A lower bound on the order of the largest induced linear forest in triangle-free planar graphs

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François Dross[a], Mickael Montassier[a], and Alexandre Pinlou[b]

[a] *Université de Montpellier, LIRMM*
[b] *Université Paul-Valéry Montpellier 3, LIRMM*

161 rue Ada, 34095 Montpellier Cedex 5, France
{francois.dross,mickael.montassier,alexandre.pinlou}@lirmm.fr

Abstract

We prove that every triangle-free planar graph of order n and size m has an induced linear forest with at least $\frac{9n-2m}{11}$ vertices, and thus at least $\frac{5n+8}{11}$ vertices. Furthermore, we show that there are triangle-free planar graphs on n vertices whose largest induced linear forest has order $\lceil \frac{n}{2} \rceil + 1$.

1 Introduction

In this extended abstract, we only consider simple finite graphs.

Albertson and Berman [3] conjectured that every planar graph admits an induced forest on at least half of its vertices. This conjecture, if true, would be tight, as shown by the disjoint union of copies of the complete graph on four vertices. Moreover, it would imply that every planar graph admits an independent set on at least one fourth of its vertices, the only known proof of which relies on the Four Colour Theorem. However, this conjecture appears to be very hard to prove. The best known result for planar graphs is that every planar graph admits an induced forest on at least two fifths of its vertices, as a consequence of the theorem of 5-acyclic colourability of planar graphs of Borodin [4].

Akiyama and Watanabe [1], and Albertson and Rhaas [2] independently conjectured that every bipartite planar graph admits an induced forest on at least five eighths of its vertices, which is tight. For triangle-free planar graphs (and thus in particular for bipartite planar graphs), it is known that every triangle-free planar graph of order n and size m admits an induced forest of order at least $(38n - 7m)/44$, and thus at least $(6n + 7)/11$ [6].

An interesting variant of this problem is to look for large induced forests with bounded maximum degree. A forest with maximum degree 2 is called a *linear forest*.

The problem for linear forests was solved for outerplanar graphs by Pelsmajer [8]: every outerplanar graph admits an induced linear forest on at least four sevenths of its vertices, and this is tight. More generally, the problem for a forest of maximum degree at most d, with $d \geq 2$, was solved for graphs with treewidth at most k for all k by Chappel and Pelsmajer [5]. Their result in particular extends the results of Hosono and Pelsmayer on outerplanar graphs to series-parallel graphs, and generalises it to graphs of bounded treewidth.

In this paper we focus on linear forests. Chappel conjectured that every planar graph admits an induced linear forest on at least four ninths of its vertices. Again, this would be tight if true. Poh [9] proved that the vertices of any planar graph can be partitioned into three sets inducing linear forests, and thus that every planar graph admits an induced linear forest on at least one third of its vertices. In this paper, we prove and strengthen Chappel's conjecture in a smaller class of graphs, the class of triangle-free planar graphs. Observe that planar graphs with arbitrarily large girth can have an arbitrarily large treewidth, so in this setting the best

result known to date is that every triangle-free planar graph admits an induced linear forest on at least one third of its vertices.

We prove the following theorem:

Theorem 1. *Every triangle-free planar graph of order n and size m admits an induced linear forest of order at least $\frac{9n-2m}{11}$.*

Thanks to Euler's formula, we can derive the following corollary:

Corollary 2. *Every triangle-free planar graph of order n admits an induced linear forest of order at least $\frac{5n+8}{11}$.*

Note that we cannot hope to get a better lower bound than $\frac{n}{2}+1$. Indeed the following claim holds:

Claim 3. *For all integer $n \geq 2$, there exists a triangle-free planar graph of order n whose largest induced linear forest has order $\lceil \frac{n}{2} \rceil + 1$.*

2 Sketch of the proof of Theorem 1

Due to the lack of space, we only give certain proofs and the proofs of Lemmas 5, 6, 7, 9, 10 and 14 are omitted.

Consider a graph $G = (V, E)$. For a set $S \subset V$, let $G[S]$ denote the subgraph of G induced by S, and let $G - S$ be the graph obtained from G by removing the vertices of S and all the edges incident to a vertex of S. If $x \in V$, then we denote the neighbourhood of x, that is the set of the vertices adjacent to x, by $N(x)$. For a set $S \subset V$, we denote the neighbourhood of S, that is the set of vertices in $V \backslash S$ that are adjacent to at least an element of S, by $N(S)$. We denote $|V|$ by $|G|$ and $|E|$ by $||G||$.

