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# Extremal Values of the Chromatic Number for a Given Degree Sequence

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## Abstract

For a degree sequence  $d : d_1 \geq \dots \geq d_n$ , we consider the smallest chromatic number  $\chi_{\min}(d)$  and the largest chromatic number  $\chi_{\max}(d)$  among all graphs with degree sequence  $d$ . We show that if  $d_n \geq 1$ , then  $\chi_{\min}(d) \leq \max \left\{ 3, d_1 - \frac{n+1}{4d_1} + 4 \right\}$ , and, if  $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq d_n \geq 1$ , then  $\chi_{\max}(d) = \max_{i \in [n]} \min \{i, d_i + 1\}$ . For a given degree sequence  $d$  with bounded entries, we show that  $\chi_{\min}(d)$ ,  $\chi_{\max}(d)$ , and also the smallest independence number  $\alpha_{\min}(d)$  among all graphs with degree sequence  $d$ , can be determined in polynomial time.

**Keywords:** Degree sequence; chromatic number; independence number

**MSC 2010:** 05C07; 05C15; 05C69

# 1 Introduction

We consider finite, simple, and undirected graphs. The *degree sequence* of a graph  $G$  with vertex set  $\{v_1, \dots, v_n\}$  is the sequence  $d_G(v_1), \dots, d_G(v_n)$  of its vertex degrees. A sequence  $d_1, \dots, d_n$  of integers is a *degree sequence* if it is the degree sequence of some graph. Repetitions within the degree sequence can be indicated by suitable exponents; the degree sequence of the star  $K_{1,r}$  of order  $r+1$ , for instance, is  $r, 1^r$ . For a given sequence  $d$ , let  $\mathcal{G}(d)$  be the set of all graphs  $G$  whose degree sequence is  $d$ ; called the *realizations* of  $d$ . For an integer  $n$ , let  $[n]$  be the set of the positive integers at most  $n$ .

In the present paper we consider

$$\chi_{\min}(d) = \min \{\chi(G) : G \in \mathcal{G}(d)\} \quad \text{and} \quad \chi_{\max}(d) = \max \{\chi(G) : G \in \mathcal{G}(d)\}.$$

Punnim [11] determined  $\chi_{\min}(d)$  and  $\chi_{\max}(d)$  for regular degree sequences  $d = r^n$  in almost all cases. The parameter  $\chi_{\max}(d)$  was also considered by Dvořák and Mohar [3], who established degree sequence versions of the Hadwiger Conjecture and even the Hajós Conjecture, see also [14].

We contribute some bounds, exact values, and algorithmic results. Further discussion of related research will be given throughout the rest of the paper.

## 2 Some bounds and exact values

For a sequence  $d$  of non-negative integers  $d_1 \geq \dots \geq d_n$ , let  $H(d)$  be the sequence

$$d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n.$$

Havel [9] and Hakimi [6] showed that  $d$  is a degree sequence if and only if  $H(d)$  is a degree sequence. In fact, they observed that if  $d$  is a degree sequence, then there is a realization  $G$  of  $d$  in which the neighbours of a vertex of degree  $d_1$  have degrees  $d_2, \dots, d_{d_1+1}$ . Iteratively applying this observation to a given degree sequence yields a realization that tends to contain a large complete subgraph on the vertices of large degrees, that is, such a realization may be expected to have high chromatic number.

In order to obtain a realization with hopefully small chromatic number, one can apply Havel and Hakimi's observation to the complement. More precisely, for a degree sequence  $d$  as above, the sequence  $\bar{d}$  defined as

$$n - 1 - d_n \geq \dots \geq n - 1 - d_1$$

is also a degree sequence; in fact, the graphs in  $\mathcal{G}(\bar{d})$  are exactly the complements  $\bar{G}$  of the graphs  $G$  in  $\mathcal{G}(d)$ . Furthermore, by the above observation of Havel and Hakimi,  $\bar{d}$  has a realization in which the neighbors of a vertex of the largest degree  $n - 1 - d_n$  have degrees  $n - 1 - d_{n-1}, \dots, n - 1 - d_{d_n+1}$ . Equivalently, as already observed by Kleitman and Wang [10] in a more general form,  $d$  has a realization in which the neighbors of a vertex of the smallest

degree  $d_n$  have degrees  $d_1, \dots, d_{d_n}$ . In summary, we obtain that  $d$  is a degree sequence if and only if the sequence  $\bar{H}(d)$  defined as

$$d_1 - 1, \dots, d_{d_n} - 1, d_{d_n+1}, \dots, d_{n-1} \quad (1)$$

is a degree sequence. Iteratively applying this observation to a given degree sequence yields a realization that tends to avoid dense subgraphs on the vertices of large degrees, that is, such a realization may be expected to have small chromatic number.

