'} w(z, \xi(z)), \quad (4)$$

where $Z' \subseteq Z$ such that $|Z' \cap Z_i| = q_i$ for $i \in \{1, \dots, s\}$ and the minimum is taken over all

injective mappings $\xi: Z' \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_h)$ such that the following conditions are satisfied:

- (a) for every $i \in \{1, \dots, h\}$, there is $v \in V(F_i)$ such that $v \in \xi(Z')$, and
- (b) if v is a unique element of $V(F_i) \cap \xi(Z')$ for some $i \in \{1, \dots, h\}$ and v is incident to a bridge vu of G , then there is $x \in X$ such that $\psi(x) \neq u$ and x is adjacent to $\xi^{-1}(v)$ in H .

If such a mapping ξ does not exist, then we assume that $\alpha_h(q_1, \dots, q_s) = +\infty$. Recall that if $x, y \in Z_i$, then $w(x, v) = w(y, v)$ for $v \in V(G) \setminus \psi(X \cup Y)$. It implies that the function $\alpha_h(q_1, \dots, q_s)$ depends only on the values of q_1, \dots, q_s .

We claim that computing $\alpha_r(p_1, \dots, p_s)$ is equivalent to finding an extension φ of ψ of minimum weight such that $F = G \oplus_{\varphi} H$ is 2-connected.

Assume that $\alpha_r(p_1, \dots, p_s) < +\infty$. Notice that $Z' = Z$ if $q_i = p_i$ for $i \in \{1, \dots, s\}$. Let $\xi: Z \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_h)$ be an injection that provides the minimum in (4), that is, $\alpha_r(p_1, \dots, p_s) = \sum_{z \in Z} w(z, \xi(z))$. We define $\varphi(z) = \xi(z)$ for $z \in Z$ and $\varphi(x) = \psi(x)$ for $x \in X \cup Y$. Clearly, φ is an extension of ψ . Because ξ is an injection, we have that φ is an injective mapping. Since ξ satisfies (a) and (b), we obtain that φ satisfies (i) and (ii) and, therefore, $F = G \oplus_{\varphi} H$ is 2-connected. Let $R = \sum_{xy \in E(H), x, y \in X \cup Y} \omega(\psi(x)\psi(y))$. Then using (3), we have that

$$\begin{aligned} \omega(\varphi) &= \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y)) \\ &= \sum_{xy \in E(H), x, y \in X \cup Y} \omega(\varphi(x)\varphi(y)) + \sum_{xz \in E(H), x \in X, z \in Z} \omega(\varphi(x)\varphi(y)) \\ &= R + \sum_{z \in Z} w(z, \varphi(z)) = R + \sum_{z \in Z} w(z, \xi(z)) = R + \alpha_r(p_1, \dots, p_s). \end{aligned} \tag{5}$$

Let $\varphi': V(H) \rightarrow V(G)$ be an injection that extends ψ such that $F' = G \oplus_{\varphi'} H$ is 2-connected. We define $\xi': Z \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_h)$ by setting $\xi'(z) = \varphi'(z)$ for $z \in Z$. Since φ' is an injection, ξ' is also an injection. Because F' is 2-connected, φ' satisfies (i) and (ii). This implies that ξ' satisfies (a) and (b). Therefore, $\sum_{z \in Z} w(z, \xi'(z)) \geq \alpha_r(p_1, \dots, p_s)$. Similarly to (5), we have that $\omega(\varphi') = R + \sum_{z \in Z} w(z, \xi'(z)) \geq R + \alpha_r(p_1, \dots, p_s)$. We conclude that φ is an extension φ of ψ of minimum weight such that $F = G \oplus_{\varphi} H$ is 2-connected.

Suppose that $\alpha_r(p_1, \dots, p_s) = +\infty$. It implies that there is no injection $\xi: Z \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_h)$ satisfying (a) and (b). But this immediately implies that there is no injective extension φ of ψ satisfying (i) and (ii). This completes the proof of the claim.

We use dynamic programming to compute α_h consequently for $h = 0, 1, \dots, r$.

