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Abstract

For a degree sequence $d : d_1 \geq \cdots \geq d_n$, we consider the smallest chromatic number $\chi_{\min}(d)$ and the largest chromatic number $\chi_{\max}(d)$ among all graphs with degree sequence d. We show that if $d_n \geq 1$, then $\chi_{\min}(d) \leq \max\left\{3, d_1 - \frac{n+1}{4d_1} + 4\right\}$, and, if $\sqrt{n+\frac{1}{4}} - \frac{1}{2} > d_1 \geq d_n \geq 1$, then $\chi_{\max}(d) = \max\limits_{i\in[n]} \min\{i, d_i + 1\}$. For a given degree sequence d with bounded entries, we show that $\chi_{\min}(d)$, $\chi_{\max}(d)$, and also the smallest independence number $\alpha_{\min}(d)$ among all graphs with degree sequence d, can be determined in polynomial time.

Keywords: Degree sequence; chromatic number; independence number

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1 Introduction

We consider finite, simple, and undirected graphs. The *degree sequence* of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is the sequence $d_G(v_1), \ldots, d_G(v_n)$ of its vertex degrees. A sequence $d_1, \ldots, d_n$ of integers is a *degree sequence* if it is the degree sequence of some graph. Repetitions within the degree sequence can be indicated by suitable exponents; the degree sequence of the star $K_{1,r}$ of order $r+1$, for instance, is $r, 1^r$. For a given sequence d, let $\mathcal{G}(d)$ be the set of all graphs G whose degree sequence is d; called the *realizations* of d. For an integer n, let $[n]$ be the set of the positive integers at most n.

In the present paper we consider

$$\chi_{\min}(d) = \min\{\chi(G) : G \in \mathcal{G}(d)\} \qquad \text{and} \qquad \chi_{\max}(d) = \max\{\chi(G) : G \in \mathcal{G}(d)\}.$$

Punnim [11] determined $\chi_{\min}(d)$ and $\chi_{\max}(d)$ for regular degree sequences $d = r^n$ in almost all cases. The parameter $\chi_{\max}(d)$ was also considered by Dvořák and Mohar [3], who established degree sequence versions of the Hadwiger Conjecture and even the Hajós Conjecture, see also [14].

We contribute some bounds, exact values, and algorithmic results. Further discussion of related research will be given throughout the rest of the paper.

2 Some bounds and exact values

For a sequence d of non-negative integers $d_1 \geq \cdots \geq d_n$, let $H(d)$ be the sequence

$$d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

Havel [9] and Hakimi [6] showed that d is a degree sequence if and only if $H(d)$ is a degree sequence. In fact, they observed that if d is a degree sequence, then there is a realization G of d in which the neighbours of a vertex of degree d_1 have degrees $d_2, \ldots, d_{d_1+1}$. Iteratively applying this observation to a given degree sequence yields a realization that tends to contain a large complete subgraph on the vertices of large degrees, that is, such a realization may be expected to have high chromatic number.

In order to obtain a realization with hopefully small chromatic number, one can apply Havel and Hakimi's observation to the complement. More precisely, for a degree sequence d as above, the sequence $\bar{d}$ defined as

$$n - 1 - d_n \geq \cdots \geq n - 1 - d_1$$

is also a degree sequence; in fact, the graphs in $\mathcal{G}\left(\bar{d}\right)$ are exactly the complements $\bar{G}$ of the graphs G in $\mathcal{G}(d)$. Furthermore, by the above observation of Havel and Hakimi, $\bar{d}$ has a realization in which the neighbors of a vertex of the largest degree $n-1-d_n$ have degrees $n-1-d_{n-1}, \ldots, n-1-d_{d_n+1}$. Equivalently, as already observed by Kleitman and Wang [10] in a more general form, d has a realization in which the neighbors of a vertex of the smallest degree d_n have degrees $d_1,\ldots,d_{d_n}$. In summary, we obtain that d is a degree sequence if and only if the sequence $\bar{H}(d)$ defined as

$$d_1-1,\ldots,d_{d_n}-1,d_{d_n+1},\ldots,d_{n-1} \tag{1}$$

is a degree sequence. Iteratively applying this observation to a given degree sequence yields a realization that tends to avoid dense subgraphs on the vertices of large degrees, that is, such a realization may be expected to have small chromatic number.