As an example consider the degree sequence  $d : r^{r+1}, 1^{r(r+1)}$  for some positive integer  $r$ . Havel and Hakimi's original observation yields the realization  $K_{r+1} \cup \binom{r}{2} K_2$ , whose chromatic number is  $r + 1$ , which equals  $\chi_{\max}(d)$ , while the above complementary version yields the realization  $(r + 1)K_{1,r}$ , whose chromatic number is 2, which equals  $\chi_{\min}(d)$ .

For a sequence  $d$  of integers  $d_1, \dots, d_n$ , let  $n$  be the *length* of  $d$ , let  $\min(d) = \min\{d_1, \dots, d_n\}$ , and let  $\max(d) = \max\{d_1, \dots, d_n\}$ . Furthermore, let  $\bar{H}^0(d) = d$ ,  $\bar{H}^1(d) = \bar{H}(d)$ , and  $\bar{H}^i(d) = \bar{H}(\bar{H}^{i-1}(d))$  for an integer  $i$  at least 2. Note that iteratively applying the reductions  $d \mapsto H(d)$  or  $d \mapsto \bar{H}(d)$  always requires reordering the constructed sequences in a non-increasing way.

**Theorem 1** *If  $d$  is a degree sequence of length  $n$ , then*

$$\chi_{\min}(d) \leq \max \left\{ \min \left( \bar{H}^{n-i}(d) \right) : i \in [n] \right\} + 1.$$

*Proof:* Iteratively applying the complementary version of Havel and Hakimi's observation to the degree sequence  $d$  yields a realization  $G$  of  $d$  with vertex set  $\{v_1, \dots, v_n\}$  such that, for  $i$  from  $n$  down to 1, the vertex  $v_i$  has degree  $\min(\bar{H}^{n-i}(d))$  in the graph  $G[\{v_1, \dots, v_i\}]$ . Greedily coloring the vertices of  $G$  in the order  $v_1, \dots, v_n$  yields a coloring that uses at most  $\max \left\{ \min \left( \bar{H}^{n-i}(d) \right) : i \in [n] \right\} + 1$  colors.  $\square$

Note that for the degree sequence  $d : r^{r+1}, 1^{r(r+1)}$  of length  $n = (r + 1)^2$  considered as an example above, we obtain  $\max \left\{ \min \left( \bar{H}^{n-i} \left( r^{r+1}, 1^{r(r+1)} \right) \right) : i \in [n] \right\} + 1 = 2$ , that is, for this degree sequence  $d$ , Theorem 1 reproduces the correct value of  $\chi_{\min}(d)$ .

Unfortunately, Theorem 1 is not very explicit. As a more explicit consequence, we quantify how small degrees may reduce the effect of large degrees on  $\chi_{\min}(d)$ .

**Corollary 2** *If  $d$  is a degree sequence  $d_1 \geq \dots \geq d_n$ , and  $k$  and  $\ell$  are positive integers such that  $d_k \geq k + \ell$  and  $d_{n-\ell+1} \leq k$ , then*

$$\chi_{\min}(d) \leq \max \left\{ d_1 - \frac{1}{k} \left( 1 + \sum_{i=n-\ell+1}^n d_i \right) + 1, d_{k+1}, k \right\} + 1.$$

*Proof:* We consider the first  $\ell$  applications of the reduction  $d \mapsto \bar{H}(d)$ . Since  $d_k \geq k + \ell$  and  $d_{n-\ell+1} \leq k$ , we obtain that, for  $i \in [\ell]$ , the degree sequence  $\bar{H}^i(d)$  arises from  $\bar{H}^{i-1}(d)$  by removing the degree  $d_{n-i+1}$ , and reducing the  $d_{n-i+1}$  largest degrees by 1. For  $i \in \{0, \dots, \ell\}$ , let  $\Delta_i = \max(\bar{H}^i(d))$ , and let  $n_i$  be the number of entries of  $\bar{H}^i(d)$  that are equal to  $\Delta_i$ . Suppose,

for a contradiction, that  $\Delta_\ell > \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$ , where  $D = \sum_{i=n-\ell+1}^n d_i$ . Note that each of the  $\ell + 1$  degree sequences  $d, \bar{H}(d), \dots, \bar{H}^\ell(d)$  contains at most  $k$  entries that are strictly larger than  $d_{k+1}$ . So, for  $i \in [\ell]$ , we have

- $(\Delta_i, n_i) = (\Delta_{i-1}, n_{i-1} - d_{n-i+1})$  if  $d_{n-i+1} < n_{i-1}$ , and
- $\Delta_i = \Delta_{i-1} - 1$  and  $n_i \leq k - (d_{n-i+1} - n_{i-1}) = n_{i-1} - d_{n-i+1} + k$  if  $d_{n-i+1} \geq n_{i-1}$ .

