Multidimensional possibility/probability domination for extending maxitive kernel based signal processing

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Abstract

Signal and image processing make intensive use of positive, bounded and centered functions that are called kernels. Kernels are used for defining the interplay between discrete and continuous domains, filtering, modeling a system through a point spread function, etc. The possible analogy between kernels and fuzzy sets has led to a wide use of fuzzy set theory for signal and image processing [1]. The possibilistic interpretation of fuzzy sets has recently been exploited to extend signal processing with the aim of accounting for poor knowledge of the appropriate kernel to be used. These *imprecise kernels* are called maxitive kernels. A maxitive kernel can be seen as a convex set of conventional kernels. Within this framework, the triangular kernel with mode 0 and spread Δ has a specific role since it can be used to represent a convex set of all bounded centered bell-shaped kernels of spread $\delta \leq \Delta$, i.e. the way kernels are usually imprecisely known (shape unknown, spread imprecise). However, this principle has yet to be extended to more than one dimension despite the fact that it is needed for image processing. An extension to higher dimensions is proposed in this paper.

Keywords: Possibility, maxitive kernels, image processing, probability.

1. Introduction

In digital image processing, positive kernels are widely used for deriving a discrete operator that is initially defined in the continuous domain

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[19]. Super-resolution, computation of derivatives, tomographic reconstruction and geometrical transformations are typical examples of such problems. Within such applications, positive kernels that sums to one (also called summative kernels in [11]) are used to model a sampling function, a point spread function (e.g. in computed tomography, image reconstruction requires explicit modeling of the point spread function of the detector), an approximation function (e.g. to compute image derivatives [16]) or an interpolation function (e.g. for geometrical transformations [10]). Most kernels used in these applications are centered, even and non-increasing with the distance to 0. Hereafter, we will call this kind of kernel: a centered bell-shaped summative kernel (cbsk). The choice of a particular kernel shape or spread is usually prompted more by practical aspects than by any theoretical purpose, since the *ideal kernel* should be spatially unbounded for unbounded signals while digital images are bounded. However, there is high dependance on this choice of obtained discrete operator.

In recent work, the obvious analogy between probability density functions (pdf) and positive kernels has been used to extend the signal processing theory to the case where the modeling is imprecisely known [11]. In these studies, possibility measures [3] are used to define *maxitive kernels* that can be seen as convex sets of summative kernels. By analogy, the core of a maxitive kernel is the (convex) set of conventional positive kernels whose associate probability measure is dominated by the possibility measure associated with the maxitive kernel.

Most maxitive-based signal processing extensions lead to interval-valued signals that represent the set of all signals that would have been obtained by the corresponding conventional method using a positive kernel that belongs to the core of the maxitive kernel [14] . This allows us to represent scant knowledge on the appropriate kernel to be used in a given application.

It can be technically stated as follows. Let $\mathbb{R}$ be the real line. Let f be a real signal, i.e. a function from $\mathbb{R}$ to $\mathbb{R}$. Let κ be the summative kernel associated to the impulse response of a filter ($\kappa : \mathbb{R} \to \mathbb{R}$). Convoluting f with κ leads to a new signal $g : \mathbb{R} \to \mathbb{R}$ which is the signal f processed by the considered filter. This operation can be written $g = f \otimes \kappa$ with $g(x) = \int_{\mathbb{R}} f(u).\kappa(x-u)du$.

The extension proposed in [11] consists in considering replacing the summative kernel κ by a maxitive kernel π that defines a convex set of summative kernels $\mathcal{M}(\pi)$. The extension of the convolution they proposed is such that $[g] = f \otimes \pi$ is an interval-valued function, i.e. $\forall x \in \mathbb{R}$, $[g](x)$ is a real interval.

Within this extension, if κ is a summative kernel belonging to $\mathcal{M}(\pi)$, then $\forall x \in \mathbb{R}$, $(f \otimes \kappa)(x)$ belongs to $(f \otimes \pi)(x)$. Moreover, $\forall x \in \mathbb{R}$, $\forall y \in (f \otimes \pi)(x)$, $\exists \kappa \in \mathcal{M}(\pi)$ such that $y = (f \otimes \kappa)(x)$. Working with π is then equivalent to working with a convex set of summative kernels which allows us to represent imprecise knowledge of the summative kernel.

As an illustration, Figure 1 presents an electrocardiographic signal (in cyan) that has been filtered by a low-pass summative kernel based filter and a maxitive kernel based filter. The output of the summative kernel based filter are plotted in black. The upper (rsp. lower) values of the output of the maxitive based filter are plotted in blue (rsp. in red). The considered summative kernel belongs to the core of the considered maxitive kernel.

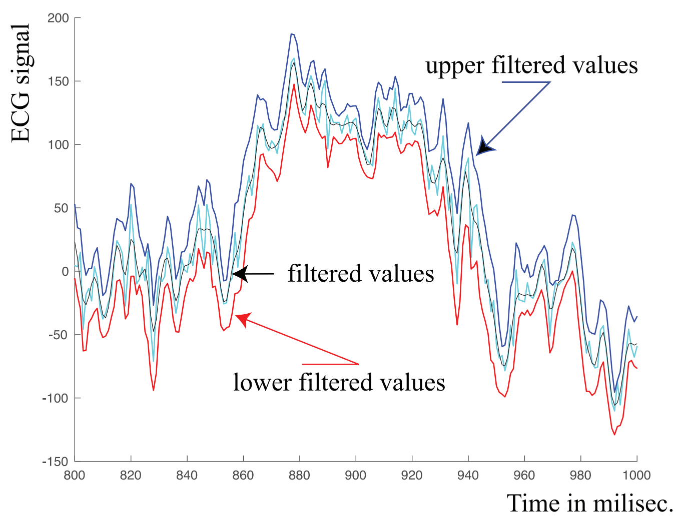

Figure 1: Electrocardiographic signal (in cyan), summative kernel based low-pass filtered signal (in black), upper (blue) and lower (red) values of the maxitive based interval-valued filtered signal.

For example, in [17] this modeling was used to perform deconvolution of a measured signal when the point spread function of the measurement device was imprecisely known. Other applications have been proposed, especially in image processing, to account for the fact that the relation between the continuous domain, where the problem is defined, and the discrete domain, where the problem has to be solved, is imprecisely known (see e.g. [12, 6, 15, 9]).

The triangular maxitive kernel plays an important role within this framework. As shown in [5], a triangular fuzzy set of mode m and spread Δ defines the most specific possibility distribution whose associate possibility measure dominates any probability whose associate pdf is symmetric, non -increasing with the distance to m and with support $[m - \delta, m + \delta]$, with $\delta \leq \Delta$ [5]. The effect of this theorem on maxitive-based signal processing is that when a centered triangular maxitive kernel of spread Δ is used, then the obtained interval-valued signal includes any signal that would have been obtained by the corresponding conventional method using any bounded cbsk (bcbsk) with a spread lower than Δ. This allows us to represent imprecise knowledge of a bcbsk shape and spread. Note that this set of kernels also includes the uniform kernel which is not really bell-shaped. In signal processing, uniform kernels can be used to represent a nearest neighbor interpolation [8] or to model the fact that the point spread function of a sensor is uniform.

However, extending this idea to image processing requires definition of a bi-dimensional maxitive kernel that has the same kind of domination property as the triangular maxitive kernel, i.e. a maxitive kernel that dominates any bi-dimensional bcbsk with a spread that can be bounded. So far, this extension has always been performed by hypothesizing separable bi-dimensional kernels [12, 7, 15, 9]. When kernels are separable, then most image processing operations has the advantage of reducing to two uni-dimensional signal processing operations w.r.t. rows and colomns. However, this hypothesis weakens the power of the obtained representation since some image processing operations cannot be achieved by considering two uni-dimensional kernels. For example in [6], the super-resolution operation requires alignement of a series of low resolution images by using a bi-dimensional kernel. Considering a separable operation restricts this alignement to pure translational movements (no rotation or zoom).

Thus, defining maxitive based image processing requires an extension of the maxitive kernel concept into two dimensions. In image processing, bi-dimensional fuzzy subsets have often been used since the outset for representing image information at different levels (seen e.g. [1] for a nice overview on fuzzy set based image processing). However, so far, no studies have been devoted to extending the work of [5] to two (or more) dimensions.

In this paper, we consider $2D$ maxitive kernels – i.e. possibility distributions – whose cores include all conventional positive centered symmetric $2D$ kernels whose support is bounded. We envisage the two most useful cases: first, the radial case, i.e. the case where the kernels – i.e. pdf(s) – are radial, and, second, the separable case, i.e. the case where the $2D$ kernels – i.e. pdf(s) – are defined by their marginals. We then consider extending the two dimensional case to higher dimensions. For the radial case, we show that extending the proposition of [5] to two or more dimensions is rather straightforward. For the separable case, we show that, contrary to the radial case, there is no single solution for the $2D$ extension, i.e. there are several possible $2D$ optimal maxitive kernels having the same specificity. Extending this proposition to higher dimension is no longer trivial. We provide some potential ways to obtain this extension. We also consider a very simple maxitive kernel that dominates all optimal maxitive kernels. Naturally, this kernel is not optimal in the sense of [5], i.e. it is not the most specific kernel that dominates the set of bcbsk.

In this paper, since maxitive kernels can be seen as possibility distributions and bcbsk as pdf, for the sake of simplicity we will only argue by considering possibility distributions and bounded bell-shaped centered probability density functions.

After this introduction, in Section 2 we propose to provide some necessary background on possibility distributions. In Section 3 we recall the main Theorem of [5] and propose to extend it to higher dimensions. We also propose another way to prove this Theorem that we will use in the sequel. Section 4 proposes a study of the *radial case*, i.e. the case where the pdf(s) are radial, in two then in higher dimensions. In section 5 we study the *separable case*, i.e. the case where the pdf(s) are defined by their marginals. We show that extending this case to higher dimensions is not trivial. We then conclude this article.

