Degree-constrained 2-partitions of graphs

Jørgen Bang-Jensen, Stéphane Bessy

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Abstract

A $(\delta \geq k_1, \delta \geq k_2)$-partition of a graph G is a vertex-partition (V_1, V_2) of G satisfying that $\delta(G[V_i]) \geq k_i$ for $i = 1, 2$. We determine, for all positive integers k_1, k_2, the complexity of deciding whether a given graph has a $(\delta \geq k_1, \delta \geq k_2)$-partition.

We also address the problem of finding a function $g(k_1, k_2)$ such that the $(\delta \geq k_1, \delta \geq k_2)$-partition problem is $\mathcal{NP}$-complete for the class of graphs of minimum degree less than $g(k_1, k_2)$ and polynomial for all graphs with minimum degree at least $g(k_1, k_2)$. We prove that $g(1, k) = k$ for $k \geq 3$, that $g(2, 2) = 3$ and that $g(2, 3)$, if it exists, has value 4 or 5.

Keywords: $\mathcal{NP}$-complete, polynomial, 2-partition, minimum degree.

1 Introduction

A **2-partition** of a graph G is a partition of $V(G)$ into two disjoint sets. Let $\mathbb{P}_1, \mathbb{P}_2$ be two graph properties, then a **$(\mathbb{P}_1, \mathbb{P}_2)$-partition** of a graph G is a 2-partition (V_1, V_2) where V_1 induces a graph with property $\mathbb{P}_1$ and V_2 a graph with property $\mathbb{P}_2$. For example a $(\delta \geq k_1, \delta \geq k_2)$-partition of a graph G is a 2-partition (V_1, V_2) where $\delta(G[V_i]) \geq k_i$, for $i = 1, 2$

There are many papers dealing with vertex-partition problems on (di)graphs. Examples (from a long list) are [1, 2, 4, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 33, 34]. Examples of 2-partition problems are recognizing bipartite graphs (those having has a 2-partition into two independent sets) and split graphs (those having a 2-partition into a clique and an independent set) [15]. It is well known and easy to show that there are linear algorithms for deciding whether a graph is bipartite, respectively, a split graph. It is an easy exercise to show that every graph G has a 2-partition (V_1, V_2) such that the degree of each vertex in $G[V_i]$, $i \in [2]$ is at most half of its original degree. Furthermore such a partition can be found efficiently by a greedy algorithm. In [16, 17] and several other papers the opposite condition for a 2-partition was studied. Here we require the that each vertex has at least half of its neighbours inside the set it belongs to in the partition. This problem, known as the satisfactory partition problem, is $\mathcal{NP}$-complete for general graphs [5].

A partition problem that has received particular attention is that of finding sufficient conditions for a graph to possess a $(\delta \geq k_1, \delta \geq k_2)$-partition. Thomassen [31] proved the existence of a function $f(k_1, k_2)$ so that every graph of minimum degree at least $f(k_1, k_2)$ has a $(\delta \geq k_1, \delta \geq k_2)$-partition. He proved that $f(k_1, k_2) \leq 12 \cdot \max\{k_1, k_2\}$. This was later improved by Hajnal [19] and Häggkvist (see [31]). Thomassen [31, 32] asked whether it would hold that $f(k_1, k_2) = k_1 + k_2 + 1$ which would be best possible because of the complete graph $K_{k_1+k_2+1}$. Stiebitz [29] proved that indeed we have $f(k_1, k_2) = k_1 + k_2 + 1$. Since this result was published, several groups of researchers have tried to find extra conditions on the graph that would allow for a smaller minimum degree requirement. Among others the following results were obtained.

Theorem 1.1 *[20] For all integers $k_1, k_2 \geq 1$ every triangle-free graph G with $\delta(G) \geq k_1 + k_2$ has a $(\delta \geq k_1, \delta \geq k_2)$-partition.*

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Theorem 1.2 *[26] For all integers $k_1, k_2 \geq 2$ every graph G with no 4-cycle and with $\delta(G) \geq k_1+k_2-1$ has a $(\delta \geq k_1, \delta \geq k_2)$-partition.*

Theorem 1.3 *[23]*

- *For all integers $k_1, k_2 \geq 1$, except for K_3, every graph G with no $K_4 - e$ and with $\delta(G) \geq k_1 + k_2$ has a $(\delta \geq k_1, \delta \geq k_2)$-partition.*
- *For all integers $k_1, k_2 \geq 2$ every triangle-free graph G in which no two 4-cycles share an edge and with $\delta(G) \geq k_1 + k_2 - 1$ has a $(\delta \geq k_1, \delta \geq k_2)$-partition.*

The original proof that $f(k_1, k_2) = k_1 + k_2 + 1$ in [29] is not constructive and neither are those of Theorems 1.2 and 1.3. In [7] Bazgan et al. gave a polynomial algorithm for constructing a $(\delta \geq k_1, \delta \geq k_2)$-partition of a graph with minimum degree at least $k_1 + k_2 + 1$ or at least $k_1 + k_2$ when the input is triangle-free.