Note that  $(k\Delta_{i-1} + n_{i-1}) - (k\Delta_i + n_i) \geq d_{n-i+1}$  in both cases. Summation over  $i \in [\ell]$  yields  $(k\Delta_0 + n_0) - (k\Delta_\ell + n_\ell) \geq D$ . Since  $\Delta_0 = d_1$ ,  $n_0 \leq k$ , and  $n_\ell \geq 1$ , this implies  $\Delta_\ell \leq d_1 - \frac{D+1}{k} + 1$ , which is a contradiction. Hence,  $\Delta_\ell \leq \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$ , and any realization  $H$  of the degree sequence  $\bar{H}^\ell(d)$  can be colored using at most  $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\} + 1$  many colors. Adding  $\ell$  further vertices of degrees  $d_{n-\ell+1}, \dots, d_n$  one by one to  $H$ , and connecting them to suitable vertices according to the previous reductions, yields a realization  $G$  of  $d$ . Since the added vertices all have degree at most  $k$ , the coloring of  $H$  can be extended greedily to a coloring of  $G$  using at most  $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}, k\right\} + 1$  different colors in total.  $\square$

For a given degree sequence  $d$  not satisfying any further restriction, one can only bound  $\chi_{\min}(d)$  from above by  $\max(d) + 1$ . In fact,  $d$  might be  $\max(d)^{\max(d)+1}, 0^{n-\max(d)-1}$ , whose only realization contains a clique of size  $\max(d) + 1$ .

Our next two results improve this trivial estimate for graphs without isolated vertices.

**Theorem 3** *If  $d$  is a degree sequence of length  $n$  with  $\max(d) \geq \sqrt{\frac{n\delta}{4}}$  and  $\min(d) \geq \delta$  for some positive integer  $\delta$ , then  $\chi_{\min}(d) \leq \max(d) - \frac{n\delta}{4\max(d)} + \delta + 3$ .*

*Proof:* Our first goal is to show that we may assume that  $d$  has a realization with a very large independent set. Therefore, among all realizations  $G$  of the degree sequence  $d$  and all (not necessarily optimal) colorings  $f$  of  $G$ , we choose  $G$  and  $f$  with color classes  $V_1, \dots, V_k$ , where  $V_i$  contains  $n_i$  vertices for  $i \in [k]$ , in such a way that

- $(n_1, \dots, n_k)$  is lexicographically maximal, and
- subject to this first condition, the number of edges between  $V_{k-1}$  and  $V_k$  is minimum.

Note that  $k$  may actually be larger than  $\chi(G)$ , and that  $n_1$  is necessarily equal to the independence number  $\alpha(G)$  of  $G$ .

Let  $\Delta = \max(d)$ . If  $k \leq \Delta - \frac{n\delta}{4\Delta} + \delta + 3$ , then  $\chi_{\min}(d) \leq \chi(G) \leq k$  implies the desired bound. Hence, we may assume that  $k > \Delta - \frac{n\delta}{4\Delta} + \delta + 3$ . Since  $\Delta \geq \sqrt{\frac{n\delta}{4}}$  and  $\delta \geq 1$ , we have  $k \geq 5$ . By the choice of the coloring  $f$ , there is an edge, say  $uv$ , between the smallest two color classes  $V_{k-1}$  and  $V_k$ . If  $G \setminus (V_{k-1} \cup V_k \cup N_G(u) \cup N_G(v))$  contains an edge  $xy$ , then removing from  $G$  the two edges  $uv$  and  $xy$ , and adding the two edges  $ux$  and  $vy$ , yields another realization  $G'$  of  $d$ . Note that  $f$  is still a coloring of  $G'$ . This implies that there is a coloring  $f'$  of  $G'$  such that either the non-increasing vector of the sizes of the color classes is lexicographically larger than the one of  $f$ , or there are fewer edges between the two smallest color classes. Since both cases imply a

contradiction to the choice of  $G$  and  $f$ , we obtain that  $V(G) \setminus (V_{k-1} \cup V_k \cup N_G(u) \cup N_G(v))$  is an independent set, which implies  $\alpha(G) \geq n - (n_{k-1} + n_k) - 2\Delta$ . Since  $V_{k-1}$  and  $V_k$  are the smallest two color classes, and  $n_2 + \dots + n_k = n - \alpha(G)$ , we obtain  $n_{k-1} + n_k \leq \frac{2}{k-1}(n - \alpha(G))$ . This implies  $\alpha(G) \geq n - \frac{2}{k-1}(n - \alpha(G)) - 2\Delta$ , and, using  $k \geq 5$ , we obtain  $\alpha(G) \geq n - \frac{k-1}{k-3} \cdot 2\Delta \geq n - 4\Delta$ .