2. Possibility measures and domination

2.1. Concave capacities

Let Ω be a bounded set of reference (e.g. bounded set of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{N}$, ...) and $\mathcal{L}(\Omega)$ be the set of all Lebesgue measurable sets of Ω. A^c denotes the complementary set of the subset $A \subset \Omega$, and $\emptyset$ the empty set of Ω.

A capacity is a confidence measure defined on Ω.

Definition 2.1.1. *A capacity ν is a set function $\nu : \mathcal{L}(\Omega) \to [0, 1]$ such that $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and $\nu(A) \leq \nu(B)$ for all $A \subset B \in \mathcal{L}(\Omega)$.*

A capacity ν such that $\forall A, B \in \mathcal{L}(\Omega)$, $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$ is said to be concave. The core of a concave capacity ν, denoted $\mathcal{M}(\nu)$, is the set of probabilities P on $\mathcal{L}(\Omega)$ such that $\forall A \in \mathcal{L}(\Omega)$, $\nu(A) \geq P(A)$. We say that ν **dominates** P.

2.2. Probability measures

A probability measure P is a special case of capacity that complies with the additivity axiom, i.e. $\forall A, B \in \mathcal{L}(\Omega)$, $P(A \cup B) + P(A \cap B) = P(A) + P(B)$. The core of a probability measure is the probability measure itself. A continuous probability measure is completely defined by its associated pdf p, which is a mapping from Ω to $\mathbb{R}^+$:

$$\forall A \in \mathcal{L}(\Omega), P(A) = \int_A p(\omega) d\omega. \tag{1}$$

2.3. Possibility measures

A possibility measure is an interesting way to represent uncertainty when information is scarce or imprecise [13]. A possibility measure Π is a special case of a concave capacity that complies with the maxitive axiom: $\forall A, B \in \mathcal{L}(\Omega)$, $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$. Here, we consider the possibility measures that are completely defined by an associated possibility distribution π, which is a mapping from Ω to $[0, 1]$:

$$\forall A \in \mathcal{L}(\Omega), \Pi(A) = \sup_{\omega \in A} \pi(\omega). \tag{2}$$

A specificity ordering of possibility measures can be obtained by comparing the integral of their distributions, i.e. $Sp(\Pi) = \int_\Omega \pi(\omega) d\omega$. A possibility measure Π_1 is at least as informative as another one Π_2 if their respective distributions π_1 and π_2 follow: $\forall \omega \in \Omega$, $\pi_1(\omega) \leq \pi_2(\omega)$ [3]. In that case, Π_1 is at least as specific as Π_2 and also $Sp(\Pi_1) \leq Sp(\Pi_2)$. If, in addition, $\exists \omega_0 \in \Omega$, such that $\pi_1(\omega_0) < \pi_2(\omega_0)$ then Π_1 is more specific than Π_2.

3. Triangular possibility distribution

In this section, $\Omega = \mathbb{R}$.

3.1. The domination Theorem

In this section, we recall the main result of [5] and formulate it in a way that makes its extension to higher dimensions easier.

Let p be a pdf and P_p its associated probability measure. As a first step, the probability to possibility transform defined in [5] allows us to define a possibility distribution π_p^* whose associated possibility measure $\Pi_{\pi_p^*}$ is the most specific that dominates P_p. The second step follows: if p is unimodal, symmetric, with spread δ and mode m and non-increasing with the distance to m then π_p^* is unimodal, symmetric, non-increasing with the distance to m with spread δ, and mode m (i.e. $\pi_p^*(m) = 1$ and $\forall x \in \mathbb{R}$, $x \neq m$, $\pi_p^*(x) < 1$).

Here, we aim to find a possibility distribution that dominates any centered even bounded pdf, non-increasing with the distance to 0 and having a spread $\delta \leq \Delta$. This problem can be reduced to finding the possibility distribution that dominates any centered even bounded pdf having a spread in $[0, 1]$. As noted in [5], such a possibility distribution is also centered and has a spread in $[0, 1]$. In the following, we call Λ the set of bounded centered even possibility distributions, non-increasing with the distance to 0 and with spread in $[0, 1]$, and Θ the set of bounded centered even pdf, non-increasing with the distance to 0 with a spread in $[0, 1]$.

Let $\pi^{\triangle}$ be the triangular possibility distribution with mode 0 and spread 1 defined by:

$$\forall x \in \mathbb{R}, \quad \pi^{\triangle}(x) = \begin{cases} 1 - |x|, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Definition 3.1.1. *A pdf p belongs to Θ iff:*

i) $\forall x \notin]-1, 1[$, $p(x) = 0$,
ii) *p is continuous on* $]-1, 1[$,
iii) *p is even,*
iv) *p is non-increasing on* $[0, 1[$.

Definition 3.1.2. *A possibility distribution π belongs to Λ iff:*

i) $\pi(0) = 1$,
ii) $\forall x \notin]-1, 1[$, $\pi(x) = 0$,
iii) *π is even,*
iv) *π is non-increasing on* $[0, 1[$.

As a shortcut, for simplification, we can take for granted that *"the pdf p is dominated by a possibility distribution π"* means *"the measure P_p is dominated by Π_π"*.

The triangular possibility distribution defined above can be considered as the most specific possibility distribution belonging to Λ that dominates the uniform pdf $p^{\square}$ defined on $\mathbb{R}$ by:

$$\forall x \in \mathbb{R}, \quad p^{\square}(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in]-1,1[\\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 3.1.1. *(optimal domination of $\pi^{\triangle}$)*
i) $\forall p \in \Theta$, $\forall A \in \mathcal{L}(\mathbb{R})$,

$$P_p(A) \leq \Pi_{\pi^{\triangle}}(A) := \sup_{x \in A} \pi^{\triangle}(x). \tag{5}$$

ii) $\Pi_{\pi^{\triangle}}$ *is the most specific possibility that dominates any probability induced by a pdf of* Θ *in the sense that, with* $\pi \in \Lambda$ *being a possibility distribution:*

$$\big(\exists x \in \mathbb{R} \mid \pi(x) < \pi^{\triangle}(x)\big) \Longrightarrow \big(\exists A \in \mathcal{L}(\mathbb{R}), \exists p \in \Theta \mid P_p(A) > \Pi_{\pi}(A)\big). \tag{6}$$

3.2. Reformulating the triangular domination

In this section, we suggest another way of presenting the proof of Theorem 3.1.1 that will be useful for extending this Theorem to higher dimensions.

This proof needs the following Lemma, where we denote $B_r =]-r, r[$.

Lemma 3.2.1.
$\forall p \in \Theta$, $\forall \pi \in \Lambda$, *we have*

$$\left(\forall r \in [0,1],\ P_p(B_r^c) \leq \Pi_\pi(B_r^c)\right) \Longleftrightarrow \left(\forall A \in \mathcal{L}(\mathbb{R}),\ P_p(A) \leq \Pi_\pi(A)\right).$$

Proof of the Lemma.
Only the right implication is non-trivial.
First, $0 \in A \Rightarrow \Pi_\pi(A) := \sup_{x \in A} \pi(x) = 1$ because $\pi(0) = 1$ (since $\pi \in \Lambda$). This yields the inequality.
Second, if $0 \notin A$, consider the positive distance $r = \inf_{x \in A} |x|$. Then $A \subset B_r^c \Rightarrow P_p(A) \leq P_p(B_r^c) \leq \Pi_\pi(B_r^c)$. Moreover, since $\pi \in \Lambda$, π is even and non-increasing on $[0,1]$. This implies that $\Pi_\pi(B_r^c) = \Pi_\pi(A)$, which terminates the proof. $\square$

Proof of the Theorem.

Proof of i). Thanks to Lemma 3.2.1, we just have to show that $\forall p \in \Theta$ and $\forall r \in [0,1]$, $P_p(B_r^c) \leq \Pi_{\pi^\triangle}(B_r^c) := \sup_{x \in B_r^c}(1-|x|) = 1-r$, that is $P_p(B_r) \geq r$. By parity of p (Definition 3.1.1) , this comes down to showing that $\varphi(r) := \int_0^r p - r/2 \geq 0$, $\forall r \in [0,1]$.
We have $\varphi'(r) = p(r) - 1/2$. Thus φ' verifies $\varphi'(0) = p(0) - 1/2 \geq 0$. Indeed $\varphi'(0) < 0$ implies $p(0) < 1/2$. Since p is non-increasing on $[0,1]$, this would make p verify $\int_0^1 p < 1/2$, which is a contradiction. We also note that, by definition, $\varphi'(1) = p(1) - 1/2 = -1/2 < 0$. Moreover, since p is a bijection on $[0,1[$, as a continuous and non-increasing function (Definition 3.1.1), so is φ'. Thus, there is a single $r_0 \in [0,1[\,|\,\varphi'(r_0) = 0$, which is the maximum of φ. Finally, we have $\varphi(0) = 0$ and $\varphi(1) = \int_0^1 p - 1/2 = 0$. Thus, $\forall r \in [0,1]$, $\varphi(r) \geq 0$, which ends the proof of point i).

Proof of ii). Consider the uniform density $p^\square \in \Theta$ defined by Expression (4). Then $P_{p^\square}(B_r^c) = 2\int_r^1 \frac{1}{2} = 1 - r = \Pi_{\pi^\triangle}(B_r^c)$. Now, suppose that $\pi \in \Lambda$ dominates any $p \in \Theta$ and $\exists\, r \in \mathbb{R}$ such that $\pi(r) < \pi^\triangle(r)$. Then we would have $P_{p^\square}(B_r^c) = \Pi_{\pi^\triangle}(B_r^c) > \Pi_\pi(B_r^c)$. Thus, by Lemma 3.2.1, we would have a subset A such that $P_{p^\square}(A) > \Pi_\pi(A)$, a contradiction that ends the proof of point ii). $\square$

4. Optimal domination: the radial case

We call *the radial case* that in which the pdf is radial. Defining a possibility distribution inducing a possibility measure that dominates any probability measure associated with a radial pdf is very similar to the previous unidimensional situation.