The main result of this paper is a full characterization of the complexity of the $(\delta \geq k_1, \delta \geq k_2)$-partition problem.

Theorem 1.4 *Let $k_1, k_2 \geq 1$ with $k_1 \leq k_2$ be integers. When $k_1 + k_2 \leq 3$ it is polynomial to decide whether a graph has a 2-partition (V_1, V_2) such that $\delta(G[V_i]) \geq k_i$ for $i = 1, 2$. For all other values of k_1, k_2 it is NP-complete to decide the existence of such a partition.*

A result in [13] implies that $(\delta \geq 3, \delta \geq 3)$-partition is $\mathcal{NP}$-complete already for 4-regular graphs and there are other results about the complexity of finding partitions with lower and/or upper bounds on the degrees inside each partition, such as [5, 6, 16, 17, 34], but we did not find anything which implies Theorem 1.4.

The result of Stiebitz [29] insures that if the minimum degree of the input graph is large enough, at least $k_1 + k_2 + 1$, then the $(\delta \geq k_1, \delta \geq k_2)$-partition always exists. We conjecture that if this minimum degree is large but less than $k_1 + k_2 + 1$ then the $(\delta \geq k_1, \delta \geq k_2)$-partition is not always trivial but can be solved in polynomial time.

Conjecture 1.5 *There exists a function $g(k_1, k_2)$ so that for all $1 \leq k_1 \leq k_2$ with $k_1 + k_2 \geq 3$ the $(\delta \geq k_1, \delta \geq k_2)$-partition problem is $\mathcal{NP}$-complete for the class of graphs of minimum degree less than $g(k_1, k_2)$ and polynomial for all graphs with minimum degree at least $g(k_1, k_2)$.*

In the next section we introduce notions and tools that will be used later. In Section 3 we give the proof of Theorem 1.4 and in Section 4 we provide some partial results concerning Conjecture 1.5. In particular we prove that $g(1, k) = k$ for $k \geq 3$, that $g(2, 2) = 3$ and that $g(2, 3)$, if it exists, has value 4 or 5. Finally in Section 5 we address some other partition problems mainly dealing with (edge-)connectivity in each part of the partition.

Notice that regarding the results that we establish in this paper the first open case of Conjecture 1.5 is the following problem.

Problem 1.6 *What is the complexity of the $(\delta \geq 2, \delta \geq 3)$-partition problem for graphs of minimum degree 4?*

2 Notation, definitions and preliminary results

Notation is standard and follows [3, 12]. In this paper graphs have no parallel edges and no loops. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$.

2.1 Special Graphs

We first define some graphs that will be used frequently in our proofs to ensure that certain vertices have a sufficiently high degree.

For all $k \geq 3$ we let $X_{k,2}$ be the graph we obtain from K_{k+1} by subdividing one edge by a vertex x. Let $X_{3,1}$ be obtained from $X_{3,2}$ by adding a vertex x' adjacent to the degree 2 vertex x of $X_{3,2}$.

Let $Y_{4,1}$ be the graph on 7 vertices which we obtain from a 5-wheel by adding a new edge e linking 2 non adjacent vertices of the outer cycle of the 5-wheel, a new vertex joined to the 3 vertices of this outer cycle not incident to e and to another new vertex y. For $k = 3$ let Z_k be the graph $X_{3,1}$ that we defined above and let $z = x'$. For $k \geq 4$ let Z_k be the graph that we obtain from $K_{k-2,k-1}$ by adding a cycle on the $k-1$ vertices of degree $k-2$ and then adding a new vertex z adjacent to all the $k-2$ vertices of degree $k-1$. And finally let W_k be the graph we obtain from K_{k+1} by deleting one edge $u'v'$ and the adding two new vertices u, v and the edges uu', vv'.
All these graphs are depicted in Figure 1.

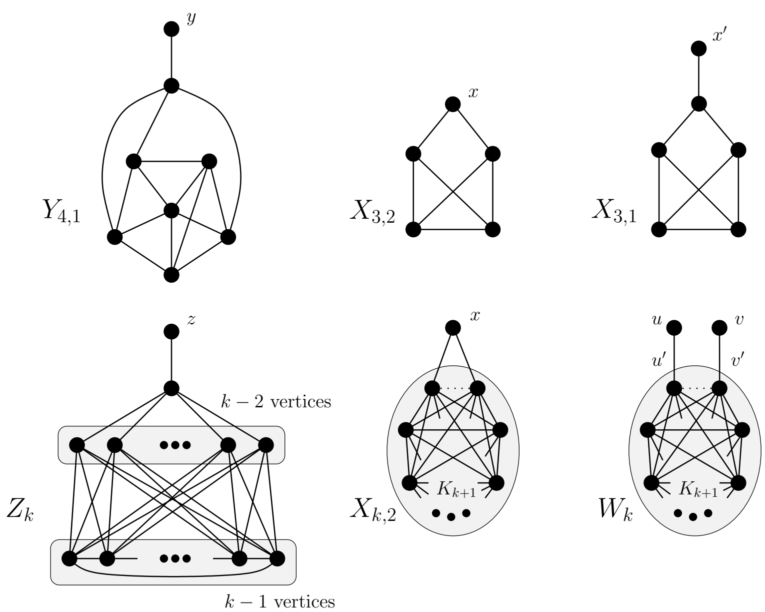

Figure 1: The graphs $Y_{4,1}$, $X_{3,2}$, $X_{3,1}$, Z_k, $X_{k,2}$ and W_k.