Altogether, we may assume that  $d$  has a realization  $G$  with an independent set  $I = \{u_1, \dots, u_\alpha\}$  of order at least  $n - 4\Delta$ . By the above-mentioned observations of Havel [9], Hakimi [6], Rao [12], and Kleitman and Wang [10], we may further assume that, for every  $i \in [\alpha]$ , the vertex  $u_i$  is adjacent to  $d_G(u_i)$  vertices in  $V(G) \setminus I$  of the largest degrees in the induced subgraph  $G - \{u_1, \dots, u_{i-1}\}$  of  $G$ . Arguing as in the proof of Corollary 2, we obtain  $\left((n - \alpha)\Delta + (n - \alpha)\right) - \left((n - \alpha)\Delta(G - I) + 1\right) \geq d_G(u_1) + \dots + d_G(u_\alpha) \geq \alpha\delta$ , where  $\Delta(G - I)$  denotes the maximum degree of  $G - I$ . This implies  $\Delta(G - I) \leq \Delta - \frac{\alpha\delta+1}{n-\alpha} + 1 \leq \Delta - \frac{(n-4\Delta)\delta+1}{4\Delta} + 1 = \Delta - \frac{n\delta+1}{4\Delta} + \delta + 1$ . Therefore, we can color  $G$  using at most  $\Delta - \frac{n\delta+1}{4\Delta} + \delta + 2$  colors on the vertices in  $V(G) \setminus I$ , and one additional color on the vertices in  $I$ , which implies  $\chi_{\min}(d) \leq \chi(G) \leq \Delta - \frac{n\delta+1}{4\Delta} + \delta + 3$ .  $\square$

For positive integers  $r, s$ , and  $\delta$  such that  $r + 1$  is a multiple of  $\delta$ , let  $d$  be the degree sequence  $(r+s)^{r+1}, \delta^{s(r+1)/\delta}$ . Since the sum of the largest  $r+1$  degrees equals exactly  $2\binom{r+1}{2} + \delta s(r+1)/\delta$ , every realization  $G$  of  $d$  contains a clique on the  $r+1$  vertices of largest degrees, and an independent set on the remaining vertices. Note that  $\chi(G) \in \{r+1, r+2\}$ , which, for  $r \gg s \gg \delta$ , is roughly  $\max(d) - \frac{n \min(d)}{\max(d)}$ , that is, up to the constants, the bound in Theorem 3 is best possible. In fact, by imposing a stronger lower bound on  $\max(d)$  or by increasing the additive constant, the factor 4 within the term  $\frac{n\delta+1}{4\Delta}$  can easily be reduced to slightly more than 2.

Our next result gives a best possible bound on  $\chi_{\min}(d)$  for degree sequences of small degrees.

**Theorem 4** *If  $n, d_1, \dots, d_n$  are integers such that  $\sqrt{\frac{n-1}{2}} \geq d_1 \geq \dots \geq d_n \geq 1$  and  $d_1 + \dots + d_n$  is even, then  $\chi_{\min}(d) \leq 3$ . (In particular,  $d_1, \dots, d_n$  is a degree sequence.)*

*Proof:* There is a partition of  $[n]$  into two sets  $X$  and  $Y$  with  $||X| - |Y|| \leq 1$  and  $0 \leq s \leq d_1 \leq \sqrt{\frac{n-1}{2}}$ , where  $s = \sum_{i \in X} d_i - \sum_{i \in Y} d_i$ ; in fact, as long as there are two equal entries  $d_i$  and  $d_j$  in the sequence  $d_1, \dots, d_n$ , we assign  $i$  to  $X$  and  $j$  to  $Y$ , and remove  $d_i$  and  $d_j$  from the sequence, and once all remaining entries are distinct, say  $d_{i_1} > \dots > d_{i_k}$ , we assign  $i_1, i_3, \dots$  to  $X$  and  $i_2, i_4, \dots$  to  $Y$ . Let  $x = |X|$  and  $y = |Y|$ . Note that  $x, y \geq \frac{n-1}{2}$ ; in particular,  $s \leq x$ . Reducing  $s$  distinct entries of the sequence  $(d_i)_{i \in X}$  by 1, and reordering yields a sequence  $a_1 \geq \dots \geq a_x$ . Reordering the sequence  $(d_i)_{i \in Y}$  yields  $b_1 \geq \dots \geq b_y$ .

By construction,  $\sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i$ ,  $\max\{a_1, b_1\} \leq \sqrt{\frac{n-1}{2}}$ , and  $b_y \geq 1$ .