Let $B_r = \{x \in \mathbb{R}^n\,;\, \|x\| < r\}$ denote the open ball of radius r, where $\|.\|$ is the Euclidean norm, and $S_r = \{x \in \mathbb{R}^n\,;\, \|x\| = r\}$ the sphere of radius r. Let $\mathcal{V}_r$ denote the volume of B_r (i.e. its n–dimensional Lebesgue measure) and $\mathcal{A}_r$ the area of S_r (i.e. its $(n-1)$–dimensional Lebesgue measure). We also set $B = B_1$, $S = S_1$, $\mathcal{V} = \mathcal{V}_1$, $\mathcal{A} = \mathcal{A}_1$, and recall that $\mathcal{V}_r = \mathcal{V}r^n$, $\mathcal{A}_r = \mathcal{A}r^{n-1}$, and $\mathcal{A} = n\mathcal{V}$ (see [2]).

We define a set of radial probability densities Θ in a very similar manner as in $1D$.

Definition 4.1. *(the set of n-D radial probabilities Θ)*
A pdf p belongs to Θ if it is defined as a function on the unit ball B verifying:

i) $p \geq 0$ *and* $\int_B p = 1$,
ii) p *is continuous,*
iii) p *is radial, i.e.* $\exists \psi_p : \mathbb{R} \to \mathbb{R}^+$, $p(x) = \psi_p(\|x\|)$,
iv) ψ_p *is non-increasing on* $[0, 1[$,
v) $\forall u \in [1, \infty[$, $\psi_p(u) = 0$.

We also define the set of radial possibility distributions Λ.

Definition 4.2. *(the set of n-D radial possibilities* Λ*)*
A distribution of possibility $\pi \in \Lambda$ *is defined as a function on* B *verifying:*

i) $\pi \geq 0$ *and* $\pi(0) = 1$,
ii) π *is radial, i.e.* $\exists \psi_\pi : \mathbb{R} \to [0, 1]$, $\pi(x) = \psi_\pi(\|x\|)$,
iii) ψ_π *is non-increasing on* $[0, 1[$,
iv) $\forall u \in [1, \infty[$, $\psi_\pi(u) = 0$.

Let $\pi^\triangle \in \Lambda$ denote the radial distribution of possibility defined by

$$\pi^\triangle(x) = \max(0, 1 - \|x\|^n), \tag{7}$$

i.e. $\forall u \in \mathbb{R}$, $\psi_{\pi^\triangle}(u) = \max(0, 1 - u^n)$.

We have the following result:

Theorem 4.1. *(optimal n-D radial domination of* $\pi^\triangle$*)*
i) $\forall p \in \Theta$, $\forall A \in \mathcal{L}(\mathbb{R}^n)$,

$$P_p(A) \leq \Pi_{\pi^\triangle}(A) := \sup_{x \in A} \pi^\triangle(x). \tag{8}$$

ii) $\Pi_{\pi^\triangle}$ *is the most specific possibility that dominates any probability induced by a pdf of* Θ *in the sense that, with* $\pi \in \Lambda$ *being a possibility distribution:*

$$\left(\exists x \in \mathbb{R}^n,\ \pi(x) < \pi^\triangle(x)\right) \Longrightarrow \left(\exists A \in \mathcal{L}(B), \exists p \in \Theta \mid P_p(A) > \Pi_\pi(A)\right). \tag{9}$$

Proving Theorem 4.1 requires a Lemma that is close to Lemma 3.2.1.

Lemma 4.1.
$\forall p \in \Theta, \forall \pi \in \Lambda$ *we have:*
$\left(\forall r \in [0, 1],\ P_p(B_r^c) \leq \Pi_\pi(B_r^c)\right) \Longleftrightarrow \left(\forall A \in \mathcal{L}(B),\ P_p(A) \leq \Pi_\pi(A)\right).$

Proof of the Theorem.
Proof of i). Again, thanks to Lemma 4.1, we just have to show that $\forall p \in \Theta$ and $\forall r \in [0,1]$, $P_p(B_r^c) \leq \Pi_{\pi^\triangle}(B_r^c) := \max_{x\in B_r^c}(1-\|x\|)^n) = 1-r^n$. That comes down to showing that $\varphi(r) := \int_{B_r} p - r^n = \mathcal{A}\int_0^r \psi_p(t)t^{n-1}\,dt - r^n \geq 0, \forall r \in [0,1]$. Its derivative is $\varphi'(r) = \mathcal{A}r^{n-1}\psi_p(r) - nr^{n-1} = \mathcal{A}r^{n-1}(\psi_p(r) - n/\mathcal{A}) = \mathcal{A}r^{n-1}(\psi_p(r) - 1/\mathcal{V})$, as $\mathcal{A} = n\mathcal{V}$. It vanishes on $]0,1[$ for the single value r_0 such that $\psi_p(r_0) = 1/\mathcal{V}$ (as ψ_p is clearly a bijection on $[0,1[$). This value r_0 still exists, as in $2D$, because $\psi_p(0) - 1/\mathcal{V} \geq 0$. Indeed, $\psi_p(0) < 1/\mathcal{V}$ implies, since ψ_p is non-increasing on $[0,1]$, that p would verify $\int_B p = \mathcal{A}\int_0^1 \psi_p(r)r^{n-1}\,dr < \mathcal{A}\int_0^1 \psi_p(0)r^{n-1}\,dr < \mathcal{A}/(n\mathcal{V}) = 1$, which is a contradiction. Finally, we have $\varphi'(0) = 0$, $\varphi'(r_0) = 0$ and $\varphi'(1) = -n < 0$. Thus, $\varphi(0) = 0$, $\varphi(1) = \int_B p - 1 = 0$ and φ is maximal for r_0: this yields $\varphi(r) \geq 0$, $\forall r \in [0,1]$, which ends the proof of point i).
Proof of ii). Consider the uniform density $p^\square$ such that $\psi_{p^\square}(r) = \frac{1}{\mathcal{V}}, \forall r \in [0,1[$ and 0 otherwise, verifying the properties of Definition 4.1. Then $P_{p^\square}(B_r^c) = \int_{B_r^c}\frac{1}{\mathcal{V}} = \frac{1}{\mathcal{V}}(\mathcal{V} - \mathcal{V}r^n) = 1 - r^n = \Pi_{\pi^\triangle}(B_r^c)$. Now, suppose that $\pi \in \Lambda$ dominates any $p \in \Theta$ and $\exists x \in \mathbb{R}^n$ such that $\pi(x) < \pi^\triangle(x)$. Then, for $r = \|x\|$, we would have $P_{p^\square}(B_r^c) = \Pi_{\pi^\triangle}(B_r^c) > \Pi_\pi(B_r^c)$. Thus, by Lemma 4.1, we would have a subset A such that $P_{p^\square}(A) > \Pi_\pi(A)$, a contradiction that ends the proof of point ii). $\square$

For example, when conventional ($2D$) image processing is concerned, the most specific radial maxitive kernel is defined by the possibility distribution $\pi^\triangle(x) = \max(0, 1 - \|x\|^2)$ (see Figure (2)).

5. Optimal domination: the separable case

In this section, we consider the case where the variables are independent, i.e. the probability density is fully defined by its marginal probability density functions. This case is more intricate than the radial case. Indeed, while the radial case in n dimensions can be seen as a direct extension of the unidimensional case, in the separable case, there are several possibility distributions that are Pareto optimal in the sense that they dominate any separable non-increasing probability density and cannot be specificity ordered. That is, if π_1 and π_2 are Pareto optimal, $\int_\Omega \pi_1(x)dx = \int_\Omega \pi_2(x)dx$ while $\pi_1 \neq \pi_2$.

We first consider the 2-dimensional case. After proposing a straightforward extension of the unidimensional case, we show that there are many other

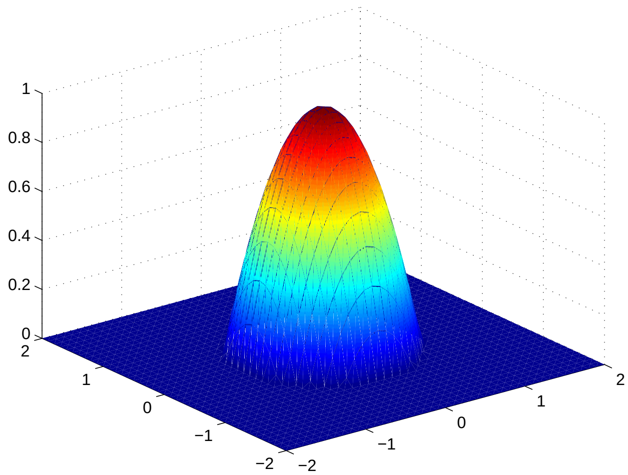

Figure 2: $\pi^{\triangle}(x) = \max(0, 1 - \|x\|^2)$

possible extensions. We attempt an interpretation based on aggregation functions. We finally propose some possible extensions in the n-dimensional case ($n > 2$).

5.1. The 2-dimensional case

5.1.1. Separate variables in dimension 2 - a straightforward extension

In this section, we show that this case can be treated as a simple extension of the $1D$ case.