2.2 Connected instances of 3-SAT

For a given instance $\mathcal{F}$ of 3-SAT with clauses $C_1, C_2, \ldots, C_m$ and variables $x_1, \ldots, x_n$, where each variable x_i occurs at least once as the literal x_i and at least once as $\bar{x}_i$, we define the bipartite graph $B(\mathcal{F})$ as the graph with vertex set $\{v_1, \bar{v_1}, v_2, \bar{v}_2, \ldots, v_n, \bar{v}_n\} \cup \{c_1, c_2, \ldots, c_m\}$, where the first set corresponds to the literals and the second one to the clauses, and edge set containing an edge between the vertex c_j and each of the 3 vertices corresponding to the literals of C_j for every $j \in [m]$. We say that $\mathcal{F}$ is a **connected** instance if $B(\mathcal{F})$ is connected.

Lemma 2.1 *3-SAT is $\mathcal{NP}$-complete for instances $\mathcal{F}$ where $B(\mathcal{F})$ is connected*

Proof: Suppose $B(\mathcal{F})$ has connected components $X_1, X_2, \ldots, X_k$, where $k \geq 2$. Fix a literal vertex $\ell_i \in X_i$ for each $i \in [k]$, add a new variable y and $k-1$ new clauses $C'_1, \ldots, C'_{k-1}$ where $C'_j = (\ell_{j-1} \vee y \vee \ell_j)$, $j \in [k-1]$. Let $\mathcal{F}'$ be the new formula obtained by adding the variable y and the clauses $C'_1, \ldots, C'_{k-1}$. It is easy to check that $\mathcal{F}'$ is equivalent to $\mathcal{F}$ and that $B(\mathcal{F}')$ is connected. $\diamond$

By adding a few extra variables, if necessary, we can also obtain an equivalent connected instance in which each literal occurs at least twice. We leave the easy details to the interested reader.

2.3 Ring graphs and 3-SAT

We first introduce an important class of graphs that will play a central role in our proofs. The directed analogue of these graphs was used in [2, 4]. A **ring graph** is the graph that one obtains by taking two or more copies of the complete bipartite graph on 4 vertices $\{a_1, a_2, b_1, b_2\}$ and edges $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ and joining these in a circular manner by adding a path $P_{i,1}$ from the vertex

$b_{i,1}$ to $a_{i+1,1}$ and a path $P_{i,2}$ from $b_{i,2}$ to $a_{i+1,2}$ where $b_{i,1}$ is the ith copy of b_1 etc and indices are 'modulo' n ($b_{n+1,j} = b_{1,j}$ for $j \in [2]$ etc). Our proofs are all reductions from $\mathcal{NP}$-complete variants of the 3-SAT problems. We call the copies of $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ **switch vertices**.

We start by showing how we can associate a ring graph to a given 3-SAT formula. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting on m clauses $C_1, \ldots, C_m$ over the same set of n boolean variables $x_1, \ldots, x_n$. Each clause C_i is of the form $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ where each $\ell_{i,j}$ belongs to $\{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ and $\bar{x}_i$ is the negation of variable x_i. By adding extra clauses to obtain an equivalent formula, if necessary, we can ensure that every literal occurs at least twice in $\mathcal{F}$. We shall use this fact in one of our proofs.

For each variable x_i the ordering of the clauses above induces an ordering of the occurrences of x_i, resp $\bar{x}_i$, in the clauses. Let q_i (resp. p_i) denote the number of times x_i (resp. $\bar{x}_i$) occurs in the clauses Let $R(\mathcal{F}) = (V, E)$ be the ring graph defined as follows. Its vertex set is

$$V = \{a_{1,1}, \ldots a_{n,1}, a_{1,2} \ldots, a_{n,2}\} \cup \{b_{1,1}, \ldots, b_{n,1}, b_{1,2}, \ldots, b_{n,2}\} \cup \bigcup_{i=1}^{n} \{v_{i,1}, \ldots, v_{i,q_i}, v'_{i,1}, \ldots, v'_{i,p_i}\}$$

Its edge set E consists of the following edges:

- $\bigcup_{i=1}^{n} \{a_{i,1}b_{i,1}, a_{i,1}b_{i,2}, a_{i,2}b_{i,1}, a_{i,2}b_{i,2}\}$
- the edges of the paths $P_{i,1}, P_{i,2}$, $i \in [n]$ where $P_{i,1} = b_{i,1}v_{i,1} \ldots v_{i,q_i}a_{i+1,1}$ and $P_{i,2} = b_{i,2}v'_{i,1} \ldots v'_{i,p_i}a_{i+1,2}$.