We call a vertex of degree d, at least d, and at most d, a d-vertex, a d^+-vertex, and a d^--vertex respectively. Similarly, a cycle of length ℓ is called an ℓ-cycle. Moreover, if G is embedded in the plane, a face of length ℓ is called an ℓ-face.

Let $\mathcal{P}_4$ be the class of triangle-free planar graphs. Let $G = (V, E)$ be a counter-example to Theorem 1 with the minimum order. Assume that G is embedded in the plane. Let $n = |G|$ and $m = ||G||$. We use the scheme presented in Observation 4 many times throughout the proof.

Observation 4. *Let α, β, γ be integers satisfying $\alpha \geq 1$, $\beta \geq 0$, $\gamma \geq 0$. Let $H^* \in \mathcal{P}_4$ be a graph with $|H^*| = n - \alpha$ and $||H^*|| \leq m - \beta$. By minimality of G, H^* admits an induced linear forest of order at least $\frac{9}{11}(n-\alpha) - \frac{2}{11}(m-\beta)$. Given an induced linear forest F^* of H^* of order $|F^*| \geq \frac{9}{11}(n-\alpha) - \frac{2}{11}(m-\beta)$, if there is an induced linear forest F of G of order $|F| \geq |F^*| + \gamma$, then as $|F| < \frac{9}{11}n - \frac{2}{11}m$, we have $\gamma < \frac{9}{11}\alpha - \frac{2}{11}\beta$.*

We prove some structural properties of the counter-example G to show that it does not eventually exist, and thus that Theorem 1 is true. First note that G is connected, otherwise one of its components would be a smaller counter-example to Theorem 1. Then note that every vertex of G has degree at most 4. Otherwise, by considering a 5^+-vertex v and Observation 4 applied to $H^* = G - v$ with $(\alpha, \beta, \gamma) = (1, 5, 0)$ and $F = F^*$, we have $0 < \frac{9}{11} - 5\frac{2}{11}$, a contradiction.

Let us define the notion of a *chain* of G (or *simple chain*) of G which is a quadruplet $C = (P, N, u, v)$ such that:

- $P \subset V$, $N \subset V \setminus P$, $u \in P$, and $v \in V \backslash (N \cup P)$;
- $G[P]$ is a linear forest;
- vertex u is a 1^--vertex of $G[P]$, and $N(v) \cap P = \{u\}$;
- $N(P) \subset N \cup \{v\}$ in G;
- vertex v is a 2^--vertex in $G - (N \cup P)$.

See Figure 1 (left) for an illustration. We will use the following notation for a chain $C = (P, N, u, v)$ of G :
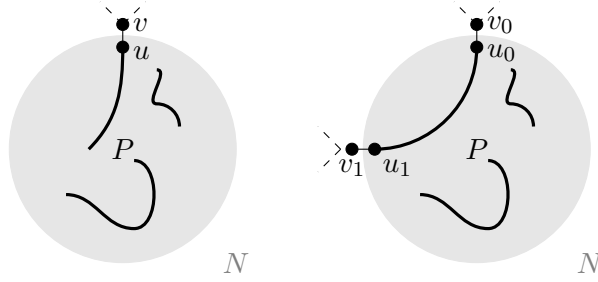
Figure 1: A simple chain (left) and a double chain (right).

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;
- $d(C)$ is the degree of v in $G - C$ (thus $d(C) \leq 2$);
- $||C|| = ||G|| - ||G - C||$.

Lemma 5. *For every chain $C = (P, N, u, v)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$.*

Let us now define a new notion quite similar to the notion of chain. A *double chain* of G is a sextuplet $C = (P, N, u_0, u_1, v_0, v_1)$ such that:

- $P \subset V$, $N \subset V \setminus P$, $u_0 \in P$, $u_1 \in P$, $v_0 \in V \setminus (N \cup P)$ and $v_1 \in V \setminus (N \cup P)$;
- $v_0 \neq v_1$;
- $G[P]$ is a linear forest;
- u_0 and u_1 are 1^--vertices of $G[P]$ if they are distinct, a 0-vertex of $G[P]$ if they are equal, and for $i \in \{0, 1\}$, $N(v_i) \cap P = \{u_i\}$;
- $N(P) \subset N \cup \{v_0\} \cup \{v_1\}$;
- v_0 and v_1 are 2^--vertices in $G - (N \cup P)$.