Let us first define Θ as the set of two-dimensional bounded probability density functions of separate variables as follows:

Definition 5.1.1. *(the set of $2D$ separate variable probabilities Θ)*
A pdf p belongs to Θ iff:

i) *there are two densities of probability p_1 and p_2 such that $p(x, y) = p_1(x)p_2(y)$.*
ii) $\forall i \in \{1, 2\}$, $\forall x \notin]-1, 1[$, $p_i(x) = 0$,
iii) $\forall i \in \{1, 2\}$, *p_i is continuous on* $]-1, 1[$,
iv) $\forall i \in \{1, 2\}$, *p_i is even,*
v) $\forall i \in \{1, 2\}$, *p_i is non-increasing on* $[0, 1[$.

Note that $\int_{[-1,1]^2} p = \int_{-1}^1 p_1 \int_{-1}^1 p_2 = 1$.

Let us now define Λ as the set of $2D$ possibility distributions that are decreasing functions of $\max(|x|,|y|)$.

Definition 5.1.2. *(The set of $2D$ possibilities Λ)*
A possibility distribution π belongs to Λ iff:

i) $\forall (x,y) \notin]-1,1[^2$, $\pi(x,y) = 0$,
ii) *There is a function ψ_π such that* $\pi(x,y) = \psi_\pi(\max(|x|,|y|))$,
iii) $\psi_\pi(0) = 1$ *and* $\psi_\pi \geq 0$,
iv) ψ_π *is non-increasing on* $[0,1[$.

As in the $1D$ case, P_p and Π_π denote the measures of probability and possibility respectively associated with the density p and the distribution π. $\pi^\triangle \in \Lambda$ also denotes the distribution of possibility depicted in Figure 3, defined by

$$\pi^\triangle(x,y) = 1 - \max(|x|,|y|)^2. \tag{10}$$

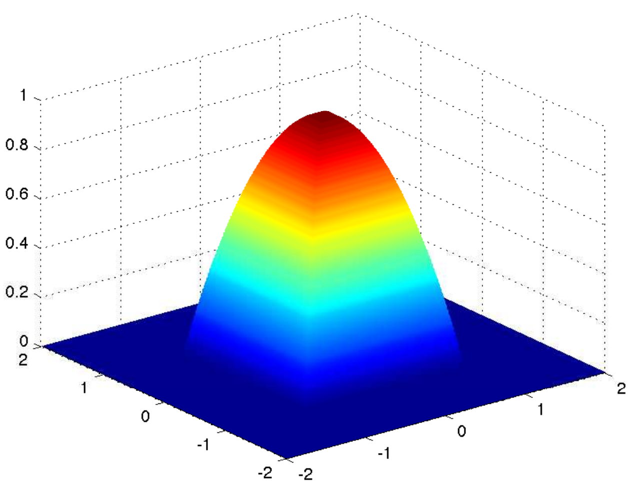

Figure 3: $\pi^\triangle(x,y) = 1 - \max(|x|,|y|)^2$

As proved below, $\pi^\triangle$ is optimal in Λ in the sense that there is no possibility distribution that belongs to Λ that is more specific than $\pi^\triangle$ and dominates any pdf $p \in \Theta$.

Theorem 5.1.1. *(optimal domination of $\pi^{\triangle}$ in Λ)*
i) $\forall p \in \Theta$, $\forall A \in \mathcal{L}(\mathbb{R}^2)$,

$$P_p(A) \leq \Pi_{\pi^{\triangle}}(A) := \max_{(x,y)\in A} \left(1 - \max(|x|,|y|)^2\right). \tag{11}$$

ii) If a possibility distribution $\pi \in \Lambda$ is dominated by $\pi^{\triangle}$, then Π_π cannot dominate any probability induced by a pdf of Θ:

$$\begin{aligned}&\left(\exists (x,y) \in \mathbb{R}^2 \mid \pi(x,y) < \pi^{\triangle}(x,y)\right) \Longrightarrow \\ &\left(\exists A \in \mathcal{L}(\mathbb{R}^2), \exists p \in \Theta \mid P_p(A) > \Pi_\pi(A)\right).\end{aligned} \tag{12}$$

To prove the Theorem, we need the following Lemma:

Lemma 5.1.1.
Let us define $\mathcal{H}_\alpha = \{(x,y) \in [-1,1]^2, \max(|x|,|y|) < \alpha\}$. Then $\forall p \in \Theta, \forall \pi \in \Lambda$ we have

$$(\forall \alpha \in [0,1],\ P_p(\mathcal{H}^c_\alpha) \leq \Pi_\pi(\mathcal{H}^c_\alpha)) \Longleftrightarrow \left(\forall A \in \mathcal{L}(\mathbb{R}^2),\ P_p(A) \leq \Pi_\pi(A)\right).$$

Proof of the Lemma.
To prove the right implication, let us consider a set $A \in \mathcal{L}(\mathbb{R}^2)$.
First, if $0 \in A$, we have by definition $\Pi_\pi(A) := \max_{(x,y)\in A} \pi(x,y) = \max_{(x,y)\in A} \psi_\pi(\max(|x|,|y|)) = \psi_\pi(0) = 1$ and thus $P_p(A) \leq \Pi_\pi(A)$.
Second, if $0 \notin A$, consider the positive value $\alpha = \sup\{\beta \mid A \subset \mathcal{H}^c_\beta\}$. Then $A \subset \mathcal{H}^c_\alpha \Rightarrow P_p(A) \leq P_p(\mathcal{H}^c_\alpha) \leq \Pi_\pi(\mathcal{H}^c_\alpha)$. As ψ_π is non-increasing on $[0,1]$, it is maximal for α on the set $\mathcal{H}^c_\alpha$, by definition. This means that $\pi(x,y)$ is maximal on the set A for at least one point (x,y) such that $\max(|x|,|y|) = \alpha$. So we have $\Pi_\pi(\mathcal{H}^c_\alpha) = \Pi_\pi(A)$, an equality that ends the proof of ii). $\square$

Remark 5.1.1. *In this part, a key point is that the value α plays the same role as the distance $r = \inf_{x\in A} \|x\|$ in the proofs of Lemma 3.2.1*

Proof of the Theorem.
Proof of i). Thanks to Lemma 5.1.1, we just have to show that $\forall p \in \Theta$ and $\forall \alpha \in [0,1]$, we have $P_p(\mathcal{H}^c_\alpha) \leq \Pi_{\pi^{\triangle}}(\mathcal{H}^c_\alpha) := \max_{(x,y)\in\mathcal{H}^c_\alpha} (1 - \max(|x|,|y|)^2) = 1 - \alpha^2$. This is equivalent to showing that $\forall \alpha \in [0,1], \varphi(\alpha) := P_p(\mathcal{H}_\alpha) - \alpha^2 \geq 0$. Let us consider the first quadrant $[0,1]^2$ of $[-1,1]^2$ and recall that $p(x,y) = p_1(x)p_2(y)$. By symmetry of p we have $\varphi(\alpha) = 4\int_0^\alpha \int_0^\alpha p(x,y)\,dxdy - \alpha^2 =$

$\left(2\int_0^\alpha p_1(x)\,dx\right)\left(2\int_0^\alpha p_2(y)\,dy\right) - \alpha^2 \geq 0$, as a direct consequence of the 1D computation made in the proof of Theorem 3.1.1. This proves point i).

Proof of ii). Consider the uniform density $p^\square$ defined by $p^\square(x,y) = \frac{1}{4}, \forall (x,y) \in]-1,1[^2$ and $p^\square(x,y) = 0$ otherwise (still verifying the properties of Definition 5.1.1). Then, by parity, $P_{p^\square}(\mathcal{H}_\alpha^c) = \int_{\mathcal{H}_\alpha^c} \frac{1}{4} = 1 - 4(\frac{1}{4})\int_0^\alpha\int_0^\alpha dydx = 1 - \alpha^2 = \Pi_{\pi^\triangle}(\mathcal{H}_\alpha^c)$. Now, suppose we have $\pi \in \Lambda$ that dominates any $p \in \Theta$ and $(x,y) \in \mathbb{R}^2$ such that $\pi(x,y) < \pi^\triangle(x,y)$. Then, for a value $\alpha \in [0,1]$, we would have $P_{p^\square}(\mathcal{H}_\alpha^c) = \Pi_{\pi^\triangle}(\mathcal{H}_\alpha^c) > \Pi_\pi(\mathcal{H}_\alpha^c)$. Thus, by Lemma 5.1.1, there would be a subset A such that $P_{p^\square}(A) > \Pi_\pi(A)$, a contradiction that ends the proof of point ii). $\square$

5.1.2. Separate variables in dimension 2: generalization

In Section 5.1.1 , we have proved that, among the possibility distributions belonging to Λ, $\pi^\triangle(x,y) = 1 - \max(|x|,|y|)^2$ is optimal.

In Definition 5.1.2, the set Λ was completely defined by the function $s(x,y) = \max(|x|,|y|)$, i.e. $\pi \in \Lambda \implies \exists \psi_\pi \,|\, \forall (x,y) \in \mathbb{R}^2,\ \pi(x,y) = \psi_\pi(s(x,y))$. A relevant question arises: what would happen if we chose another function $s(x,y)$ for defining Λ?

For instance, function $s(x,y) = \min(|x|,|y|)$ has contour lines that differ from those of $\max(|x|,|y|)$ (four squares in the corners instead of a single square centered at the origin). A similar approach would show that

$$\pi^\triangle(x,y) = (1 - \min(|x|,|y|))^2 \tag{13}$$

is optimal in that case. As a last example, choosing $s(x,y) = |xy|$ (with hyperbolic contour lines) yields

$$\pi^\triangle(x,y) = 1 - |xy| + |xy|\log|xy| \tag{14}$$

to be optimal. In fact, we will prove that a set Λ_s corresponds to each function s.

A separate variable probability following Definition 5.1.1 has a pdf $p(x,y) = p_1(x)p_2(y)$ symmetric w.r.t. both x and y axes. It is therefore obvious that s should have the same symmetries.