We start with computing $\alpha_0(q_1, \dots, q_s)$ for each s -tuple (q_1, \dots, q_s) . Notice that the conditions (a) and (b) are irrelevant in this case, because they concern only $h \geq 1$. We construct the auxiliary complete bipartite graph \mathcal{G}_0 with the bipartition $(V(F_0) \setminus \psi(X \cup Y), Z')$ of its vertex set and define the weight of each edge zv for $z \in Z'$ and $v \in V(F_0) \setminus \psi(X \cup Y)$ as $w(z, v)$. We find a matching M in \mathcal{G}_0 that saturates every vertex of Z' and has the minimum weight using the Hungarian algorithm [7, 11] in time $\mathcal{O}(|V(G)|^3 \cdot \log W)$. If there is no matching saturating Z' , we set $\alpha_0(q_1, \dots, q_s) = +\infty$. Otherwise, $\alpha_0(q_1, \dots, q_s) = w(M)$. It is straightforward to verify the correctness of computing $\alpha_0(q_1, \dots, q_s)$ by the definition of this function.

Assume that $h \geq 1$ and we already computed the table of values of $\alpha_{h-1}(q_1, \dots, q_s)$. We explain how to construct the table of values of $\alpha_h(q_1, \dots, q_s)$. The computation is based on

the observation that we can see an injective mapping $\xi: Z' \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_h)$ as the union of two injections $\xi': Z'' \rightarrow (V(F_0) \setminus \psi(X \cup Y)) \cup V(F_1) \cup \dots \cup V(F_{h-1})$ and $\lambda: Z''' \rightarrow V(F_h)$ for the appropriate partition (Z'', Z''') of Z' .

For each s -tuple of integers (q_1, \dots, q_s) such that $0 \leq q_i \leq p_i$ for $i \in \{1, \dots, s\}$, we define

$$\alpha'_h(q_1, \dots, q_s) = \min_{\lambda} \sum_{z \in Z'} w(z, \xi(z)), \quad (6)$$

where $Z' \subseteq Z$ such that $|Z' \cap Z_i| = q_i$ for $i \in \{1, \dots, s\}$ and the minimum is taken over all injective mappings $\lambda: Z' \rightarrow V(F_h)$ such that the following conditions are fulfilled:

- (a*) there is $v \in V(F_h)$ such that $v \in \lambda(Z')$, and
- (b*) if v is the unique element of $V(F_h) \cap \lambda(Z')$ and v is incident to a bridge vu of G , then there is $x \in X$ such that $\psi(x) \neq u$ and x is adjacent to $\lambda^{-1}(v)$ in H .

If such a mapping λ does not exist, then we assume that $\alpha'_h(q_1, \dots, q_s) = +\infty$. As for $\alpha_h(q_1, \dots, q_s)$, $\alpha'_h(q_1, \dots, q_s)$ depends only on the values of q_1, \dots, q_s , because if $x, y \in Z_i$, then $w(x, v) = w(y, v)$ for $v \in V(G) \setminus \psi(X \cup Y)$.

Let uv be the unique bridge of G with $v \in V(F_h)$. Suppose that for an s -tuple (q_1, \dots, q_s) , we obtain that $|Z'| = 1$ and for the unique vertex $z \in Z'$, z has a unique neighbor $x \in X$ in H and $\psi(x) = u$. Then we set $\alpha'_h(q_1, \dots, q_s) = +\infty$ if $|V(F_h)| = 1$ and $\alpha'_h(q_1, \dots, q_s) = \min\{w(zv') \mid v' \in V(F_h) \setminus \{v\}\}$ otherwise. For other s -tuples (q_1, \dots, q_s) , we compute $\alpha'_h(q_1, \dots, q_s)$ as follows. We construct the auxiliary complete bipartite graph \mathcal{G}_h with the bipartition $(V(F_h), Z')$ of its vertex set and define the weight of each edge zv for $z \in Z'$ and $v \in V(F_0) \setminus \psi(X \cup Y)$ as $w(zv)$. We find a matching M in \mathcal{G}_h that saturates every vertex of Z' and has the minimum weight using the Hungarian algorithm [7, 11] in time $\mathcal{O}(|V(G)|^3 \cdot \log W)$. If there is no matching saturating Z' , we set $\alpha'_h(q_1, \dots, q_s) = +\infty$. Otherwise, $\alpha'_h(q_1, \dots, q_s) = w(M)$. It is again straightforward to verify the correctness of computing $\alpha'_h(q_1, \dots, q_s)$ using the definition of this function.