As an example consider the degree sequence $d : r^{r+1}, 1^{r(r+1)}$ for some positive integer r. Havel and Hakimi's original observation yields the realization $K_{r+1} \cup \binom{r}{2} K_2$, whose chromatic number is $r+1$, which equals $\chi_{\max}(d)$, while the above complementary version yields the realization $(r+1)K_{1,r}$, whose chromatic number is 2, which equals $\chi_{\min}(d)$.

For a sequence d of integers $d_1,\ldots,d_n$, let n be the *length* of d, let $\min(d)=\min\{d_1,\ldots,d_n\}$, and let $\max(d)=\max\{d_1,\ldots,d_n\}$. Furthermore, let $\bar{H}^0(d)=d$, $\bar{H}^1(d)=\bar{H}(d)$, and $\bar{H}^i(d)=\bar{H}\left(\bar{H}^{i-1}(d)\right)$ for an integer i at least 2. Note that iteratively applying the reductions $d\mapsto H(d)$ or $d\mapsto \bar{H}(d)$ always requires reordering the constructed sequences in a non-increasing way.

Theorem 1 *If d is a degree sequence of length n, then*

$$\chi_{\min}(d) \leq \max\left\{\min\left(\bar{H}^{n-i}(d)\right) : i\in[n]\right\}+1.$$

Proof: Iteratively applying the complementary version of Havel and Hakimi's observation to the degree sequence d yields a realization G of d with vertex set $\{v_1,\ldots,v_n\}$ such that, for i from n down to 1, the vertex v_i has degree $\min\left(\bar{H}^{n-i}(d)\right)$ in the graph $G[\{v_1,\ldots,v_i\}]$. Greedily coloring the vertices of G in the order $v_1,\ldots,v_n$ yields a coloring that uses at most $\max\left\{\min\left(\bar{H}^{n-i}(d)\right) : i\in[n]\right\}+1$ colors. $\Box$

Note that for the degree sequence $d : r^{r+1}, 1^{r(r+1)}$ of length $n=(r+1)^2$ considered as an example above, we obtain $\max\left\{\min\left(\bar{H}^{n-i}\left(r^{r+1},1^{r(r+1)}\right)\right) : i\in[n]\right\}+1=2$, that is, for this degree sequence d, Theorem 1 reproduces the correct value of $\chi_{\min}(d)$.

Unfortunately, Theorem 1 is not very explicit. As a more explicit consequence, we quantify how small degrees may reduce the effect of large degrees on $\chi_{\min}(d)$.

Corollary 2 *If d is a degree sequence $d_1\geq\ldots\geq d_n$, and k and ℓ are positive integers such that $d_k\geq k+\ell$ and $d_{n-\ell+1}\leq k$, then*

$$\chi_{\min}(d) \leq \max\left\{d_1-\frac{1}{k}\left(1+\sum_{i=n-\ell+1}^{n} d_i\right)+1, d_{k+1}, k\right\}+1.$$

Proof: We consider the first ℓ applications of the reduction $d\mapsto\bar{H}(d)$. Since $d_k\geq k+\ell$ and $d_{n-\ell+1}\leq k$, we obtain that, for $i\in[\ell]$, the degree sequence $\bar{H}^i(d)$ arises from $\bar{H}^{i-1}(d)$ by removing the degree d_{n-i+1}, and reducing the d_{n-i+1} largest degrees by 1. For $i\in\{0,\ldots,\ell\}$, let $\Delta_i=\max\left(\bar{H}^i(d)\right)$, and let n_i be the number of entries of $\bar{H}^i(d)$ that are equal to Δ_i. Suppose, for a contradiction, that $\Delta_\ell > \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$, where $D = \sum\limits_{i=n-\ell+1}^{n} d_i$. Note that each of the $\ell + 1$ degree sequences $d, \bar{H}(d), \ldots, \bar{H}^\ell(d)$ contains at most k entries that are strictly larger than d_{k+1}. So, for $i \in [\ell]$, we have

- $(\Delta_i, n_i) = (\Delta_{i-1}, n_{i-1} - d_{n-i+1})$ if $d_{n-i+1} < n_{i-1}$, and
- $\Delta_i = \Delta_{i-1} - 1$ and $n_i \leq k - (d_{n-i+1} - n_{i-1}) = n_{i-1} - d_{n-i+1} + k$ if $d_{n-i+1} \geq n_{i-1}$.