For $1 \leq j \leq m$, we associate the clause $C_j = (\ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3})$ with the set O_j consisting of three vertices of $R(\mathcal{F})$ representing the occurrences of the literals of C_j in $\mathcal{F}$: if $\ell_{j,1} = x_i$ for some $i \in [n]$ and this is the r'th occurrence of x_i in the clauses, then O_j contains the vertex $v_{i,r}$. If $\ell_{j,1} = \bar{x}_i$ for some $i \in [n]$ and this is the r'th occurrence of $\bar{x}_i$ in the clauses, then O_j contains the vertex $v'_{i,r}$. The other two vertices of O_j are defined similarly. In our proofs below we will often add a vertex c_i adjacent to all the vertices of O_i for $i \in [n]$. An example is depicted in Figure 2.

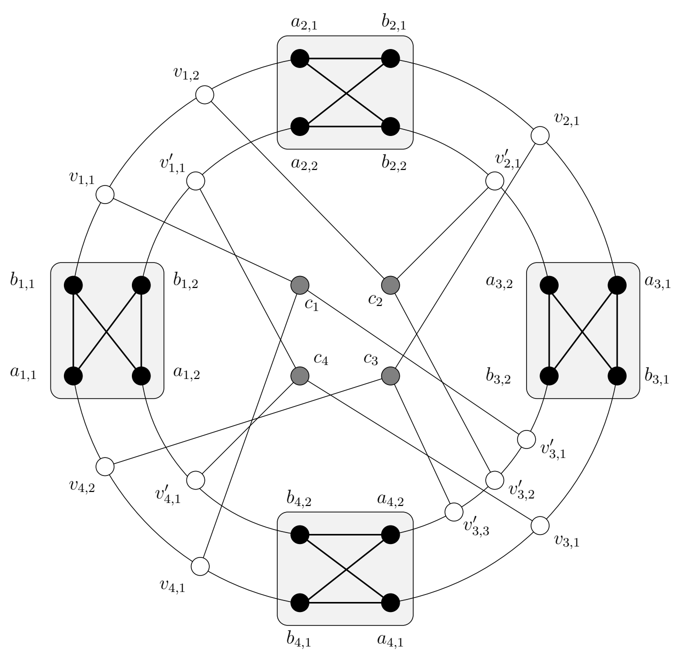

Figure 2: The ring graph $R(\mathcal{F})$ corresponding to the formula $\mathcal{F} = (x_1 \vee x_4 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_4)$. The grey boxes contain the switch vertices, the white vertices are the variable vertices and we added the clauses vertices c_1, c_2, c_3 and c_4 (these are not part of the ring graph $R(\mathcal{F})$).

The following observation which forms the base of many of our proofs is easy to prove (for a proof of result a very similar to this see [4]).

Theorem 2.2 *Let $\mathcal{F}$ be a 3-SAT formula and let $R(\mathcal{F})$ be the corresponding ring graph. Then $R = R(\mathcal{F})$ contains a cycle C which intersects all the sets $O_1, \ldots, O_m$ so that $R - C$ is a cycle C' if and only if $\mathcal{F}$ is a 'Yes'-instance of 3-SAT.*

3 Proof of Theorem 1.4

3.1 The case $k_1 + k_2 \leq 3$

We start with a trivial observation.

Proposition 3.1 *Every graph G with $\delta(G) \geq 1$ and at least 4 vertices has a $(\delta \geq 1, \delta \geq 1)$-partition except if G is a star.*

Proposition 3.2 *There is a polynomial algorithm for testing whether a graph has a $(\delta \geq 1, \delta \geq 2)$-partition*

Proof: We try for every choice of adjacent vertices u, v whether there is a solution with $u, v \in V_1$. Clearly G is a 'yes'-instance if and only if at least one of these $O(n^2)$ attempts will succeed. Hence by starting with $V_1 = \{u, v\}$ and then moving vertices with at most one neighbour in $V - V_1$ to V_1 we either end with a good partition or $V_1 = V$ in which case no partition exists for that choice $\{u, v\}$. $\diamond$

3.2 The case $k_1 = 1$ and $k_2 \geq 3$

The following variant of satisfiability, which we call $\leq$ 3-SAT(3), is known to be NP-complete: Given a boolean CNF formula $\mathcal{F}$ consisting of clauses $C_1, C_2, \ldots, C_m$ over variables $x_1, x_2, \ldots, x_n$ such that each clause has 2 or 3 literals, no variable occurs in more than 3 clauses and no literal appears more than twice; decide whether $\mathcal{F}$ can be satisfied.