Definition 5.1.3. *(the set of sink functions Υ)*
A function s belongs to Υ iff:

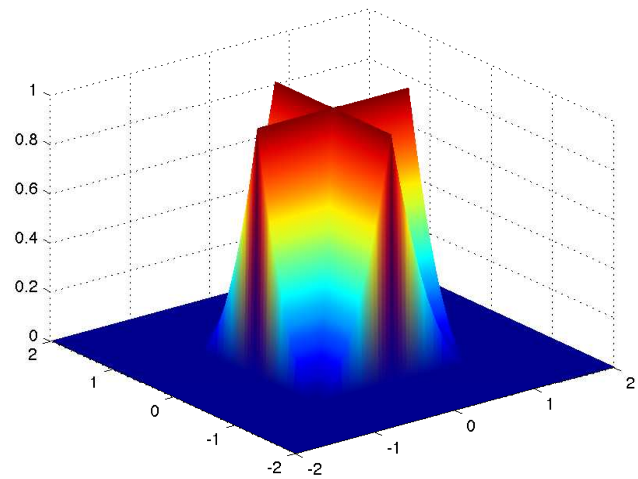

Figure 4: $\pi^{\triangle}(x,y) = (1 - \min(|x|,|y|))^2$

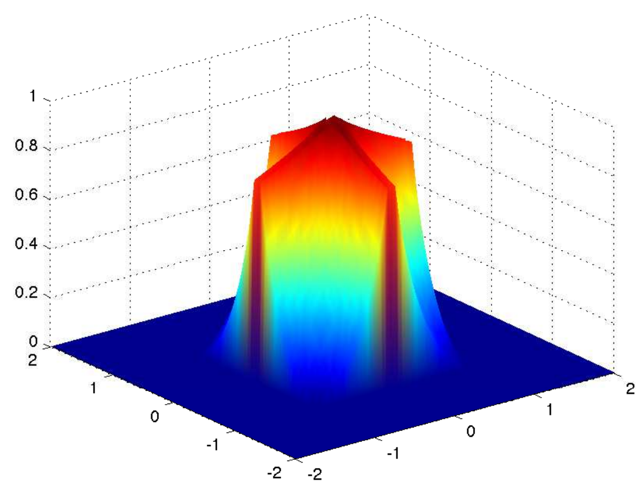

Figure 5: $\pi^{\triangle}(x,y) = 1 - |xy| + |xy| \log |xy|$

i) $\forall(x,y) \notin]-1,1[^2,\ s(x,y)=0$,
ii) $s(1,1)=1$,
iii) $s(0,0)=0$,
iv) $\forall(x,y) \in [-1,1]^2\ s(x,y)=s(y,x)$
v) s *is non-decreasing w.r.t.* $|x|$ *and* $|y|$.

In the following, we show that many sink functions $s \in \Upsilon$ define a set Λ_s that has a single optimal possibility distribution $\pi_s^{\triangle}$. The above properties are required for $\psi_\pi(s(x,y))$ to be an aggregation function as defined in [4]. We also show that the optimal distributions for two different sink-shaped functions s cannot be compared in the sense that none of them dominates the other. Moreover, $\forall s \in \Upsilon$, with $\pi_s^{\triangle}$ being the optimal distribution in Λ_s, its specificity $Sp(\pi_s^{\triangle})=2$.

With s being a sink-shaped function as defined above, we define Λ_s.

Definition 5.1.4. *(the set of possibilities Λ_s)*
Let $s \in \Upsilon$ and $\psi_\pi : \mathbb{R} \mapsto [0,1]$ be a continuous function. A possibility distribution π belongs to Λ_s iff:

i) $\pi(x,y)=\psi_\pi(s(x,y))$,
ii) $\pi \geq 0$ *and* $\pi(0)=1$,
iii) ψ_π *is non-increasing on* $[0,1[$,
iv) $\forall u \in]1,\infty[,\ \psi_\pi(u)=0$.

Definition 5.1.5. *(contour lines and level sets)*
For $\alpha \in [0,1]$, we define the contour lines and their associated level sets:

i) $\mathcal{C}_\alpha = \{(x,y) \in [-1,1]^2, s(x,y)=\alpha\}$,
ii) $\mathcal{H}_\alpha = \{(x,y) \in [-1,1]^2, s(x,y)<\alpha\}$,
iii) $\mathcal{H}_\alpha^c = \{(x,y) \in [-1,1]^2, s(x,y)\geq\alpha\}$.

Since s is symmetric w.r.t. the x and y axis, we only consider working in the first quadrant $[0,1]^2$.

In many cases, contour lines associated with a sink function s, can be defined by a function $\ell_\alpha : x \to \ell_\alpha(x)$. In these cases, under some other conditions, there exists an optimal possibility distribution $\pi_s^{\triangle}$ associated to s. Before stating the main result, we first need to define the contour function ℓ_α.

Definition 5.1.6. *(contour line function in the first quadrant)*
For $\alpha \in [0,1]$*, we define* $\ell_\alpha : [0,1] \to [0,1], x \to \ell_\alpha(x)$ *by*
$\mathcal{C}_\alpha = \{(x, \ell_\alpha(x)), x \in [0,1]\} = \{(x,y) \in [0,1]^2, s(x,y) = \alpha\}$.

Theorem 5.1.2. *(*$\pi_s^\triangle$ *is the optimal dominating possibility in* Λ_s*)*
Let $s \in \Upsilon$ *be a sink-shaped function. Let* $\mathcal{H}_\alpha$ *and* ℓ_α *be respectively the level sets and the contour line functions of* s*.*

Let us define $\psi_s^\triangle$ *by:*

$$\psi_s^\triangle(\alpha) = P_{p^\square}\left(\mathcal{H}_\alpha^c\right) = \frac{1}{4}\mathscr{L}_2\left(\mathcal{H}_\alpha^c\right), \tag{15}$$

where $p^\square$ *is the uniform probability density function on* $[-1,1]^2$ *(*$p^\square(x,y) = \frac{1}{4}, \forall (x,y) \in]-1,1[^2$ *and* $p^\square(x,y) = 0$ *otherwise) and* $\mathscr{L}_2$ *is the two dimensional Lebesgue measure.*

If ℓ_α *is piecewise derivable and verifies one of the following properties:*
1) ℓ_α *maps* $[0,\alpha]$ *into* $[0,\alpha]$*, with* $\ell_\alpha(0) = \alpha$*,* $\ell_\alpha(\alpha) = 0$*,*
2) ℓ_α *maps* $[\alpha,1]$ *into* $[\alpha,1]$*, with* $\ell_\alpha(\alpha) = 1$*,* $\ell_\alpha(1) = \alpha$*,*

then the possibility function $\pi_s^\triangle \in \Lambda_s$ *defined by* $\forall (x,y) \in [-1,1]^2$*,* $\pi_s^\triangle(x,y) = \psi_s^\triangle(s(x,y))$
i) is the most specific possibility in Λ_s *that dominates any probability induced by a pdf of* Θ*:* $\forall p \in \Theta$*,* $\forall A \in \mathcal{L}(\mathbb{R}^2)$*,* $P_p(A) \le \Pi_{\pi_s^\triangle}(A) := \max_{(x,y)\in A} \pi_s^\triangle(x,y)$*, and* $\forall \pi_s \in \Lambda_s$ *we have:*

$$\left(\exists (x,y) \in \mathbb{R}^2 | \pi_s(x,y) < \pi_s^\triangle(x,y)\right) \Rightarrow \left(\exists A \in \mathcal{L}(\mathbb{R}^2), \exists p \in \Theta | P_p(A) > \Pi_{\pi_s}(A)\right),$$

ii) has a specificity equal to 2*.*

Examples.
a) Optimal possibility derived from $s(x,y) = |xy|$ (namely $\pi_s^\triangle(x,y) = 1 - |xy| + |xy| \log |xy|$) has hyperbolic contour lines that verify 1).
b) $\pi_s^\triangle(x,y) = 1 - \max(|x|,|y|)^2$, considered as the natural generalization (made in previous section 5.1.1) of $\pi^\triangle(x) = 1 - |x|$, and derived from $s(x,y) = \max(|x|,|y|)$ is a limit case of possibilities verifying 1).
c) Optimal possibility derived from $s(x,y) = 1 - (1-|x|)(1-|y|)$ (namely $\pi_s^\triangle(x,y) = 1 - |xy| + (1-|xy|)\log(1-|xy|)$) has hyperbolic contour lines that verify 2).

d) $\pi_s^{\triangle}(x,y) = (1-\min(|x|,|y|))^2$, derived from $s(x,y) = \min(|x|,|y|)$, is a limit case of possibilities verifying 2).

Proof of the Theorem.

Proof of i).
By construction of $\mathcal{H}_\alpha$, both in situations 1) and 2), the proof of the preliminary result: $\forall p \in \Theta, \forall \pi \in \Lambda_s$,

$$(\forall \alpha \in [0,1],\ P_p(\mathcal{H}_\alpha^c) \leq \Pi_\pi(\mathcal{H}_\alpha^c)) \Longleftrightarrow \big(\forall A \in \mathcal{L}([-1,1]^2),\ P_p(A) \leq \Pi_\pi(A)\big)$$

is straightforward, i.e. being similar to the proof of Lemma 5.2.1 in the case of $s(x,y) = \max(|x|,|y|)$.