Note that $(k\Delta_{i-1} + n_{i-1}) - (k\Delta_i + n_i) \geq d_{n-i+1}$ in both cases. Summation over $i \in [\ell]$ yields $(k\Delta_0 + n_0) - (k\Delta_\ell + n_\ell) \geq D$. Since $\Delta_0 = d_1$, $n_0 \leq k$, and $n_\ell \geq 1$, this implies $\Delta_\ell \leq d_1 - \frac{D+1}{k} + 1$, which is a contradiction. Hence, $\Delta_\ell \leq \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$, and any realization H of the degree sequence $\bar{H}^\ell(d)$ can be colored using at most $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\} + 1$ many colors. Adding ℓ further vertices of degrees $d_{n-\ell+1}, \ldots, d_n$ one by one to H, and connecting them to suitable vertices according to the previous reductions, yields a realization G of d. Since the added vertices all have degree at most k, the coloring of H can be extended greedily to a coloring of G using at most $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}, k\right\} + 1$ different colors in total. □

For a given degree sequence d not satisfying any further restriction, one can only bound $\chi_{\min}(d)$ from above by $\max(d)+1$. In fact, d might be $\max(d)^{\max(d)+1}, 0^{n-\max(d)-1}$, whose only realization contains a clique of size $\max(d) + 1$.

Our next two results improve this trivial estimate for graphs without isolated vertices.

Theorem 3 *If d is a degree sequence of length n with $\max(d) \geq \sqrt{\frac{n\delta}{4}}$ and $\min(d) \geq \delta$ for some positive integer δ, then $\chi_{\min}(d) \leq \max(d) - \frac{n\delta}{4\max(d)} + \delta + 3$.*

Proof: Our first goal is to show that we may assume that d has a realization with a very large independent set. Therefore, among all realizations G of the degree sequence d and all (not necessarily optimal) colorings f of G, we choose G and f with color classes $V_1, \ldots, V_k$, where V_i contains n_i vertices for $i \in [k]$, in such a way that

- $(n_1, \ldots, n_k)$ is lexicographically maximal, and
- subject to this first condition, the number of edges between V_{k-1} and V_k is minimum.

Note that k may actually be larger than $\chi(G)$, and that n_1 is necessarily equal to the independence number $\alpha(G)$ of G.

Let $\Delta = \max(d)$. If $k \leq \Delta - \frac{n\delta}{4\Delta} + \delta + 3$, then $\chi_{\min}(d) \leq \chi(G) \leq k$ implies the desired bound. Hence, we may assume that $k > \Delta - \frac{n\delta}{4\Delta} + \delta + 3$. Since $\Delta \geq \sqrt{\frac{n\delta}{4}}$ and $\delta \geq 1$, we have $k \geq 5$. By the choice of the coloring f, there is an edge, say uv, between the smallest two color classes V_{k-1} and V_k. If $G \setminus (V_{k-1} \cup V_k \cup N_G(u) \cup N_G(v))$ contains an edge xy, then removing from G the two edges uv and xy, and adding the two edges ux and vy, yields another realization G' of d. Note that f is still a coloring of G'. This implies that there is a coloring f' of G' such that either the non-increasing vector of the sizes of the color classes is lexicographically larger than the one of f, or there are fewer edges between the two smallest color classes. Since both cases imply a contradiction to the choice of G and f, we obtain that $V(G)\setminus(V_{k-1}\cup V_k\cup N_G(u)\cup N_G(v))$ is an independent set, which implies $\alpha(G)\geq n-(n_{k-1}+n_k)-2\Delta$. Since V_{k-1} and V_k are the smallest two color classes, and $n_2+\cdots+n_k=n-\alpha(G)$, we obtain $n_{k-1}+n_k\leq\frac{2}{k-1}(n-\alpha(G))$. This implies $\alpha(G)\geq n-\frac{2}{k-1}(n-\alpha(G))-2\Delta$, and, using $k\geq 5$, we obtain $\alpha(G)\geq n-\frac{k-1}{k-3}\cdot 2\Delta\geq n-4\Delta$.