Let  $k \in [x]$ . If  $k \leq \sqrt{\frac{n-1}{2}}$ , then  $a_1 \leq \sqrt{\frac{n-1}{2}}$  and  $b_n \geq 1$  imply

$$\sum_{i \in [k]} a_i \leq ka_1 \leq \frac{n-1}{2} \leq y \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

If  $k > \sqrt{\frac{n-1}{2}}$ , then  $b_1 \leq \sqrt{\frac{n-1}{2}}$  implies

$$\sum_{i \in [k]} a_i \leq \sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i = \sum_{i \in [y]} \min\{k, b_i\}.$$

By the Gale-Ryser Theorem [5, 15], there is a bipartite graph  $H$  with partite sets  $X$  and  $Y$  with  $|X| = x$  and  $|Y| = y$  such that the vertices in  $X$  have degrees  $a_1, \dots, a_x$  and the vertices in  $Y$  have degrees  $b_1, \dots, b_y$ . Since  $s$  has the same parity as  $\sum_{i \in X} d_i + \sum_{i \in Y} d_i = d_1 + \dots + d_n$ , it is an even integer, and adding to  $H$  a matching of size  $s/2$  incident to those vertices in  $X$  corresponding to the entries of  $(d_i)_{i \in X}$  that were previously reduced by 1, results in a graph  $G$  with degree sequence  $d_1, \dots, d_n$ . Clearly,  $\chi(G) \leq 3$ , and the upper bound on  $\chi_{\min}(d)$  follows.  $\square$

The conclusion of Theorem 4 is best possible, because there might not be a subset  $X$  of  $[n]$  with  $\sum_{i \in X} d_i = \sum_{i \in [n] \setminus X} d_i$ , which is a necessary condition for the existence of a bipartite realization. The complexity of deciding the existence of a bipartite realization for a given degree sequence is unknown.

Note that together, Theorem 3 and Theorem 4 imply

$$\chi_{\min}(d) \leq \max \left\{ 3, \max(d) - \frac{n+1}{4 \max(d)} + 4 \right\}$$

for every degree sequence  $d$  with  $\min(d) \geq 1$ .

Theorem 4 has the following variant where the essential assumption is that  $\max(d) - \min(d)$  is small. Note that this next result also covers regular degree sequences of sufficient length.

**Theorem 5** *If  $n, d_1, \dots, d_n$  are integers and  $\epsilon > 0$  is such that  $\frac{n-1}{2}\epsilon \geq d_1 \geq \dots \geq d_n \geq 1$ ,  $d_1 - d_n \leq \sqrt{\frac{n-1}{2}}(1 - \epsilon)$ , and  $d_1 + \dots + d_n$  is even, then  $\chi_{\min}(d) \leq 3$ .*

*Proof:* We may assume that  $d_1 > \sqrt{\frac{n-1}{2}}$ ; otherwise Theorem 4 implies the result. Furthermore, we have  $\epsilon \leq 1$ . Exactly as in the proof of Theorem 4, we obtain the existence of a partition of  $[n]$  into two sets  $X$  and  $Y$  with  $||X| - |Y|| \leq 1$  and  $0 \leq s \leq d_1 \leq \frac{n-1}{2}\epsilon$ , where  $s = \sum_{i \in X} d_i - \sum_{i \in Y} d_i$ . Setting  $x = |X|$  and  $y = |Y|$ , we obtain, as above, that  $x, y \geq \frac{n-1}{2}$ ,  $s \leq x$ , and  $s$  is even. Let  $a_1 \geq \dots \geq a_x$  and  $b_1 \geq \dots \geq b_y$  be as in the proof of Theorem 4. By construction,  $\sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i$ ,  $\max\{a_1, b_1\} \leq d_1$ , and  $b_y \geq d_n$ .

Notice that as  $d_1 > \sqrt{\frac{n-1}{2}}$ , we have

$$\frac{d_n}{d_1} \geq \frac{d_1 - \sqrt{\frac{n-1}{2}}(1 - \epsilon)}{d_1} \geq 1 - (1 - \epsilon) = \epsilon.$$

Let  $k \in [x]$ . If  $k \leq d_n$ , then

$$\sum_{i \in [k]} a_i \leq kd_1 \leq k \frac{n-1}{2} \leq ky \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

If  $d_n < k < d_1$ , then

$$\sum_{i \in [k]} a_i \leq kd_1 \leq d_1^2 \leq \frac{n-1}{2} \epsilon d_1 \leq \frac{n-1}{2} d_n \leq yd_n \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

And, if  $k \geq d_1$ , then

$$\sum_{i \in [k]} a_i \leq \sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i = \sum_{i \in [y]} \min\{k, b_i\}.$$

At this point, the proof can be completed exactly as the proof of Theorem 4.  $\square$

For a graph  $G$  with degree sequence  $d_1 \geq \dots \geq d_n$ , Welsh and Powell [16] observed

$$\chi(G) \leq \max_{i \in [n]} \min\{i, d_i + 1\}, \quad (2)$$

which is an immediate consequence of applying the natural greedy coloring algorithm to the vertices of  $G$  in an order of non-increasing degrees. If  $d_1 \geq \dots \geq d_n$  is a degree sequence such that  $d_p - d_{p+1} \geq p - 2$  for  $p = \max_{i \in [n]} \min\{i, d_i + 1\}$ , then Havel and Hakimi's observation explained above implies the existence of a realization  $G$  of  $d$  for which the vertices of degrees  $d_1, \dots, d_p$  form a clique. This implies  $p \leq \chi(G) \leq \chi_{\max}(d) \leq p$ , that is,  $\chi_{\max}(d) = \max_{i \in [n]} \min\{i, d_i + 1\}$  for such degree sequences.