According to the definition of the possibility given by formula (15), $\pi_s(x,y) = \psi_\pi(s(x,y))$ with $\psi_\pi(\alpha) = P_{p^\square}(\mathcal{H}_\alpha^c)$, showing that, $\forall \alpha \in [0,1]$, $P_p(\mathcal{H}_\alpha^c) \leq \Pi_{\pi_s}(\mathcal{H}_\alpha^c)$ is equivalent to showing that $g(\alpha) := 4 \iint\limits_{\mathcal{H}_\alpha^c} (p - \frac{1}{4}) \leq 0$. Let us focus on the first quadrant Q of $[-1,1]^2$. As $\iint\limits_{[-1,1]^2} (p-\frac{1}{4}) = 0$ and by symmetry of $\mathcal{H}_\alpha$ and p, we have to show that $\varphi(\alpha) := -\frac{1}{4} g(\alpha) = \iint\limits_{\mathcal{H}_\alpha \cap Q} (p - \frac{1}{4}) \geq 0$.

Recalling that $p(x,y) = p_1(x)p_2(y)$ and formulating $p_1 = \frac{1}{2} + q_1$, $p_2 = \frac{1}{2} + q_2$, we have $p - \frac{1}{4} = \frac{1}{2}q_1 + \frac{1}{2}q_2 + q_1 q_2 = \frac{1}{2}q_1(1+q_2) + \frac{1}{2}q_1(1+q_2)$. Then, by symmetry, $\varphi(\alpha) = \frac{1}{2} \iint\limits_{\mathcal{H}_\alpha \cap Q} q_1(1+q_2) + \frac{1}{2} \iint\limits_{\mathcal{H}_\alpha \cap Q} q_2(1+q_1) = \iint\limits_{\mathcal{H}_\alpha \cap Q} q_1(1+q_2)$.

Consider situation 1). On Q, level sets are determined by a non-increasing, continuous and piecewise derivable function ℓ_α such that: ℓ_α maps $[0,\alpha]$ into $[0,\alpha]$, with $\ell_\alpha(0) = \alpha$, $\ell_\alpha(\alpha) = 0$. Then $\varphi(\alpha) = \int_0^\alpha \int_0^{\ell_\alpha(x)} q_1(x)(1+q_2(y))\,dxdy = \int_0^\alpha q_1(x)(\ell_\alpha(x) + \rho_2(\ell_\alpha(x)))\,dx$, where $\rho_2(x) = \int_0^x q_2$. Integrating by parts, we get $\varphi(\alpha) = [\rho_1(\alpha)(\ell_\alpha(\alpha) + \rho_2(\ell_\alpha(\alpha))) - \rho_1(0)(\ell_\alpha(0) + \rho_2(\ell_\alpha(0)))] - \int_0^\alpha \rho_1(x)(\ell'_\alpha(x) + \ell'_\alpha(x) q_2(\ell_\alpha(x)))\,dx$, where $\rho_1(x) = \int_0^x q_1$. As $\rho_1(0) = \rho_2(0) = 0$ and $\ell_\alpha(\alpha) = 0$, φ reduces to $\varphi(\alpha) = -\int_0^\alpha \rho_1(x)\ell'_\alpha(x)(1 + q_2(\ell_\alpha(x)))\,dx$. As p_1 is even and non-increasing and as $\rho_1(1) = \int_0^1 (p_1 - \frac{1}{2}) = 0$, then $\rho_1(x) = \int_0^x (p_1 - \frac{1}{2}) \geq 0$. Moreover, $q_2 \geq -\frac{1}{2}$ and $\ell'_\alpha \leq 0$. Thus, $\varphi(\alpha) \geq 0\,, \forall \alpha \in [0,1]$.

Consider situation 2). Now, ℓ_α maps $[\alpha,1]$ into $[\alpha,1]$, with $\ell_\alpha(\alpha) = 1$, $\ell_\alpha(1) = \alpha$. Let us make the decomposition $\varphi(\alpha) =$

$\iint\limits_{\mathcal{H}_\alpha\cap Q} q_2(1+q_1) = \int_0^\alpha\int_0^1 q_1(x)(1+q_2(y))\,dxdy + \int_\alpha^1\int_0^{\ell_\alpha(x)} q_1(x)(1+q_2(y))\,dxdy = \rho_1(\alpha)(1+\rho_2(1)) + \int_\alpha^1 q_1(x)(\ell_\alpha(x)+\rho_2(\ell_\alpha(x)))\,dx$. Noting that $\rho_2(1) = \int_0^1(p_2-\frac{1}{2}) = 0$ and integrating by parts yields $\varphi(\alpha) = \rho_1(\alpha) + [\rho_1(1)(\ell_\alpha(1)+\rho_2(\ell_\alpha(1))) - \rho_1(\alpha)(\ell_\alpha(\alpha)+\rho_2(\ell_\alpha(\alpha)))] - \int_\alpha^1 \rho_1(x)\ell'_\alpha(x)(1+q_2(\ell_\alpha(x)))\,dx = \rho_1(\alpha) - \rho_1(\alpha) - \int_\alpha^1 \rho_1(x)\ell'_\alpha(x)(1+q_2(\ell_\alpha(x)))\,dx \geq 0$, as in situation 1).

Finally, if contour line functions present a configuration mixing situations 1) and 2) – which is the case for Examples (d) and (e) – then the above properties still hold. Those properties can be proved in the same manner by using the proofs of 1) or 2), depending on the situation.

Proof of ii).
Let $\mathcal{G}_\beta$ be the β-level set of $\pi_s^\triangle$. Relation (15) leads to $\beta = \frac{1}{4}\mathscr{L}_2(\mathcal{G}_\beta^c) = \frac{1}{4}(4-\mathscr{L}_2(\mathcal{G}_\beta))$. Then we get $\mathscr{L}_2(\mathcal{G}_\beta) = 4(1-\beta)$ which entails $\iint\limits_{[-1,1]^2} \pi_s^\triangle = \int_0^1 \mathscr{L}_2(\mathcal{G}_\beta)\,d\beta = 4\int_0^1(1-\beta)\,d\beta = 2$. Therefore $Sp(\pi_s^\triangle) = 2$. $\square$

Corollary 5.1.1. *Let $s_1, s_2 \in \Upsilon$, then the two optimal possibilities $\pi_{s_1}^\triangle$ and $\pi_{s_2}^\triangle$ cannot be compared by the relation $\pi_{s_1}^\triangle \leq \pi_{s_2}^\triangle$ since they are continuous functions verifying $Sp(\pi_{s_1}^\triangle) = Sp(\pi_{s_2}^\triangle) = 2$.*

Proposition 5.1.1. *(π_{dom} dominates every $\pi_s^\triangle$)*
The possibility π_{dom} defined by: $\forall(x,y) \in [-1,1]^2$, $\pi_{dom}(x,y) := 1-|xy|$ dominates any $\pi_s^\triangle$ whatever the sink-shaped function s (say $\Pi_{\pi_s^\triangle}(A) \leq \Pi_{\pi_{dom}}(A)$, $\forall A \in \mathcal{L}(\mathbb{R}^2)$), but is not optimal.

Proof of the Proposition.
We just need to verify that, for any sink-shaped function s, $\pi_s^\triangle \leq \pi_{dom}$ on the contour lines of s in the first quadrant Q of $[-1,1]^2$, i.e. $\pi_s^\triangle(x,\ell_\alpha(x)) \leq 1 - x\ell_\alpha(x)$, $\forall x \in [0,1]$. As, by definition, $\pi_s^\triangle(x,\ell_\alpha(x)) = \psi_\pi(s(x,\ell_\alpha(x))) = \psi_\pi(\alpha) = \frac{1}{4}\mathscr{L}_2(\mathcal{H}_\alpha^c) = \mathscr{L}_2(\mathcal{H}_\alpha^c\cap Q)$, this is equivalent to verifying that $x\ell_\alpha(x) \leq \mathscr{L}_2(\mathcal{H}_\alpha\cap Q)$, $\forall x \in [0,1]$. In both configurations 1) and 2) in Theorem 5.1.2, the fact that ℓ_α is non-increasing implies that $[0,x]\times[0,\ell_\alpha(x)] \subset \mathcal{H}_\alpha\cap Q$ (with $x \in [0,\alpha]$ in case 1) and $x \in [\alpha,1]$ in case 2)). Note that in case 1) we naturally have the relation $\Pi_{\pi_{dom}}(\mathcal{H}_\alpha\cap Q) := \max_{\mathcal{H}_\alpha\cap Q}(1-xy) = 1 \geq \Pi_{\pi_s^\triangle}(\mathcal{H}_\alpha\cap Q)$ so the result is immediate.

The specificity of π_{dom} is equal to 4. Therefore π_{dom} is not optimal, otherwise the comparison $\pi_s^{\triangle} \leq \pi_{dom}$ would lead to a contradiction. $\square$

As an illustration, in Figure 6 we give a plot of π_{dom}. This possibility distribution has avantages over other functions due to its very simple computation and the fact that it avoids choosing a particular sink-shaped function. Moreover, its specificity equals 3, which is just at the mid-point between the specificity of the optimal dominating possibility distributions and the least specific one $\pi^{\square}$ defined by $\forall(x,y) \in [-1,1]^2$, $\pi^{\square}(x,y) = 1$ and 0 otherwise whose specificity is $Sp(\pi^{\square}) = 4$.