See Figure 1 (right) for an illustration. We will use the following notation for a double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G:

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;
- $d_0(C)$ is the degree of v_0 in $G - C$ (thus $d_0(C) \leq 2$);
- $d_1(C)$ is the degree of v_1 in $G - C$ (thus $d_1(C) \leq 2$);
- $||C|| = ||G|| - ||G - C||$.

A double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G such that v_0 and v_1 belong to different components of $G - C$ is called a *separating double chain* of G.

Lemma 6. *For every double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$.*

Lemma 7. *For every separating double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$.*

We prove some structural properties of G.

Lemma 8. *Graph G has no 1^--vertex.*

Proof. As G is connected, if it has a 0-vertex, then G is the graph with one vertex and it satisfies Theorem 1, a contradiction.

By contradiction, suppose $u \in V$ is a 1-vertex. Let v be the neighbour of u. If v is a 3^--vertex in G, then $(\{u\}, \emptyset, u, v)$ is a chain of G, thus by Lemma 5, we have $1 < \frac{9}{11} - \frac{2}{11}\frac{1}{2}$, a contradiction.

Therefore v is a 4-vertex. Let $H^* = G - \{u, v\}$. Graph H^* has $n - 2$ vertices and $m - 4$ edges. Adding vertex u to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (2, 4, 1)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction. $\square$

Lemma 9. *Graph G has no 2^--vertex.*

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Lemma 10. *Graph G has no 3-vertex adjacent to another 3-vertex and two 4-vertices.*

Lemma 11. *Graph G has no 3-vertex adjacent to two other 3-vertices and a 4-vertex.*

Proof. Let u be a 3-vertex adjacent to two 3-vertices v_0 and v_1, and to a 4-vertex w. Let x_0 and x_1 be the two neighbours of v_0 distinct from u. Note that x_0 and x_1 are 3^+-vertices in G by Lemma 9, and thus 1^+-vertices in $G' = G - \{u, w, v_0\}$ since they are not adjacent to u.

Suppose that either x_0 and x_1 are 2^+-vertices in G', or one is a 3-vertex and the other a 1^+-vertex. We have a simple chain $(\{u, v_0\}, \{x_0, x_1, w\}, u, v_1)$. By Lemma 5, we have $2 < \frac{9}{11}5 - \frac{2}{11}\frac{23}{2}$, a contradiction.

Suppose one of the x_i's, say x_0, is a 2^+-vertex in G', and the other one is a 1-vertex in G'. We have a double chain $(\{u, v_0\}, \{w, x_0\}, u, v_0, v_1, x_1)$. By Lemma 6, we have $2 < \frac{9}{11}4 - \frac{2}{11}7$, a contradiction.

Now the x_i's are 1-vertices in G'. By Lemma 9, the x_i's are 3-vertices in G and they are both adjacent to w. By planarity of G, one of the x_i's, say x_0, is not adjacent to v_1. Let y be the neighbour of x_0 in G'. By Lemmas 9 and 10, y is a 3-vertex in G. We have a simple chain $(\{u, v_0, x_0\}, \{w, v_1, x_1\}, x_0, y)$. By Lemma 5, we have $3 < \frac{9}{11}6 - \frac{2}{11}\frac{21}{2}$, a contradiction. $\square$

Lemma 12. *Graph G has no two adjacent 3-vertices in G.*

Proof. By Lemma 9, every vertex in G has degree 3 or 4. By Lemmas 10 and 11, there is no 3-vertex adjacent to a 3-vertex and a 4-vertex in G. Suppose by contradiction that there are two adjacent 3-vertices in G. Then G only has 3-vertices since it is connected.

Suppose there is a 4-cycle $u_0u_1u_2u_3$ in G and let v_i be the third neighbour of u_i. Since G has no triangle, the $v_i \neq v_{i+1}$ (indices are taken modulo 3). Suppose $v_0 = v_2$ and $v_1 = v_3$. Therefore v_0 and v_1 are separated by $u_0u_1u_2u_3$ since G is planar (one of v_0 and v_1 is outside of the 4-cycle and the other is inside). Let $H^* = G - \{u_0, u_1, u_2, u_3\}$. Graph H^* has $n - 4$ vertices and $m - 8$ edges. Since v_0 and v_1 are 1-vertices in H^* and $u_0u_1u_2u_3$ is a separating 4-cycle in G, adding vertices u_0 and u_1 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (4, 8, 2)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}8$, a contradiction. Now w.l.o.g. v_0 and v_2 are distinct. We have a double chain $(\{u_0, u_1, u_2\}, \{u_3, v_1\}, u_0, u_2, v_0, v_2)$. By Lemma 6, $3 < \frac{9}{11}5 - \frac{2}{11}6$, a contradiction.