Our next result shows that the Welsh-Powell bound (2) also gives the correct value of  $\chi_{\max}(d)$  for degree sequences  $d$  of small degrees.

**Theorem 6** *If  $n, d_1, \dots, d_n$  are integers such that  $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq \dots \geq d_n \geq 1$  and  $d_1 + \dots + d_n$  is even, then  $\chi_{\max}(d) = \max_{i \in [n]} \min\{i, d_i + 1\}$ .*

*Proof:* Let  $p = \max_{i \in [n]} \min\{i, d_i + 1\}$ . Note that  $p \leq d_p + 1 \leq d_1 + 1$ .

By the Welsh-Powell bound (2), every graph  $G$  with degree sequence  $d_1, \dots, d_n$  satisfies  $\chi(G) \leq p$ , which implies  $\chi_{\max}(d) \leq p$ . In order to establish equality, we show the existence of a realization that contains a clique of size  $p$ .

Let  $k \in [n]$ . We obtain  $\sum_{i \in [k]} d_i \leq kd_1$  and  $k(k-1) + \sum_{i \in [n] \setminus [k]} \min\{k, d_i\} \geq k(k-1) + n - k$ . Therefore,  $\sum_{i \in [k]} d_i$  is at most  $k(k-1) + \sum_{i \in [n] \setminus [k]} \min\{k, d_i\}$  if  $kd_1 \leq k(k-1) + n - k$ , which is equivalent to  $k(d_1 + 2 - k) \leq n$ . Since  $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq 1$  implies  $n \geq 3$  and  $k(d_1 + 2 - k) \leq \left(\frac{d_1 + 2}{2}\right)^2 \leq n$ , the Erdős-Gallai Theorem [4] implies the existence of a graph with degree sequence  $d_1, \dots, d_n$ . Among all such graphs with vertex set  $\{v_1, \dots, v_n\}$ , where  $v_i$  has degree  $d_i$  for  $i \in [n]$ , we choose  $G$  such that the number  $m(G[\{v_1, \dots, v_p\}])$  of edges of the subgraph of  $G$  induced by  $\{v_1, \dots, v_p\}$  is as large as possible.

Suppose, for a contradiction, that  $G[\{v_1, \dots, v_p\}]$  is not a clique, that is,  $v_i$  and  $v_j$  are not adjacent in  $G$  for distinct  $i$  and  $j$  in  $[p]$ . By the choice of  $p$ , we have  $d_i, d_j \geq p - 1$ , which implies that  $v_i$  and  $v_j$  both have at least one neighbor in  $R = \{v_{p+1}, \dots, v_n\}$ .

First, we assume that  $v_i$  and  $v_j$  both have the same unique neighbor  $v_r$  in  $R$ , that is,  $\{v_r\} = N_G(v_i) \cap R = N_G(v_j) \cap R$ . Since there are at most  $1 + d_1^2$  vertices at distance at most 2



from  $v_r$ , including, in particular,  $v_i$  and  $v_j$ , and  $n - (p - 2) - (1 + d_1^2) \geq n - d_1^2 - d_1 > 0$ , there is a vertex  $v_s$  in  $R$  with a neighbor  $v_t$  such that  $v_s$  and  $v_t$  are both not adjacent to  $v_r$ . Now, removing from  $G$  the edges  $v_i v_r$ ,  $v_j v_r$ , and  $v_s v_t$ , and adding the edges  $v_i v_j$ ,  $v_r v_s$ , and  $v_r v_t$  yields a realization  $G'$  of  $d_1, \dots, d_n$  with  $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$ , which contradicts the choice of  $G$ .

Now, we may assume that  $v_i$  is adjacent to some vertex  $v_r$  in  $R$ , and that  $v_j$  is adjacent to a different vertex  $v_s$  in  $R$ . If  $v_r$  is not adjacent to  $v_s$ , then removing from  $G$  the edges  $v_i v_r$  and  $v_j v_s$ , and adding the edges  $v_i v_j$  and  $v_r v_s$  yields a realization  $G'$  of  $d_1, \dots, d_n$  with  $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$ , which contradicts the choice of  $G$ . Hence, we may assume that  $v_r$  and  $v_s$  are adjacent. Since there are at most  $1 + d_1^2$  vertices at distance at most 2 from  $v_r$ , including, in particular,  $v_i$ ,  $v_s$ , and  $v_j$ , and  $n - (p - 2) - (1 + d_1^2) \geq n - d_1^2 - d_1 > 0$ , there is a vertex  $v_p$  in  $R$  with a neighbor  $v_q$  such that  $v_p$  is not adjacent to  $v_s$ , and  $v_q$  is not adjacent to  $v_r$ . Note that  $v_q$  may be  $v_j$ , in which case,  $v_j$  has distance 2 from  $v_r$ . Now, removing from  $G$  the edges  $v_i v_r$ ,  $v_j v_s$ , and  $v_p v_q$ , and adding the edges  $v_i v_j$ ,  $v_s v_p$ , and  $v_r v_q$  yields a realization  $G'$  of  $d_1, \dots, d_n$  with  $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$ , which contradicts the choice of  $G$ .