Altogether, we may assume that d has a realization G with an independent set $I=\{u_1,\ldots,u_\alpha\}$ of order at least $n-4\Delta$. By the above-mentioned observations of Havel [9], Hakimi [6], Rao [12], and Kleitman and Wang [10], we may further assume that, for every $i\in[\alpha]$, the vertex u_i is adjacent to $d_G(u_i)$ vertices in $V(G)\setminus I$ of the largest degrees in the induced subgraph $G-\{u_1,\ldots,u_{i-1}\}$ of G. Arguing as in the proof of Corollary 2, we obtain $\Big((n-\alpha)\Delta+(n-\alpha)\Big)-\Big((n-\alpha)\Delta(G-I)+1\Big)\geq d_G(u_1)+\cdots+d_G(u_\alpha)\geq\alpha\delta$, where $\Delta(G-I)$ denotes the maximum degree of $G-I$. This implies $\Delta(G-I)\leq\Delta-\frac{\alpha\delta+1}{n-\alpha}+1\leq\Delta-\frac{(n-4\Delta)\delta+1}{4\Delta}+1=\Delta-\frac{n\delta+1}{4\Delta}+\delta+1$. Therefore, we can color G using at most $\Delta-\frac{n\delta+1}{4\Delta}+\delta+2$ colors on the vertices in $V(G)\setminus I$, and one additional color on the vertices in I, which implies $\chi_{\min}(d)\leq\chi(G)\leq\Delta-\frac{n\delta+1}{4\Delta}+\delta+3$. $\square$

For positive integers r, s, and δ such that $r+1$ is a multiple of δ, let d be the degree sequence $(r+s)^{r+1},\delta^{s(r+1)/\delta}$. Since the sum of the largest $r+1$ degrees equals exactly $2\binom{r+1}{2}+\delta s(r+1)/\delta$, every realization G of d contains a clique on the $r+1$ vertices of largest degrees, and an independent set on the remaining vertices. Note that $\chi(G)\in\{r+1,r+2\}$, which, for $r\gg s\gg\delta$, is roughly $\max(d)-\frac{n\min(d)}{\max(d)}$, that is, up to the constants, the bound in Theorem 3 is best possible. In fact, by imposing a stronger lower bound on $\max(d)$ or by increasing the additive constant, the factor 4 within the term $\frac{n\delta+1}{4\Delta}$ can easily be reduced to slightly more than 2.

Our next result gives a best possible bound on $\chi_{\min}(d)$ for degree sequences of small degrees.

Theorem 4 *If $n,d_1,\ldots,d_n$ are integers such that $\sqrt{\frac{n-1}{2}}\geq d_1\geq\cdots\geq d_n\geq 1$ and $d_1+\cdots+d_n$ is even, then $\chi_{\min}(d)\leq 3$. (In particular, $d_1,\ldots,d_n$ is a degree sequence.)*

Proof: There is a partition of $[n]$ into two sets X and Y with $\Big||X|-|Y|\Big|\leq 1$ and $0\leq s\leq d_1\leq\sqrt{\frac{n-1}{2}}$, where $s=\sum\limits_{i\in X}d_i-\sum\limits_{i\in Y}d_i$; in fact, as long as there are two equal entries d_i and d_j in the sequence $d_1,\ldots,d_n$, we assign i to X and j to Y, and remove d_i and d_j from the sequence, and once all remaining entries are distinct, say $d_{i_1}>\cdots>d_{i_k}$, we assign $i_1,i_3,\ldots$ to X and $i_2,i_4,\ldots$ to Y. Let $x=|X|$ and $y=|Y|$. Note that $x,y\geq\frac{n-1}{2}$; in particular, $s\leq x$. Reducing s distinct entries of the sequence $(d_i)_{i\in X}$ by 1, and reordering yields a sequence $a_1\geq\cdots\geq a_x$. Reordering the sequence $(d_i)_{i\in Y}$ yields $b_1\geq\cdots\geq b_y$.

By construction, $\sum\limits_{i\in[x]}a_i=\sum\limits_{i\in[y]}b_i$, $\max\{a_1,b_1\}\leq\sqrt{\frac{n-1}{2}}$, and $b_y\geq 1$.

Let $k\in[x]$. If $k\leq\sqrt{\frac{n-1}{2}}$, then $a_1\leq\sqrt{\frac{n-1}{2}}$ and $b_n\geq 1$ imply

$$\sum_{i\in[k]}a_i\leq ka_1\leq\frac{n-1}{2}\leq y\leq\sum_{i\in[y]}\min\{k,b_i\}.$$

If $k > \sqrt{\frac{n-1}{2}}$, then $b_1 \leq \sqrt{\frac{n-1}{2}}$ implies

$$\sum_{i\in[k]} a_i \leq \sum_{i\in[x]} a_i = \sum_{i\in[y]} b_i = \sum_{i\in[y]} \min\{k, b_i\}.$$