As we could not find a proper reference for a proof that $\leq$ 3-SAT(3) is $\mathcal{NP}$-complete, we give one here as it is presented on pages 281-283 in a set of course notes[1] by Prof. Yuh-Dauh Lyuu, National Taiwan University:

Assume that $\mathcal{F}$ is an instance of 3-SAT in which the variable x occurs a total of $r \geq 4$ times in the formula (as x or $\bar{x}$) in clauses $C_{i_1}, \ldots, C_{i_r}$. Introduce new variables $x_1, \ldots, x_r$ and replace the first occurrence of x (in C_{i_1}) by x_1 is x is not negated in C_{i_1} and otherwise replace it by $\bar{x}_1$ in C_{i_1}. Similarly we replace the occurrence of x in C_{i_j}, $j \geq 2$ by x_j or $\bar{x}_j$. Finally we add the new clauses $(\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge \ldots \wedge (\bar{x}_r \vee x_1)$. These clauses (which have size 2) will force all the variables $x_1, \ldots, x_r$ to take the same value under any satisfying truth assignment. Repeating this replacement for all variables of the original formula $\mathcal{F}$ we obtain an equivalent instance $\mathcal{F}'$ of $\leq$ 3-SAT(3).

Below we will need another variant which we call $\leq$ 3-SAT(5) where clauses still have size 2 or 3 and each variable is allowed to occur at most 5 times and at most 3 times as the same literal. By following the scheme above and for each original variable occurring at least 4 times adding r extra clauses $(x_1 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_3) \wedge \ldots \wedge (x_r \vee \bar{x}_1)$ we obtain an equivalent instance $\mathcal{F}''$ and because the $2r$ new clauses will form a cycle in the bipartite graph $B(\mathcal{F}'')$ of $\mathcal{F}''$ it is easy to see that $\mathcal{F}''$ is a connected instance of $\leq$ 3-SAT(5) if $\mathcal{F}$ is a connected instance of 3-SAT. Hence, by Lemma 2.1, connected $\leq$ 3-SAT(5) is $\mathcal{NP}$-complete.

Theorem 3.3 *For all $k \geq 3$ it is $\mathcal{NP}$-complete to decide whether a graph has a $(\delta \geq 1, \delta \geq k)$-partition.*

Proof: We show how to reduce an instance of connected $\leq$ 3-SAT(5) to $(\delta \geq 1, \delta \geq k)$-partition where $k \in \{3, 4\}$ in polynomial time and then show how to extend the construction to higher values of k. We start the construction for $k = 3$.

Below we will use several disjoint copies of the graphs $X_{3,1}$ and $X_{3,2}$ to achieve our construction. Let $\mathcal{F}$ be a connected instance of $\leq$ 3-SAT(5) with clauses $C_1, \ldots, C_m$ and variables $x_1, \ldots x_n$. We

[1] `https://www.csie.ntu.edu.tw/~lyuu/complexity/2008a/20080403.pdf`

may assume that each of the $2n$ literals occur at least once in $\mathcal{F}$ (this follows from the fact that we may assume this for any instance of normal 3-SAT and the reduction above to $\leq$ 3-SAT(5) preserves this property). We will construct $G = G(\mathcal{F})$ as follows:

- For each variable x_i, $i \in [n]$ we introduce three new vertices $y_i, v_i, \bar{v}_i$ and two the edges $y_iv_i, y_i\bar{v}_i$.
- For each $i \in [n]$: if the literal x_i ($\bar{x}_i$) occurs precisely once in $\mathcal{F}$, then we identify v_i ($\bar{v}_i$) with the vertex x in a private copy of $X_{3,2}$. If x_i ($\bar{x}_i$) occurs precisely twice, then we identify v_i ($\bar{v}_i$) with the vertex x' in a private copy of $X_{3,1}$.
- Now we add new vertices $c_1, \dots, c_m$, where c_i corresponds to the clause C_i, $i \in [m]$, and join each c_j by an edge to those (2 or 3) vertices from $\{v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n\}$ which correspond to its literals. If c_j gets only two edges this way, we identify it with the vertex x' in a private copy of $X_{3,1}$.
- Add $2m$ new vertices $z_1, z_2, \dots, z_{2m}$ and the edges of the $2m$-cycle $z_1z_2, z_2z_3, \dots, z_{2m-1}z_{2m}, z_{2m}z_1$.
- Finally we add, for each $j \in [m]$ the edges c_jz_{2j-1}, c_jz_{2j}.

We claim that $G(\mathcal{F})$ has a $(\delta \geq 1, \delta \geq 3)$-partition if and only if $\mathcal{F}$ can be satisfied. Suppose first that t is a satisfying truth assignment. Then it is easy to check that (V_1, V_2) is a good 2-partition if we take V_1 to be the union of $\{y_1, \dots, y_n\}$ and the n vertices from $\{v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n\}$ which corresponds to the false literals. Note that if a vertex v_i ($\bar{v}_i$) is in V_2 then it will have degree 3 via its private copy of one of the graphs X_1, X_2 or because the corresponding literal occurred 3 times in $\mathcal{F}$.