Altogether, we obtain that  $G$  contains a clique of order  $p$ , which completes the proof.  $\square$

### 3 Algorithmic aspects

One way to establish that  $\chi_{\max}(d)$  is large is to show the existence of a realization of  $d$  that contains a large clique. Dvořák and Mohar [3] proved the best possible statement that for every degree sequence  $d$ , some realization of  $d$  has a clique of size at least  $5/6(\chi_{\max}(d) - 3/5)$ . Since Rao [12, 13] efficiently characterized the largest clique size  $\omega_{\max}(d)$  of any realization of a given degree sequence  $d$ , and, trivially,  $\chi_{\max}(d) \geq \omega_{\max}(d)$ , we immediately obtain that  $\chi_{\max}(d)$  can be approximated in polynomial time for a given  $d$  within an asymptotic factor of  $6/5$ .

Our next two results show that  $\chi_{\max}(d)$  and  $\chi_{\min}(d)$  can both be determined in polynomial time for given degree sequences with bounded entries.

**Corollary 7** *Let  $\Delta$  be a fixed positive integer.*

*For a given degree sequence  $d$  with  $\max(d) \leq \Delta$ , one can determine  $\chi_{\max}(d)$  in polynomial time.*

*Proof:* Let  $d$  have length  $n$ . Clearly, we may assume  $\min(d) \geq 1$ . If  $\sqrt{n-2} \geq \Delta$ , then Theorem 6 implies that  $\chi_{\max}(d)$  coincides with the Welsh-Powell bound (2). If  $\sqrt{n-2} < \Delta$ , then, as  $\Delta$  is fixed, there are only constantly many realizations of  $d$ , which can all be generated and optimally colored by brute force in constant time.  $\square$

**Theorem 8** *Let  $k$  and  $p$  be fixed positive integers.*

*For a given degree sequence  $d$  with at most  $p$  distinct entries, one can decide in polynomial time whether  $\chi_{\min}(d) \leq k$ .*

*Proof:* Let  $d : d_1^{n_1}, \dots, d_p^{n_p}$  and  $n = n_1 + \dots + n_p$ . There are  $\prod_{i=1}^p \binom{n_i+k-1}{k-1} \leq \left(\frac{n}{p} + k\right)^{kp}$  distinct matrices  $(n_i^j)_{(i,j) \in [p] \times [k]}$  with non-negative integral entries  $n_i^j$  such that  $\sum_{j=1}^k n_i^j = n_i$  for  $i \in [p]$ . It is easy to see that  $\chi_{\min}(d) \leq k$  if and only if there is such a matrix  $(n_i^j)_{(i,j) \in [p] \times [k]}$  for which the complete  $k$ -partite graph whose  $j$ th partite set  $V_j$  has order  $\sum_{i=1}^p n_i^j$  for  $j \in [k]$ , has a factor  $G$  such that  $V_j$  contains exactly  $n_i^j$  vertices of degree  $d_i$  in  $G$  for every  $i \in [p]$  and  $j \in [k]$ . Since the existence of such a factor can be decided in polynomial time using matching methods, and, for fixed  $k$  and  $p$ , there are only polynomially many different suitable matrices, the desired statement follows.  $\square$

It seems plausible to wonder whether  $\chi_{\max}(d)$  is linked to  $\alpha_{\min}(d)$ , the minimum independence number of a realization of  $d$ . While  $\alpha_{\max}(d) = \omega_{\max}(\bar{d})$  can be determined efficiently using the results of Rao [12, 13], Bauer, Hakimi, Kahl, and Schmeichel [1] conjectured that it is computationally hard to determine  $\alpha_{\min}(d)$  for a given degree sequence  $d$ .

Our next goal is to show that also  $\alpha_{\min}(d)$  can be determined in polynomial time for given degree sequences  $d$  with bounded entries.

For a degree sequence  $d_1, \dots, d_n$ , let  $\alpha_{CW}(d) = \sum_{i=1}^n \frac{1}{d_i+1}$ . Caro [2] and Wei [17] proved that  $\alpha(G) \geq \alpha_{CW}(d)$  for every graph  $G$  with degree sequence  $d$ . For a connected graph  $G$  with degree sequence  $d$ , Harant and Rautenbach [7] showed  $\alpha(G) \geq k \geq \sum_{u \in V(G)} \frac{1}{d_G(u)-f(u)+1}$ , where  $k$  is an integer, and, for every vertex  $u$  of  $G$ ,  $f(u)$  is a non-negative integer at most  $d_G(u)$  such that  $\sum_{u \in V(G)} f(u) \geq 2(k-1)$ . This improved an earlier result of Harant and Schiermeyer [8].