By the Gale-Ryser Theorem [5, 15], there is a bipartite graph H with partite sets X and Y with $|X| = x$ and $|Y| = y$ such that the vertices in X have degrees $a_1, \ldots, a_x$ and the vertices in Y have degrees $b_1, \ldots, b_y$. Since s has the same parity as $\sum\limits_{i\in X} d_i + \sum\limits_{i\in Y} d_i = d_1 + \cdots + d_n$, it is an even integer, and adding to H a matching of size $s/2$ incident to those vertices in X corresponding to the entries of $(d_i)_{i\in X}$ that were previously reduced by 1, results in a graph G with degree sequence $d_1, \ldots, d_n$. Clearly, $\chi(G) \leq 3$, and the upper bound on $\chi_{\min}(d)$ follows. $\square$

The conclusion of Theorem 4 is best possible, because there might not be a subset X of $[n]$ with $\sum\limits_{i\in X} d_i = \sum\limits_{i\in[n]\setminus X} d_i$, which is a necessary condition for the existence of a bipartite realization. The complexity of deciding the existence of a bipartite realization for a given degree sequence is unknown.

Note that together, Theorem 3 and Theorem 4 imply

$$\chi_{\min}(d) \leq \max\left\{3, \max(d) - \frac{n+1}{4\max(d)} + 4\right\}$$

for every degree sequence d with $\min(d) \geq 1$.

Theorem 4 has the following variant where the essential assumption is that $\max(d) - \min(d)$ is small. Note that this next result also covers regular degree sequences of sufficient length.

Theorem 5 *If $n, d_1, \ldots, d_n$ are integers and $\epsilon > 0$ is such that $\frac{n-1}{2}\epsilon \geq d_1 \geq \cdots \geq d_n \geq 1$, $d_1 - d_n \leq \sqrt{\frac{n-1}{2}}(1-\epsilon)$, and $d_1 + \cdots + d_n$ is even, then $\chi_{\min}(d) \leq 3$.*

Proof: We may assume that $d_1 > \sqrt{\frac{n-1}{2}}$; otherwise Theorem 4 implies the result. Furthermore, we have $\epsilon \leq 1$. Exactly as in the proof of Theorem 4, we obtain the existence of a partition of $[n]$ into two sets X and Y with $\big||X| - |Y|\big| \leq 1$ and $0 \leq s \leq d_1 \leq \frac{n-1}{2}\epsilon$, where $s = \sum\limits_{i\in X} d_i - \sum\limits_{i\in Y} d_i$. Setting $x = |X|$ and $y = |Y|$, we obtain, as above, that $x, y \geq \frac{n-1}{2}$, $s \leq x$, and s is even. Let $a_1 \geq \cdots \geq a_x$ and $b_1 \geq \cdots \geq b_y$ be as in the proof of Theorem 4. By construction, $\sum\limits_{i\in[x]} a_i = \sum\limits_{i\in[y]} b_i$, $\max\{a_1, b_1\} \leq d_1$, and $b_y \geq d_n$.

Notice that as $d_1 > \sqrt{\frac{n-1}{2}}$, we have

$$\frac{d_n}{d_1} \geq \frac{d_1 - \sqrt{\frac{n-1}{2}}(1-\epsilon)}{d_1} \geq 1 - (1-\epsilon) = \epsilon.$$

Let $k \in [x]$. If $k \leq d_n$, then

$$\sum_{i\in[k]} a_i \leq kd_1 \leq k\frac{n-1}{2} \leq ky \leq \sum_{i\in[y]} \min\{k, b_i\}.$$

If $d_n < k < d_1$, then

$$\sum_{i\in[k]} a_i \leq kd_1 \leq d_1^2 \leq \frac{n-1}{2}\epsilon d_1 \leq \frac{n-1}{2}d_n \leq yd_n \leq \sum_{i\in[y]} \min\{k, b_i\}.$$

And, if $k \geq d_1$, then

$$\sum_{i\in[k]} a_i \leq \sum_{i\in[x]} a_i = \sum_{i\in[y]} b_i = \sum_{i\in[y]} \min\{k, b_i\}.$$

At this point, the proof can be completed exactly as the proof of Theorem 4. $\square$

For a graph G with degree sequence $d_1 \geq \cdots \geq d_n$, Welsh and Powell [16] observed

$$\chi(G) \leq \max_{i\in[n]} \min\{i, d_i + 1\}, \tag{2}$$

which is an immediate consequence of applying the natural greedy coloring algorithm to the vertices of G in an order of non-increasing degrees. If $d_1 \geq \cdots \geq d_n$ is a degree sequence such that $d_p - d_{p+1} \geq p-2$ for $p = \max_{i\in[n]} \min\{i, d_i+1\}$, then Havel and Hakimi's observation explained above implies the existence of a realization G of d for which the vertices of degrees $d_1, \ldots, d_p$ form a clique. This implies $p \leq \chi(G) \leq \chi_{\max}(d) \leq p$, that is, $\chi_{\max}(d) = \max_{i\in[n]} \min\{i, d_i+1\}$ for such degree sequences.