Remark 5.1.2. *It would have been interesting to find a possibility distribution that is the supremum of all optimal possibilities (in the sense above). In fact, it would avoid the user to question him-/her-self about the appropriate sink function to consider. Unfortunately, a Monte-Carlo simulation shows that this supremum has smaller specificity.*

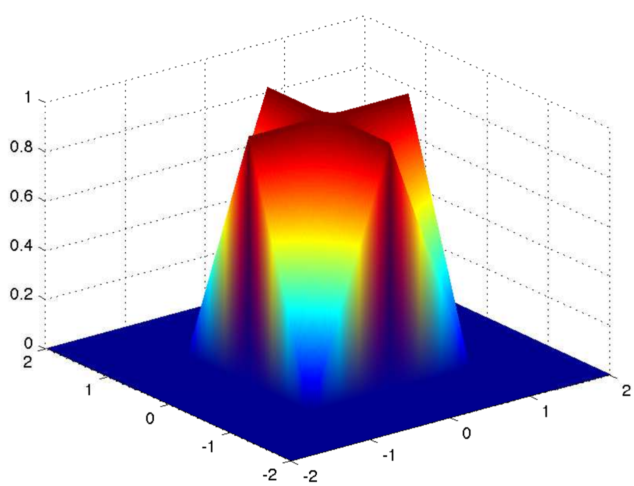

Figure 6: $\pi_{dom}(x,y) = 1 - |xy|$

5.1.3. Discussion about the sink function and its relation to aggregation operators

In Section 5.1.2, the notion of *sink function* has been introduced, which leads to a multiple choice for the most specific possibility distribution that dominates any pdf belonging to Θ. In fact, this is in line with the comment of Sudkamp in [18]: each sink function can be associated with an aggregation function of the two fuzzy triangular set associated with each variable. Let $\pi_X^\triangle$ and $\pi_Y^\triangle$ respectively be the triangular possibility function associated with x and y. Those are defined by: $\pi_X^\triangle(x) = 1 - |x|$ for $\forall x \in [-1, 1]$ and 0 elsewhere (the same for y). Thus, $\forall u \in [-1, 1]$, $|x| = 1 - \pi_X^\triangle(x)$ and $|y| = 1 - \pi_Y^\triangle(y)$. Now let us define $\mathcal{T}_s$ byt $\mathcal{T}_s(\pi_X^\triangle(x), \pi_Y^\triangle(y)) = 1 - s(x, y) = 1 - s(1 - \pi_X^\triangle(x), 1 - \pi_Y^\triangle(y))$. By construction, if $s \in \Upsilon$, $\mathcal{T}_s$ follows:

i) $\mathcal{T}_s(0, 0) = 1$,
ii) $\mathcal{T}_s(1, 1) = 0$,
iii) $\forall(\alpha, \beta) \in [0, 1]^2$ $\mathcal{T}_s(\alpha, \beta) = \mathcal{T}_s(\beta, \alpha)$,
iv) $\mathcal{T}_s(\alpha, \beta)$ is non-increasing w.r.t. α and β.

Therefore $\mathcal{T}_s$ is an aggregation operator [4]. Thus choosing s amounts to choosing a function to aggregate the two triangular marginal possibility distributions. Let us review the examples of Section 5.1.2.

a) $s(x, y) = |xy| \Longleftrightarrow \mathcal{T}_s(\alpha, \beta) = 1 - \alpha - \beta + \alpha\beta$, i.e. T-conorm,
b) $s(x, y) = max(|x|, |y|) \Longleftrightarrow \mathcal{T}_s(\alpha, \beta) = min(\alpha, \beta)$, i.e. a T-norm,
c) $s(x, y) = 1 - (1 - |x|)(1 - |y|) \Longleftrightarrow \mathcal{T}_s(\alpha, \beta) = \alpha\beta$, i.e. a T-norm,
d) $s(x, y) = min(|x|, |y|) \Longleftrightarrow \mathcal{T}_s(\alpha, \beta) = max(\alpha, \beta)$, i.e. T-conorm,
e) $s(x, y) = \frac{|x|+|y|}{2} \Longleftrightarrow \mathcal{T}_s(\alpha, \beta) = \frac{\alpha+\beta}{2}$, is another possible example using a mean operator,
f) etc.

This could be an avenue to decide whether a sink function could be more adapted to a problem than another. In an epistemic interpretation of the possibility distribution, the optimal possibility distribution $\pi_s^\triangle$ is sought in a family of distributions that can be seen as functions of a combination the two marginal distributions $\pi_X^\triangle$ and $\pi_Y^\triangle$ by using the aggregation operator $\mathcal{T}_s$: where $\pi_s^\triangle$ is the optimal possibility distributions among all those that can be written as $\varphi\big(\mathcal{T}_s(\pi_X^\triangle, \pi_Y^\triangle)\big)$. For example, considering $\mathcal{T}_s(\alpha, \beta) = max(\alpha, \beta)$ can be interpreted as an attempt to search for a pessimistic disjonctive based possibility distribution. However, this interpretation may be limited w.r.t.

the possibility theory. The use of this interpretation to choose an appropriate maxitive kernel w.r.t. specific application is a very interesting avenue for future work.

5.2. The n-dimensional case

In this section, we propose to extend the previous results to more than 2 dimensions. The good news is that for many n-dimensional sink-shaped functions this extension is rather straightforward. The bad news is that contrary to the 2-dimensional case we were able to prove this for only a few sink-shaped functions in n dimension.

In section 5.2.1, we propose an n-dimensional extension for $s(x_1, \ldots, x_n) = \max(|x_1|, \ldots, |x_n|)$. This proof can be easily derived for e.g. $s(x_1, \ldots, x_n) = \min(|x_1|, \ldots, |x_n|)$ and $s(x_1, \ldots, x_n) = |x_1||x_2| \ldots |x_n|$. Section 5.2.2 is an attempt to prove this for any sink-shaped function. Some partial results will be presented.

5.2.1. Separate variables in dimension n - a straightforward extension

This section is the natural extension of the 1D, and is consequently very similar to the 2D natural extension presented in Section 5.1.1.

We define Θ for n separate variables (n marginal densities of probability), and a set of possibilities Λ similar to the $2D$ case. We denote $x = (x_1, \ldots, x_n)$.

Definition 5.2.1. *(the set of n-D separate variables probabilities Θ)*
A pdf p belongs to Θ iff:

i) *there are n densities of probability $p_1, \ldots, p_n$ such that $p(x) = p_1(x_1) \ldots p_n(x_n)$,*
ii) $\forall i \in \{1, \ldots, n\}$, $\forall t \notin]-1, 1[$, $p_i(t) = 0$,
iii) $\forall i \in \{1, \ldots, n\}$, *$p_i$ is continuous on* $]-1, 1[$,
iv) $\forall i \in \{1, \ldots, n\}$, *$p_i$ is even,*
v) $\forall i \in \{1, \ldots, n\}$, *$p_i$ is non-increasing on* $[0, 1[$.

Note that $\int_{[-1,1]^n} p = \int_{-1}^{1} p_1 \cdots \int_{-1}^{1} p_n = 1$.

Definition 5.2.2. *(the set of n-D separate variables possibilities Λ)*
A possibility distribution π belongs to Λ iff:

i) $\forall x \notin]-1, 1[^n$, $\pi(x) = 0$,
ii) *There is a function ψ_π such that* $\pi(x) = \psi_\pi(\max(|x_1|, \ldots, |x_n|))$,

iii) $\psi_\pi(0) = 1$ *and* $\psi_\pi \geq 0$,
iv) ψ_π *is non-increasing on* $[0, 1[$.

Let $\pi^\triangle \in \Lambda$ denote the distribution of possibility defined by

$$\pi^\triangle(x) = 1 - \max(|x_1|, \ldots, |x_n|)^n. \tag{16}$$

Then, a similar result follows: as proved below, $\pi^\triangle$ is the most specific possibility distribution that belongs to Λ that dominates any pdf belonging to Θ.

Theorem 5.2.1. *(optimal domination of $\pi^\triangle$ in Λ)*
i) $\forall p \in \Theta$, $\forall A \in \mathcal{L}(\mathbb{R}^n)$,

$$P_p(A) \leq \Pi_{\pi^\triangle}(A) := \max_{x \in A} \left(1 - \max(|x_1|, \ldots, |x_n|)^n\right).$$

ii) $\Pi_{\pi^\triangle}$ *is the most specific possibility that dominates any probability induced by a pdf of* Θ *in the sense that, with* $\pi \in \Lambda$ *being a possibility distribution, we have:*

$$\big(\exists x \in \mathbb{R}^n \mid \pi(x) < \pi^\triangle(x)\big) \Longrightarrow \big(\exists A \in \mathcal{L}(\mathbb{R}^n), \exists p \in \Theta \mid P_p(A) > \Pi_\pi(A)\big).$$

To prove this Theorem, we need the following Lemma, very close to Lemma 5.1.1, whose very similar proof is left to the reader.

Lemma 5.2.1.
Let us define $\mathcal{H}_\alpha = \{x \in [-1, 1]^n, \max(|x_1|, \ldots, |x_n|) < \alpha\}$. *Then* $\forall p \in \Theta, \forall \pi \in \Lambda$ *we have*

$$(\forall \alpha \in [0, 1],\ P_p(\mathcal{H}_\alpha^c) \leq \Pi_\pi(\mathcal{H}_\alpha^c)) \Longleftrightarrow (\forall A \in \mathcal{L}(\mathbb{R}^n),\ P_p(A) \leq \Pi_\pi(A)).$$

The proof of the Theorem follows directly:

Proof of the Theorem.
Proof of i). Thanks to Lemma 5.2.1, we just have to show that $\forall p \in \Theta$ and $\forall \alpha \in [0, 1]$, we have $P_p(\mathcal{H}_\alpha^c) \leq \Pi_{\pi^\triangle}(\mathcal{H}_\alpha^c) := \max_{x \in \mathcal{H}_\alpha^c} (1 - \max(|x_1|, \ldots, |x_n|)^n) = 1 - \alpha^n$. This is equivalent to showing that $\forall \alpha \in [0, 1], \varphi(\alpha) := P_p(\mathcal{H}_\alpha) - \alpha^n \geq 0$. Let us consider the first quadrant $[0, 1]^n$ of $[-1, 1]^n$ and recall that $p(x) = p_1(x_1) \ldots p_n(x_n)$.