If  $\alpha_{CW}(d) \geq 2$ , then  $k \geq \alpha_{CW}(d)$  implies  $2(k-1) \geq k \geq \alpha_{CW}(d)$ , and, hence,

$$\begin{aligned} \alpha(G) &\geq \sum_{u \in V(G)} \frac{1}{d_G(u) - f(u) + 1} \\ &= \alpha_{CW}(d) + \sum_{u \in V(G)} \left( \frac{1}{d_G(u) - f(u) + 1} - \frac{1}{d_G(u) + 1} \right) \\ &\geq \alpha_{CW}(d) + \frac{1}{(\max(d) + 1)^2} \sum_{u \in V(G)} f(u) \\ &\geq \left( 1 + \frac{1}{(\max(d) + 1)^2} \right) \alpha_{CW}(d). \end{aligned}$$

**Theorem 9** *Let  $\Delta$  be a fixed positive integer.*

*For a given degree sequence  $d$  with  $\max(d) \leq \Delta$ , every component of every realization  $G$  of  $d$  with  $\alpha(G) = \alpha_{\min}(d)$  has order at most  $((\Delta + 1)^3 + 1) \left( \left( \frac{\Delta+2}{2} \right)^2 + \binom{\Delta+1}{2} \right)$ . In particular, one can determine  $\alpha_{\min}(d)$  in polynomial time.*

*Proof:* Let  $d$  be a degree sequence with  $\max(d) \leq \Delta$ . Let  $G$  be a realization of  $d$  with  $\alpha(G) = \alpha_{\min}(d)$ . Suppose, for a contradiction, that some component  $K$  of  $G$  has order  $n(K)$  more than the stated value. Let  $R$  be a set of  $\left( \frac{\Delta+2}{2} \right)^2$  vertices of  $K$ . For  $i \in [\Delta]$ , let  $V_i$  be the set of vertices of degree  $i$  in  $V(K) \setminus R$ , and let  $n_i = |V_i|$ . Let  $p_i = \lfloor \frac{n_i}{i+1} \rfloor$ , and let  $S_i$  arise by removing

$p_i(i+1)$  vertices from  $V_i$  for each  $i \in [\Delta]$ . Note that  $|S| \leq \sum_{i=1}^{\Delta} i = \binom{\Delta+1}{2}$ , where  $S = S_1 \cup \dots \cup S_{\Delta}$ , that is,  $R \cup S$  is a set of at least  $\left(\frac{\Delta+2}{2}\right)^2$  and at most  $\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}$  many vertices of  $K$ . Let  $d'$  be the sequence of the degrees of the vertices in  $R \cup S$ , and let  $d''$  be the sequence of the degrees of the vertices in  $V(K) \setminus (R \cup S)$ . Note that  $\alpha_{CW}(d'') \geq \frac{(n(K)-|R \cup S|)}{\Delta+1}$ . Hence, the lower bound on  $n(K)$  implies  $\left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d'') = \frac{1}{(\Delta+1)^2} \alpha_{CW}(d'') + \alpha_{CW}(d'') > |R \cup S| + \alpha_{CW}(d'')$ . As observed in the proof of Theorem 6, the Erdős-Gallai Theorem implies that the sequence  $d'$ , which is a sequence of positive integers at most  $\Delta$  that is of length at least  $\left(\frac{\Delta+2}{2}\right)^2$ , is a degree sequence. Let  $K'_0$  be a realization of  $d'$ . By construction, the graph  $K' = K'_0 \cup \bigcup_{i=1}^{\Delta} p_i K_{i+1}$  has exactly the same degree sequence as  $K$ . By the result of Harant and Rautenbach mentioned above,

$$\begin{aligned}
\alpha(K') &= \alpha(K'_0) + \sum_{i=1}^{\Delta} p_i \alpha(K_{i+1}) \\
&= \alpha(K'_0) + \alpha_{CW}(d'') \\
&\leq |R \cup S| + \alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d) \\
&\leq \alpha(K).
\end{aligned}$$

Therefore, replacing  $K$  by  $K'$  within  $G$  yields a realization  $G'$  of  $d$  with  $\alpha(G') < \alpha(G)$ , contradicting the choice of  $G$ . This completes the proof of the first part of the statement.

Since, as  $\Delta$  is fixed, there are only finitely many graphs of maximum degree at most  $\Delta$  and order at most  $((\Delta+1)^3 + 1) \left(\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}\right)$ . Listing, for each of these graphs, the degree sequence and the independence number, it is a routine matter to determine  $\alpha_{\min}(d)$  for a given degree sequence  $d$  with  $\max(d) \leq \Delta$  by dynamic programming in polynomial time.  $\square$

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