Conversely assume that (V_1, V_2) is a $(\delta \geq 1, \delta \geq 3)$-partition. Then we claim that we must have all the vertices $z_1, z_2, \dots, z_{2m}$ in V_2: If one of these is in V_1, then they all are as they have degree exactly 3. Clearly we also have $\{y_1, \dots, y_n\} \subset V_1$. However, by construction, the literal and clause vertices all have degree 3 and they induce a connected graph (here we use that the instance $\mathcal{F}$ has a connected bipartite graph $B(\mathcal{F})$). Thus all of these vertices must be in V_2, but then each vertex y_i is isolated, contradiction. Hence all the vertices $z_1, z_2, \dots, z_{2m}$ are in V_2 and this implies that all of $c_1, \dots, c_m$ are also in V_2. The vertices $y_1, \dots, y_n$ are in V_1 and hence, for each $i \in [n]$, at least one of the vertices $v_i, \bar{v}_i$ is also in V_1. Now define a truth assignment a follows. For each $i \in [n]$: If both v_i and $\bar{v}_i$ are in V_1, or v_i is in V_2 we put x_i true; otherwise we put x_i false. Since each c_j must have a neighbour from $\{v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n\}$ in V_2 this is a satisfying truth assignment.

To obtain the construction for $k = 4$ we replace each copy of $X_{3,1}$ above by a copy of $Y_{4,1}$, each copy of $X_{3,2}$ by a copy of $X_{4,2}$ and identify each of the literal and clause vertices with the vertex y in an extra private copy of $Y_{4,1}$. Finally we identify each vertex z_t, $t \in [2m]$ with the vertex y in a private copy of $Y_{4,1}$. Now it is easy to see that we can complete the proof as we did for the case $k = 3$.

For all $k \in \{3 + 2a, 4 + 2a | a \geq 1\}$ we can increase the degree of all literal, clause and z_j vertices by $2a$ by identifying these with the x vertices of a private copies of $X_{k,2}$ and repeat the proof above. $\diamond$

Corollary 3.4 *The $(\delta \geq 1, \delta \geq k)$-partition problem is $\mathcal{NP}$-complete for graphs of minimum degree $k - 1$.*

Proof: Recall that in our proof above the vertices corresponding to clauses must always belong to V_2 in any good partition (V_1, V_2) hence if we connect each vertex y_i, $i \in [n]$ to $c_1, \dots, c_{k-3}$ by edges we obtain a graph of minimum degree $k - 1$ which has a good partition if and only if $\mathcal{F}$ is satisfiable (The vertices y_i, $i \in [n]$ must belong to V_1 as they have degree $k - 1$). $\diamond$

3.3 $(\delta \geq k_1, \delta \geq k_2)$-partition when $2 \leq k_1 \leq k_2$

Theorem 3.5 *For every choice of natural numbers $2 \leq k_1 \leq k_2$ it is $\mathcal{NP}$-complete to decide whether a graph has a $(\delta \geq k_1, \delta \geq k_2)$-partition.*

Proof: We show how to reduce 3-SAT to $(\delta \geq k_1, \delta \geq k_2)$-partition. Given an instance $\mathcal{F}$ of 3-SAT with clauses $C_1, C_2, \dots, C_m$ and variables $x_1, \dots, x_n$ we proceed as follows. Start from a copy of the ring graph $R = R(\mathcal{F})$ and then add the following:

- If $k_2 \geq 3$ we identify each vertex of R with the vertex z in a private copy of Z_{k_2}.
- For each $i \in [n]$ add a new vertex u_i and join it by edges to the vertices $a_{i,1}, a_{i,2}$. If $k_1 \geq 3$ we identify u_i with the vertex z in a private copy of Z_{k_1}.
- For each $i \in [n]$ add a new vertex u'_i and join it by edges to the vertices $b_{i,1}, b_{i,2}$. If $k_1 \geq 3$ we identify u'_i with the vertex z in a private copy of Z_{k_1}.
- For each $j \in [m]$ we add a new vertex c_j, identify c_j with the vertex z in a private copy of Z_{k_1} if $k_1 \geq 3$ and add three edges from c_j to the three vertices of R which correspond to the literals of C_m.
- Finally add a new vertex r and join this to all of the vertices in $\{u_1, \dots, u_n, u'_1, \dots, u'_n, c_1, \dots, c_m\}$ via private copies of W_{k_1} by identifying the vertex u with r and v with the chosen vertex from $\{u_1, \dots, u_n, u'_1, \dots, u'_n c_1, \dots, c_m\}$.

We claim that the final graph G has a $(\delta \geq k_1, \delta \geq k_2)$-partition if and only if $\mathcal{F}$ is satisfiable. First we make some observations about G:

- Every vertex v of R which is not a switch vertex has degree exactly $k_2 + 1$ as it has degree 2 in $R(\mathcal{F})$, is adjacent to exactly one c_j, $j \in [m]$ and if $k_2 \geq 3$ then v has been identified with one vertex z of a private copy of Z_{k_2}.
- Switch vertices all have degree exactly $k_2 + 2$.
- The vertices $u_1, \dots, u_n, u'_1, \dots, u'_n$ have degree exactly $k_1 + 1$.
- All vertices in copies of Z_a have degree exactly a when $a \geq 3$.
- All vertices in copies of W_{k_1} have degree exactly k_1
- All vertices c_j, $j \in [m]$ have degree $k_1 + 2$.
- The vertex r has degree $m + 2n$ which we may clearly assume is at least k_1.