Therefore, there is no 4-cycle in G. Suppose there is a 5-cycle $u_0u_1u_2u_3u_4$ in G. For all i, let v_i be the third neighbour of u_i. Now all the v_i's are distinct, otherwise there is a 4-cycle and we fall into the previous case. We have a double chain $(\{u_0, u_1, u_2, u_3\}, \{u_4, v_1, v_2\}, u_0, u_3, v_0, v_3)$. By Lemma 6, we have $4 < \frac{9}{11}7 - \frac{2}{11}11$, a contradiction.

Thus G is a 3-regular planar graph with girth at least 6, which is impossible by Euler's formula. $\square$

Lemma 13. *There is no 4-cycle with at least two 3-vertices in G.*

Proof. By contradiction, suppose there is such a 4-cycle $u_0u_1u_2u_3$. By Lemmas 9 and 12, this cycle has exactly two 3-vertices and two 4-vertices, and the two 3-vertices are not adjacent. W.l.o.g. u_0 and u_2 are 3-vertices, and u_1 and u_3 are 4-vertices. Let v_0 and v_2 be the third neighbours of u_0 and u_2 respectively. By Lemma 12, v_0 and v_2 are 4-vertices.

Suppose that $v_0 = v_2$. Let $H^* = G - \{u_0, u_1, u_2, u_3, v_0\}$. Graph H^* has $n - 5$ vertices and $m - 12$ edges. Adding vertices u_0 and u_2 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (5, 12, 2)$, we have $2 < \frac{9}{11}5 - \frac{2}{11}12$, a contradiction.

Therefore $v_0 \neq v_2$. Suppose that $v_0v_2 \in E$. We have a chain $(\{u_0, u_2\}, \{u_1, u_3, v_2\}, u_0, v_0)$. By Lemma 5, we have $2 < \frac{9}{11}5 - \frac{2}{11}\frac{25}{2}$, a contradiction.

Therefore $v_0v_2 \notin E$. Let $H^* = G - \{u_0, u_1, u_2, u_3, v_0, v_2\}$. Graph H^* has $n - 6$ vertices and $m - 16$ edges. Adding vertices u_0 and u_2 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (6, 16, 2)$, we have $2 < \frac{9}{11}6 - \frac{2}{11}16$, a contradiction. $\square$

Lemma 14. *There is no 4-face with exactly one 3-vertex in G.*

Let F be the set of faces of G. For every face $f \in F$, let $\ell(f)$ denote the length of f, and let $c_4(f)$ denote the number of 4-vertices in f. For every vertex v, let $d(v)$ be the degree of v. Let k be the number of faces of G, and for every $3 \leq d \leq 4$ and every $4 \leq \ell$, let k_ℓ be the number of faces of length ℓ and n_d the number of d-vertices in G.

Each 4-vertex is in the boundary of at most four faces. Therefore the sum of the $c_4(f)$ over all the 4-faces and 5-faces is $\sum_{f,4\leq l(f)\leq 5} c_4(f) \leq 4n_4$. Now, by Lemmas 9, 13, and 14, every 4-face of G has only 4-vertices in its boundary, so for each 4-face f, $c_4(f) = 4$. By Lemma 12, every 5-face of G has at least three 4-vertices, so for each 5-face f, we have $c_4(f) \geq 3$. Thus $\sum_{f,l(f)=4} c_4(f) + \sum_{f,l(f)=5} c_4(f) \geq 4k_4 + 3k_5 \geq 4k_4 + 2k_5$. Thus $4n_4 \geq 4k_4 + 2k_5$, and thus $2n_4 \geq 2k_4 + k_5$. By Euler's formula, we have:

$$
\begin{aligned}
-12 &= 6m - 6n - 6k \\
&= 2\sum_{v\in V} d(v) + \sum_{f\in F} \ell(f) - 6n - 6k \\
&= \sum_{d\geq 3}(2d-6)n_d + \sum_{\ell\geq 4}(\ell-6)k_l \\
&\geq 2n_4 - 2k_4 - k_5 \\
&\geq 0
\end{aligned}
$$

That contradiction implies that G does not exist and thus ends the proof of Theorem 1.

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