Our next result shows that the Welsh-Powell bound (2) also gives the correct value of $\chi_{\max}(d)$ for degree sequences d of small degrees.

Theorem 6 *If $n, d_1, \ldots, d_n$ are integers such that $\sqrt{n+\frac{1}{4}} - \frac{1}{2} > d_1 \geq \cdots \geq d_n \geq 1$ and $d_1 + \cdots + d_n$ is even, then $\chi_{\max}(d) = \max_{i\in[n]} \min\{i, d_i+1\}$.*

Proof: Let $p = \max_{i\in[n]} \min\{i, d_i+1\}$. Note that $p \leq d_p + 1 \leq d_1 + 1$.

By the Welsh-Powell bound (2), every graph G with degree sequence $d_1, \ldots, d_n$ satisfies $\chi(G) \leq p$, which implies $\chi_{\max}(d) \leq p$. In order to establish equality, we show the existence of a realization that contains a clique of size p.

Let $k \in [n]$. We obtain $\sum_{i\in[k]} d_i \leq kd_1$ and $k(k-1) + \sum_{i\in[n]\setminus[k]} \min\{k, d_i\} \geq k(k-1) + n - k$. Therefore, $\sum_{i\in[k]} d_i$ is at most $k(k-1) + \sum_{i\in[n]\setminus[k]} \min\{k, d_i\}$ if $kd_1 \leq k(k-1) + n - k$, which is equivalent to $k(d_1 + 2 - k) \leq n$. Since $\sqrt{n+\frac{1}{4}} - \frac{1}{2} > d_1 \geq 1$ implies $n \geq 3$ and $k(d_1+2-k) \leq \left(\frac{d_1+2}{2}\right)^2 \leq n$, the Erdős-Gallai Theorem [4] implies the existence of a graph with degree sequence $d_1, \ldots, d_n$. Among all such graphs with vertex set $\{v_1, \ldots, v_n\}$, where v_i has degree d_i for $i \in [n]$, we choose G such that the number $m(G[\{v_1, \ldots, v_p\}])$ of edges of the subgraph of G induced by $\{v_1, \ldots, v_p\}$ is as large as possible.

Suppose, for a contradiction, that $G[\{v_1, \ldots, v_p\}]$ is not a clique, that is, v_i and v_j are not adjacent in G for distinct i and j in $[p]$. By the choice of p, we have $d_i, d_j \geq p-1$, which implies that v_i and v_j both have at least one neighbor in $R = \{v_{p+1}, \ldots, v_n\}$.

First, we assume that v_i and v_j both have the same unique neighbor v_r in R, that is, $\{v_r\} = N_G(v_i) \cap R = N_G(v_j) \cap R$. Since there are at most $1 + d_1^2$ vertices at distance at most 2 from v_r, including, in particular, v_i and v_j, and $n-(p-2)-(1+d_1^2)\geq n-d_1^2-d_1>0$, there is a vertex v_s in R with a neighbor v_t such that v_s and v_t are both not adjacent to v_r. Now, removing from G the edges v_iv_r, v_jv_r, and v_sv_t, and adding the edges v_iv_j, v_rv_s, and v_rv_t yields a realization G' of $d_1,\ldots,d_n$ with $m(G'[\{v_1,\ldots,v_p\}])>m(G[\{v_1,\ldots,v_p\}])$, which contradicts the choice of G.

Now, we may assume that v_i is adjacent to some vertex v_r in R, and that v_j is adjacent to a different vertex v_s in R. If v_r is not adjacent to v_s, then removing from G the edges v_iv_r and v_jv_s, and adding the edges v_iv_j and v_rv_s yields a realization G' of $d_1,\ldots,d_n$ with $m(G'[\{v_1,\ldots,v_p\}])>m(G[\{v_1,\ldots,v_p\}])$, which contradicts the choice of G. Hence, we may assume that v_r and v_s are adjacent. Since there are at most $1+d_1^2$ vertices at distance at most 2 from v_r, including, in particular, v_i, v_s, and v_j, and $n-(p-2)-(1+d_1^2)\geq n-d_1^2-d_1>0$, there is a vertex v_p in R with a neighbor v_q such that v_p is not adjacent to v_s, and v_q is not adjacent to v_r. Note that v_q may be v_j, in which case, v_j has distance 2 from v_r. Now, removing from G the edges v_iv_r, v_jv_s, and v_pv_q, and adding the edges v_iv_j, v_sv_p, and v_rv_q yields a realization G' of $d_1,\ldots,d_n$ with $m(G'[\{v_1,\ldots,v_p\}])>m(G[\{v_1,\ldots,v_p\}])$, which contradicts the choice of G.