Now, to compute $\alpha_h(q_1, \dots, q_s)$, we use the equation:

$$\alpha_h(q_1, \dots, q_s) = \min\{\alpha_{h-1}(q'_1, \dots, q'_s) + \alpha'_h(q''_1, \dots, q''_s)\}, \quad (7)$$

where the minimum is taken over all s -tuples (q'_1, \dots, q'_s) and (q''_1, \dots, q''_s) such that $q_i = q'_i + q''_i$ for $i \in \{1, \dots, s\}$.

To evaluate the running time, observe that there are at most $|V(G)|^s$ s -tuples (q_1, \dots, q_s) . Since $s \leq 2^t$, it implies that the table of values of $\alpha_0(q_1, \dots, q_s)$ can be computed in time $|V(G)|^{\mathcal{O}(2^t)} \cdot \log W$. Similarly, the table of values of $\alpha'_h(q_1, \dots, q_s)$ for each $h \in \{1, \dots, r\}$ can be computed in the same time. To compute $\alpha_h(q_1, \dots, q_s)$ for a given s -tuple (q_1, \dots, q_s) using (7), we have to consider at most $|V(G)|^s$ pairs of s -tuples (q'_1, \dots, q'_s) and (q''_1, \dots, q''_s) . Hence, we can compute the table of values $\alpha_h(q_1, \dots, q_s)$ from the tables of values of $\alpha_{h-1}(q_1, \dots, q_s)$ and $\alpha'_h(q_1, \dots, q_s)$ in time $|V(G)|^{\mathcal{O}(2^t)} \cdot \log W$ for each $h \in \{1, \dots, r\}$. We conclude that the total running time is $|V(G)|^{\mathcal{O}(2^t)} \cdot \log W$. \blacktriangleleft

3.2 Hardness of structured augmentation

In this section we show that Theorems 5 and 7 are tight in the sense that if the vertex-cover number of graphs in a hereditary graph class \mathcal{C} is unbounded, then both structured augmentation problems are NP-complete. Our hardness proof actually holds for any k -edge connectivity augmentation. For a positive integer k , we define the following problem:

STRUCTURED k -CONNECTIVITY AUGMENTATION

Input: Graphs G and H such that G is edge $(k-1)$ -connected, a weight function $\omega: \binom{V(G)}{2} \rightarrow \mathbb{N}_0$ and a nonnegative integer W .

Task: Decide whether there is an injective $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is edge k -connected and the *weight* of the mapping $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y)) \leq W$.

Let us note that for $k = 1$ this is STRUCTURED CONNECTIVITY AUGMENTATION and for $k = 2$ this is STRUCTURED 2-CONNECTIVITY AUGMENTATION.

► **Theorem 8.** *Let k be a positive integer. Let also \mathcal{C} be a hereditary graph class. Then if the vertex-cover number of \mathcal{C} is unbounded, then STRUCTURED k -CONNECTIVITY AUGMENTATION is NP-complete for $H \in \mathcal{C}$ in the strong sense.*

Also we observe that it is unlikely that we can avoid the dependency on t in the exponents of polynomial bounding the running time when solving STRUCTURED k -CONNECTIVITY AUGMENTATION for H with $\beta(H) \leq t$.

► **Proposition 9.** *For every positive integer k , STRUCTURED k -CONNECTIVITY AUGMENTATION is W[1]-hard when parameterized by $\beta(H)$ even if the weight of every pair of vertices of G is restricted to be either 0 or 1.*

This proposition implies that unless $\text{FPT} = \text{W}[1]$, we cannot solve STRUCTURED k -CONNECTIVITY AUGMENTATION for $k = 1, 2$ in time $f(\beta(H)) \cdot |V(G)|^{\mathcal{O}(1)}$. Hence the running time of the form $|V(G)|^{f(t)}$ of algorithms solving STRUCTURED k -CONNECTIVITY AUGMENTATION for graphs H with $\beta(H) \leq t$ is probably unavoidable.