For convenience in writing, below we define Z_2 to be the empty graph so that we can talk about Z_a's without having to condition this on a being at least 3. Suppose first that $\mathcal{F}$ is satisfiable. By Theorem 2.2 this means that $R(\mathcal{F})$ has a cycle C which intersects the neighbourhood of each c_j, $j \in [m]$ and so that $R - C$ is another cycle C'. Now we let V_1 consist of the vertices of C, their corresponding private copies of Z_{k_2}, all the vertices $\{u_1, \dots, u_n, u'_1, \dots, u'_n, c_1, \dots, c_m\}$ along with their private copies of Z_{k_1} and finally the vertex r and the vertices of all copies of W_{k_1} that we used. Let $V_2 = V(G) - V_1$, that is V_2 contains the vertices of C' and their private copies of Z_{k_2}. It is easy to check that $\delta(G[V_1]) \geq k_1$ and that $\delta(G[V_2]) \geq k_2$ so (V_1, V_2) is a good partition. Now assume that G has a good 2-partition (V_1, V_2). The way we connected r to the vertices in $\{u_1, \dots, u_n, u'_1, \dots, u'_n, c_1, \dots, c_m\}$ via copies of W_{k_1} implies that these must belong to the same set V_i as r. If $k_1 < k_2$ this must be V_1 and otherwise we can rename the sets so that $i = 1$. Since each c_j, $j \in [m]$ has degree $k_1 + 2$ at least one of the vertices corresponding to a literal of C_j must belong to V_1. Suppose that some vertex corresponding a literal ℓ is in V_1, then all the vertices of the path in R corresponding to that literal, including the two end vertices which are switch vertices, must belong to V_1. This follows from the fact that all these vertices have degree $k_2 + 1$ and have a neighbour in $\{u_1, \dots, u_n, u'_1, \dots, u'_n, c_1, \dots, c_m\} \subset V_1$. Moreover if $a_{i,j}$ (resp. $b_{i,j}$) belongs to V_2 then, as $a_{i,j}$ (resp. $b_{i,j}$) has degree $k_2 + 2$ and u_i (resp. u'_i) belongs to V_1, at least one of the vertices of $\{b_{i,1}, b_{i,2}\}$ (resp. $\{a_{i,1}, a_{i,2}\}$) belongs to V_2. And as u_i (resp. u'_i) belongs to V_1, one of $\{a_{i,1}, a_{i,2}\}$ (resp. $\{b_{i,1}, b_{i,2}\}$) must lie in V_1. So since V_2 is not empty this implies that the restriction of V_2 to R is a cycle consisting of paths $Q_1, \dots, Q_n$ where Q_i is either the path $P_{i,1}$ or the path $P_{i,2}$. Hence $R[V(R) \cap V_1]$ is a cycle intersecting each of the neighbourhoods of the vertices c_j, $j \in [m]$ and hence $\mathcal{F}$ is satisfiable by Theorem 2.2. $\diamond$

Combining the results of this section concludes the proof of Theorem 1.4.

4 Higher degrees

We study the borderline between polynomial and $\mathcal{NP}$-complete instances of the partition problems. That is, we try to see how close we can get to the bound $k_1 + k_2 + 1$ on the minimum degree and still have an $\mathcal{NP}$-complete instance. For $k_1 = 1$ we can give the precise answer by combining Corollary 3.4 and the result below.

Proposition 4.1 *There is a polynomial algorithm for checking whether a graph G of minimum degree at least k has a $(\delta \geq 1, \delta \geq k)$-partition.*

Proof: It suffices to see that we can test for a given edge uv of G whether there is a $(\delta \geq 1, \delta \geq k)$-partition (V_1, V_2) with $u, v \in V_1$. This is done by starting with $V_1 = \{u, v\}$ and then moving vertices from $V - V_1$ to V_1 when these vertices do not have at least k neighbours in $V - V_1$. Note that this process preserves the invariant that $\delta(G[V_1]) \geq 1$. Hence if the process terminates before $V_1 = V$ we have found the desired partition and otherwise we proceed to the next choice for an edge to start from. $\diamond$

For the $(\delta \geq 2, \delta \geq 2)$-partition problem we can also give the precise borderline between polynomial and $\mathcal{NP}$-complete instances.

Proposition 4.2 *There exists a polynomial algorithm for checking whether a given graph of minimum degree at least 3 has a $(\delta \geq 2, \delta \geq 2)$-partition.*

Proof: First test whether G has two disjoint cycles C_1, C_2. This can be done in polynomial time [9, 25]. If no such pair exists G is a 'no'-instance, so assume that we found a pair of disjoint cycles C_1, C_2. Now put the vertices of C_1 in V_1 and continue to move vertices of $V - V_1 - V(C_2)$ to V_1 if they have at least two neighbours in the current V_1. When this process stops the remaining set $V_2 = V - V_1$ induces a graph of minimum degree at least 2, since the vertices we did not move have at most one neighbour in V_1. $\diamond$

We now proceed to partitions where $2 \leq k_1 \leq k_2$ and try to raise the minimum degree above k_1 to see whether we can still prove $\mathcal{NP}$-completeness.