Altogether, we obtain that G contains a clique of order p, which completes the proof. □

3 Algorithmic aspects

One way to establish that $\chi_{\max}(d)$ is large is to show the existence of a realization of d that contains a large clique. Dvořák and Mohar [3] proved the best possible statement that for every degree sequence d, some realization of d has a clique of size at least $5/6(\chi_{\max}(d)-3/5)$. Since Rao [12, 13] efficiently characterized the largest clique size $\omega_{\max}(d)$ of any realization of a given degree sequence d, and, trivially, $\chi_{\max}(d)\geq\omega_{\max}(d)$, we immediately obtain that $\chi_{\max}(d)$ can be approximated in polynomial time for a given d within an asymptotic factor of $6/5$.

Our next two results show that $\chi_{\max}(d)$ and $\chi_{\min}(d)$ can both be determined in polynomial time for given degree sequences with bounded entries.

Corollary 7 *Let Δ be a fixed positive integer.*

For a given degree sequence d with $\max(d)\leq\Delta$, one can determine $\chi_{\max}(d)$ in polynomial time.

Proof: Let d have length n. Clearly, we may assume $\min(d)\geq 1$. If $\sqrt{n-2}\geq\Delta$, then Theorem 6 implies that $\chi_{\max}(d)$ coincides with the Welsh-Powell bound (2). If $\sqrt{n-2}<\Delta$, then, as Δ is fixed, there are only constantly many realizations of d, which can all be generated and optimally colored by brute force in constant time. □

Theorem 8 *Let k and p be fixed positive integers.*

For a given degree sequence d with at most p distinct entries, one can decide in polynomial time whether $\chi_{\min}(d)\leq k$.

Proof: Let $d : d_1^{n_1}, \ldots, d_p^{n_p}$ and $n = n_1 + \cdots + n_p$. There are $\prod\limits_{i=1}^{p} \binom{n_i+k-1}{k-1} \leq \left(\frac{n}{p} + k\right)^{kp}$ distinct matrices $(n_i^j)_{(i,j)\in[p]\times[k]}$ with non-negative integral entries n_i^j such that $\sum\limits_{j=1}^{k} n_i^j = n_i$ for $i \in [p]$. It is easy to see that $\chi_{\min}(d) \leq k$ if and only if there is such a matrix $(n_i^j)_{(i,j)\in[p]\times[k]}$ for which the complete k-partite graph whose jth partite set V_j has order $\sum\limits_{i=1}^{p} n_i^j$ for $j \in [k]$, has a factor G such that V_j contains exactly n_i^j vertices of degree d_i in G for every $i \in [p]$ and $j \in [k]$. Since the existence of such a factor can be decided in polynomial time using matching methods, and, for fixed k and p, there are only polynomially many different suitable matrices, the desired statement follows. $\Box$

It seems plausible to wonder whether $\chi_{\max}(d)$ is linked to $\alpha_{\min}(d)$, the minimum independence number of a realization of d. While $\alpha_{\max}(d) = \omega_{\max}\left(\bar{d}\right)$ can be determined efficiently using the results of Rao [12, 13], Bauer, Hakimi, Kahl, and Schmeichel [1] conjectured that it is computationally hard to determine $\alpha_{\min}(d)$ for a given degree sequence d.

Our next goal is to show that also $\alpha_{\min}(d)$ can be determined in polynomial time for given degree sequences d with bounded entries.

For a degree sequence $d_1, \ldots, d_n$, let $\alpha_{CW}(d) = \sum\limits_{i=1}^{n} \frac{1}{d_i+1}$. Caro [2] and Wei [17] proved that $\alpha(G) \geq \alpha_{CW}(d)$ for every graph G with degree sequence d. For a connected graph G with degree sequence d, Harant and Rautenbach [7] showed $\alpha(G) \geq k \geq \sum\limits_{u\in V(G)} \frac{1}{d_G(u)-f(u)+1}$, where k is an integer, and, for every vertex u of G, $f(u)$ is a non-negative integer at most $d_G(u)$ such that $\sum\limits_{u\in V(G)} f(u) \geq 2(k-1)$. This improved an earlier result of Harant and Schiermeyer [8].