By symmetry of p, we have $\varphi(\alpha) = 2^n \int_0^\alpha \cdots \int_0^\alpha p(x)\,dx - \alpha^n = \left(2\int_0^\alpha p_1(x_1)\,dx_1\right)\ldots\left(2\int_0^\alpha p_n(x_n)\,dx_n\right) - \alpha^n \geq 0$, also as a direct consequence of the 1D computation made in the proof of Theorem 3.1.1. This proves point i).

Proof of ii). Consider the uniform density $p^\square$ such that $p^\square(x) = \frac{1}{2^n}, \forall x \in]-1,1[^n$ and $p^\square = 0$ otherwise (still satisfying the properties of Definition 5.2.1). Then, by parity, $P_{p^\square}(\mathcal{H}_\alpha^c) = \Pi_{\pi^\triangle}(\mathcal{H}_\alpha^c)$. Now, suppose that $\pi \in \Lambda$ dominates any $p \in \Theta$ and $x \in \mathbb{R}^n$ such that $\pi(x) < \pi^\triangle(x)$. Then, for an $\alpha \in [0,1]$, we would have $P_{p^\square}(\mathcal{H}_\alpha^c) = \Pi_{\pi^\triangle}(\mathcal{H}_\alpha^c) > \Pi_\pi(\mathcal{H}_\alpha^c)$. Thus, by Lemma 5.2.1, $P_{p^\square}(A) > \Pi_\pi(A)$, a contradiction that ends the proof of point ii). $\square$

5.2.2. Separate variables in dimension n: an attempt at generalization

Our aim is to extend the $2D$ generalizations in Section 5.1.2 to the n-dimensional case. We show in this section that this construction is rather easy for several interesting sink-shaped functions.

We follow the same path and denote $x = (x_1, \ldots, x_n)) \in \mathbb{R}^n$. A separate variable probability following Definition 5.2.1 has a pdf $p(x) = p_1(x_1)\ldots p_n(x_n)$ where each p_i $(i = 1 \ldots n)$ follows Definition 3.1.1. A proof like that of Theorem 5.2.1 can easily be derived for many other sink-shaped functions, such as those provided below. Let s be a sink-shaped function, as previously we define Λ_s:

Definition 5.2.3. *(the set of possibilities Λ_s)*
Let $s : [-1,1]^n \mapsto [0,1]$ and $\psi_\pi : \mathbb{R} \mapsto [0,1]$ be continuous functions. A possibility distribution π belongs to Λ_s iff:

i) $\pi(x) = \psi_\pi(s(x))$,
ii) $\pi \geq 0$ *and* $\pi(0) = 1$,
iii) ψ_π *is non-increasing on* $[0,1[$,
iv) $\forall u \in]1,\infty[$, $\psi_\pi(u) = 0$.

We now consider five different examples of sink-shaped functions with their associate optimal possibility distribution. As previously, we combine each sink-shaped s with an aggregation operator $\mathcal{T}_s$. Proving the optimality of each $\pi_s^\triangle(x)$ in their respective sets Λ_s follows the same path as in the 2-dimensional case.

Examples.
a) Optimal possibility derived from $s(x) = |x_1 \dots x_n|$ (namely $\pi_s^{\triangle}(x) = 1 - |x_1 \dots x_n| + |x_1 \dots x_n| \log |x_1 \dots x_n|$) has contour lines that verify 1) with $\mathcal{T}_s(\alpha_1 \dots \alpha_n) = 1 - (1-\alpha_1) \dots (1-\alpha_n)$.
b) $\pi_s^{\triangle}(x) = 1 - \max(|x_1|, \dots, |x_n|)^n$, considered as the natural generalization (as shown in previous section 5.2.1) of $\pi^{\triangle}(x) = 1 - |x|$, and derived from $s(x) = \max(|x_1|, \dots, |x_n|)$ is a limit case of possibilities verifying 1) with $\mathcal{T}_s(\alpha_1 \dots \alpha_n) = \min(\alpha_1, \dots, \alpha_n)$.
c) Optimal possibility derived from $s(x) = 1 - (1-|x_1|) \dots (1-|x_n|)$ (namely $\pi_s^{\triangle}(x) = 1 - |x_1 \dots x_n| + (1 - |x_1 \dots x_n|) \log(1 - |x_1 \dots x_n|)$) has contour lines that verify 2) with $\mathcal{T}_s(\alpha_1 \dots \alpha_n) = \alpha_1 \dots \alpha_n$.
d) $\pi_s^{\triangle}(x) = (1 - \min(|x_1|, \dots, |x_n|))^n$, derived from $s(x) = \min(|x_1|, \dots, |x_n|)$, is a limit case of possibilities verifying 2) with $\mathcal{T}_s(\alpha_1 \dots \alpha_n) = \max(\alpha_1, \dots, \alpha_n)$.

Proposition 5.2.1. *Let s be a sink-shaped function and $\pi_s^{\triangle}$ its associate optimal possibility distribution. Let $\mathcal{H}_\alpha$ be the level set function of s.*

Let us define $\psi_s^{\triangle}$ by: '

$$\psi_s^{\triangle}(\alpha) = P_{p^{\square}}\left(\mathcal{H}_\alpha^c\right) = \frac{1}{2^n} \mathscr{L}_n\left(\mathcal{H}_\alpha^c\right), \tag{17}$$

where $p^{\square}$ is the uniform probability density function on $[-1,1]^n$, ($p^{\square}(x) = \frac{1}{2^n}, \forall x \in]-1,1[^n$ and $p^{\square}(x) = 0$ otherwise) and $\mathscr{L}_n$ is the n–dimensional Lebesgue measure.

Then $\pi_s^{\triangle}$ has a specificity equal to 2^{n-1}.

Proof of the Proposition.
Let $\mathcal{G}_\beta$ be the β-level set of $\pi_s^{\triangle}$. Relation (17) leads to $\beta = \frac{1}{2^n}\mathscr{L}_n(\mathcal{G}_\beta^c) = \frac{1}{2^n}(2^n - \mathscr{L}_n(\mathcal{G}_\beta))$, that entails $\mathscr{L}_n(\mathcal{G}_\beta) = 2^n(1-\beta)$. Thus, $\int\limits_{[-1,1]^n} \pi_s^{\triangle} = \int_0^1 \mathscr{L}_n(\mathcal{G}_\beta)\, d\beta = 2^n \int_0^1 (1-\beta)\, d\beta = 2^{n-1}$.

A very significant corollary follows:

Corollary 5.2.1. *Let s_1 and s_2 be two sink-shaped functions, then the two optimal possibilities $\pi_{s_1}^{\triangle}$ and $\pi_{s_2}^{\triangle}$ cannot be compared by the relation $\pi_{s_1}^{\triangle} \leq \pi_{s_2}^{\triangle}$ since they are continuous functions verifying $Sp(\pi_{s_1}^{\triangle}) = Sp(\pi_{s_2}^{\triangle}) = 2^{n-1}$.*

Therefore, it would be of significant interest to be able to interpret the sink-shaped functions either w.r.t. the possibility theory or w.r.t. maxitive kernel signal processing to be able to choose the possibility distribution that is optimal w.r.t. a particular application. The analogy mentioned in Section 5.1.3 could be a nice option for this interpretation.

Remark 5.2.1. *As in the 2D case (Theorem 5.1.2 and end of Section 5.1.2), the very simple possibility distribution π_{dom} defined by $\forall x \in [-1,1]^n$, $\pi_{dom}(x) := 1 - |x_1 \dots x_n|$ dominates any optimal possibility distribution $\pi_s^{\triangle}$ whatever the sink-shaped function s (say $\Pi_{\pi_s^{\triangle}}(A) \leq \Pi_{\pi_{dom}}(A)$, $\forall A \in \mathcal{L}(\mathbb{R}^n)$), but is not optimal (proving this domination and the non-optimality can be conducted easily following the same avenue).*
However, choosing π_{dom} is really not appropriate in high dimension. In fact, its specificity is equal to $2^n - 1$, i.e. very close to 2^n, the specificity of the binary possibility distribution $\pi^{\square}$ defined by $\forall x \in [-1,1]^n$, $\pi^{\square}(x) = 1$ and 0 otherwise.

6. Conclusion

Using maxitive kernel based signal processing to achieve image processing requires extending some unidimensional concepts into higher dimensions. Among all the concepts associated with maxitive kernels, the possibility that this technique offers to model a kernel whose shape is poorly known and whose spread is imprecisely known is significant. This possibility comes from the fact that a centered triangular maxitive kernel of spread Δ dominates any bounded centered bell-shaped kernel of spread $\delta \leq \Delta$. It is based on the main Theorem of [5].

In this paper, we have proposed to extend this Theorem to higher dimensions. As a main result, we found that, contrary to the unidimensional case, there was no single solution. We have considered two useful cases, i.e. the radial case (the case where the pdf – i.e. the kernel – is radial) and the separable case (the case where the pdf is completely defined by its marginals, i.e. the variables are independent) which are the most relevant for the signal processing applications we would like to consider. For the radial case, we have extended the Theorem for any dimension. This is very interesting since many kernels used in image processing are radial. For the separable case, we have shown that equivalent solutions (in terms of the distribution specificity) can be obtained by considering particular shape functions that we have named *sink-shaped functions* that are related to some aggregation functions used in fuzzy logic. This has been proved for any sink-functions in the 2-dimensional case. In higher dimension, this has only been proved for some sink-shaped functions whose associate aggregation function is classical in fuzzy logic.

There are many different interesting follow ups to this work. First, for the separable case, the main Theorem for any dimension and any sink-shaped function still requires a general proof. Second, it would be necessary to have an interpretation of the sink-shaped functions. This would help to choose a maxitive kernel that is appropriate for each application. Third, it would be interesting to consider the situation where the variable dependance is known or even better when this dependance exists but is unknown.

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