4 Augmenting unweighted graphs

In this section we investigate unweighted STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION. Let us remind that in the unweighted cases of the structured augmentation problems the task is to identify whether there is a superposition of graphs G and H of edge connectivity 1 or 2, correspondingly. In other words, we have the weight $\omega(uv) = 0$ for every pair of vertices of G and $W = 0$. We obtain structural characterizations of yes-instances for both problems.

For STRUCTURED CONNECTIVITY AUGMENTATION, we show the following theorem.

► **Theorem 10.** *Let G and H be graphs such that H has no isolated vertices and $|V(H)| \leq |V(G)|$. Then there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is connected if and only if $c(G) \leq |V(H)| - c(H) + 1$ and one of the following holds:*

- (i) H is connected,
- (ii) H is disconnected graph and $i(G) \leq |V(H)| - c(H)$.

Now we consider the case STRUCTURED 2-CONNECTIVITY AUGMENTATION.

► **Theorem 11.** *Let G and H be graphs such that G is connected, H has no isolated vertices and $|V(H)| \leq |V(G)|$. Then there is an injective mapping $\varphi: V(H) \rightarrow V(G)$ such that $F = G \oplus_{\varphi} H$ is 2-connected if and only if one of the following holds:*

- (i) G is 2-connected,
- (ii) G is not 2-connected and $p(G) \leq |V(H)|$,
unless G is a star $K_{1,n}$ where n is odd and H is a matching graph.

Theorems 10 and 11 immediately imply the next corollary.

► **Corollary 12.** *Unweighted STRUCTURED 1-CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION are solvable in time $\mathcal{O}(|V(G)| + |E(G)| + |E(H)|)$.*

5 Conclusion

We initiated the investigation of the structured connectivity augmentation problems where the aim is to increase the edge connectivity of the input graphs by adding edges when the added edges compose a given graph. In particular, we proved that STRUCTURED CONNECTIVITY AUGMENTATION and STRUCTURED 2-CONNECTIVITY AUGMENTATION are solvable in polynomial time when H is from a graph class \mathcal{C} with bounded vertex-cover number. It is natural to ask about increasing connectivity of a $(k - 1)$ -connected graph to a k -connected graph for every positive integer k . For the “traditional” edge connectivity augmentation problem (see [6, 12]), the augmentation algorithms are based on the classic work of Dinits, Karzanov, and Lomonosov [1] about the structure of minimum edge separators. However, for the structural augmentation, the structure of the graph H is an obstacle for implementing this approach directly. Due to this, we could not push further our approach to establish the complexity of STRUCTURED k -CONNECTIVITY AUGMENTATION for $k > 2$ when H is of bounded vertex cover. This remains a natural open question. Recall that our hardness results showing that it is NP-hard to increase the connectivity of a $(k - 1)$ -connected graph to a k -connected graph when H belongs to a class with unbounded vertex cover number are proved for every k .

As the first step, it could be interesting to consider the variant of the problem for multigraphs. In this case, we allow parallel edges and assume that for a mapping $\phi: V(H) \rightarrow V(G)$, the multiplicity of $\phi(x)\phi(y)$ in $G \oplus_{\phi} H$ is the sum of the multiplicities of $\phi(x)\phi(y)$ in G and xy in H . Notice that all our algorithmic and hardness results can be restated for this variant of the problem. Actually, some of the proofs for this variant of the problem become even simpler.

The question of obtaining a k -connected graph for $k \geq 3$ is also open for the unweighted problem. Here we ask whether it is possible to derive structural necessary and sufficient conditions for a $(k - 1)$ -connected graph G and a graph H such that there exists an injective mapping $\phi: V(H) \rightarrow V(G)$ such that $G \oplus_{\phi} H$ is k -connected.

Another direction of the research is to consider vertex connectivity. As it is indicated by the existing results about vertex connectivity augmentation (see, e.g., [8, 9]), this variant of the problem could be more complicated.

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