If $\alpha_{CW}(d) \geq 2$, then $k \geq \alpha_{CW}(d)$ implies $2(k-1) \geq k \geq \alpha_{CW}(d)$, and, hence,

$$\begin{aligned}
\alpha(G) &\geq \sum_{u\in V(G)} \frac{1}{d_G(u) - f(u) + 1} \\
&= \alpha_{CW}(d) + \sum_{u\in V(G)} \left(\frac{1}{d_G(u) - f(u) + 1} - \frac{1}{d_G(u) + 1} \right) \\
&\geq \alpha_{CW}(d) + \frac{1}{(\max(d) + 1)^2} \sum_{u\in V(G)} f(u) \\
&\geq \left(1 + \frac{1}{(\max(d) + 1)^2}\right) \alpha_{CW}(d).
\end{aligned}$$

Theorem 9 *Let Δ be a fixed positive integer.*

For a given degree sequence d with $\max(d) \leq \Delta$, every component of every realization G of d with $\alpha(G) = \alpha_{\min}(d)$ has order at most $((\Delta+1)^3 + 1)\left(\binom{\Delta+2}{2}^2 + \binom{\Delta+1}{2}\right)$. In particular, one can determine $\alpha_{\min}(d)$ in polynomial time.

Proof: Let d be a degree sequence with $\max(d) \leq \Delta$. Let G be a realization of d with $\alpha(G) = \alpha_{\min}(d)$. Suppose, for a contradiction, that some component K of G has order $n(K)$ more than the stated value. Let R be a set of $\binom{\Delta+2}{2}^2$ vertices of K. For $i \in [\Delta]$, let V_i be the set of vertices of degree i in $V(K) \setminus R$, and let $n_i = |V_i|$. Let $p_i = \left\lfloor \frac{n_i}{i+1} \right\rfloor$, and let S_i arise by removing

$p_i(i+1)$ vertices from V_i for each $i \in [\Delta]$. Note that $|S| \leq \sum_{i=1}^{\Delta} i = \binom{\Delta+1}{2}$, where $S = S_1 \cup \cdots \cup S_\Delta$, that is, $R \cup S$ is a set of at least $\left(\frac{\Delta+2}{2}\right)^2$ and at most $\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}$ many vertices of K. Let d' be the sequence of the degrees of the vertices in $R \cup S$, and let d'' be the sequence of the degrees of the vertices in $V(K) \setminus (R \cup S)$. Note that $\alpha_{CW}(d'') \geq \frac{(n(K)-|R\cup S|)}{\Delta+1}$. Hence, the lower bound on $n(K)$ implies $\left(1 + \frac{1}{(\Delta+1)^2}\right)\alpha_{CW}(d'') = \frac{1}{(\Delta+1)^2}\alpha_{CW}(d'') + \alpha_{CW}(d'') > |R \cup S| + \alpha_{CW}(d'')$. As observed in the proof of Theorem 6, the Erdős-Gallai Theorem implies that the sequence d', which is a sequence of positive integers at most Δ that is of length at least $\left(\frac{\Delta+2}{2}\right)^2$, is a degree sequence. Let K_0' be a realization of d'. By construction, the graph $K' = K_0' \cup \bigcup_{i=1}^{\Delta} p_i K_{i+1}$ has exactly the same degree sequence as K. By the result of Harant and Rautenbach mentioned above,

$$
\begin{aligned}
\alpha(K') &= \alpha(K_0') + \sum_{i=1}^{\Delta} p_i \alpha(K_{i+1}) \\
&= \alpha(K_0') + \alpha_{CW}(d'') \\
&\leq |R \cup S| + \alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right)\alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right)\alpha_{CW}(d) \\
&\leq \alpha(K).
\end{aligned}
$$

Therefore, replacing K by K' within G yields a realization G' of d with $\alpha(G') < \alpha(G)$, contradicting the choice of G. This completes the proof of the first part of the statement.

Since, as Δ is fixed, there are only finitely many graphs of maximum degree at most Δ and order at most $((\Delta+1)^3+1)\left(\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}\right)$. Listing, for each of these graphs, the degree sequence and the independence number, it is a routine matter to determine $\alpha_{\min}(d)$ for a given degree sequence d with $\max(d) \leq \Delta$ by dynamic programming in polynomial time. □

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