Theorem 4.3 *For every $a \geq 3$ it is $\mathcal{NP}$-complete to decide whether a graph of minimum degree $a+1$ has a $(\delta \geq a, \delta \geq a)$-partition.*

Proof: We give the proof for $a = 3$ and then explain how to extend it to larger a. Let $\mathcal{F}$ be an instance of 3-SAT with n variable and m clauses $C_1, \ldots, C_m$. Let $R' = R'(\mathcal{F})$ be obtained from $R(\mathcal{F})$ by adding, for all $i \in [n]$, an edge between all vertices at distance 2 in one of the paths $P_{i,1}, P_{i,2}$, $i \in [n]$ (that is, we replace each of these paths by their square). Now we construct the graph $H = H(\mathcal{F})$ starting from R' as follows:

- For each $j \in [m]$: add two vertices $c_{j,1}, c_{j,2}$ and join them to the vertices in R' which correspond to the literals of C_j.
- add the vertices of a $2m$ -cycle $y_1 y_2 \ldots y_{2m} y_1$.
- For each $j \in [m]$ add the two edges $y_{2j-1}c_{j,1}, y_{2j-1}c_{j,2}$ and the two edges $y_{2j}c_{j,2}, y_{2j}, c_{j+1,1}$, where $c_{m+1,1} = c_{1,1}$.

The resulting graph H has minimum degree 4 and we claim that H has a $(\delta \geq 3, \delta \geq 3)$-partition if and only if $\mathcal{F}$ is satisfiable, which we know, by Theorem 2.2 and the previous proofs, where we used the same approach, is if and only if the vertex set of R (which is the same as that of R') can be partitioned into two cycles C, C' so that C contains a neighbour of each of the vertices $c_{j,1}, c_{j,2}$, $j \in [m]$.

Again the proof is easy when $\mathcal{F}$ is satisfiable: Let C, C' be as above and let $V_2 = V(C')$ and $V_1 = V(H) - V_2$. It is easy to check that $\delta(H[V_i]) \geq 3$ for $i = 1, 2$, because C contains a neighbour of each $c_{j,1}, c_{j,2}$, $j \in [m]$ (and we assume that each literal appears at least twice in $\mathcal{F}$ to insure that $\delta(H[V_2]) \geq 3$). Suppose now that H has a $(\delta \geq 3, \delta \geq 3)$-partition (V_1, V_2). Since adjacent vertices in $\{y_1, y_2, \ldots, y_{2m}\}$ have degree 4 and share a neighbour they must all belong to the same set V_i, $i \in [2]$

and this set must also contain all the vertices $c_{j,1}, c_{j,2}$, $j \in [m]$. Without loss of generality we have $i = 1$. Thus V_2 is a subset of $V(R')$. The vertices of R have degree at most 4 in R' and the initial and terminal vertex of each path $P_{i,1}$ or $P_{i,2}$ has degree 3. Using this is not difficult to see that if some vertex of a path $P_{i,1}$ or $P_{i,2}$ is in V_2 then all the vertices of that path and the two adjacent switch vertices are in V_2. If there is some $i \in [n]$ so that both of the vertices $a_{i,1}, a_{i,2}$ or both of the vertices $b_{i,1}, b_{i,2}$ are in in V_2, then, using the observation we just made, all vertices of R' would be in V_2 which is impossible. Similarly we can show that V_1 cannot contain both of the vertices $a_{i,1}, a_{i,2}$ or both of the vertices $b_{i,1}, b_{i,2}$ for some $i \in [n]$. Hence for each switch $\{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}\}$, exactly one of the vertices $a_{i,1}, a_{i,2}$ and exactly one of the vertices $b_{i,1}, b_{i,2}$ is in V_2. Now we see that the vertices in V_1 and V_2 both induces a cycle in R and as the vertices $c_{j,1}, c_{j,2}$ have degree 2 outside R, the cycle in R which is in V_1 must contain a neighbour of each of $c_{j,1}, c_{j,2}$, $j \in [m]$. Hence, by Theorem 2.2, $\mathcal{F}$ is satisfiable.

We obtain the result for higher values of a by induction where we just proved the base case $a = 3$ above. Assume we have already constructed $H_a = H_a(\mathcal{F})$ with $\delta(H_a) \geq a + 1$ such that H_a has a $(\delta \geq a, \delta \geq a)$-partition if and only if $\mathcal{F}$ is satisfiable. Construct H_{a+1} from two copies of H_a by joining copies of the same vertex by an edge. It is easy to check that H_{a+1} has a $(\delta \geq a+1, \delta \geq a+1)$-partition if and only if H_a has a $(\delta \geq a, \delta \geq a)$-partition. ◇

Theorem 4.4 *Deciding whether a graph of minimum degree 3 has a $(\delta \geq 2, \delta \geq 3)$-partition is $\mathcal{NP}$-complete.*

Proof: Let $X = X(\mathcal{F})$ be the graph we obtain by starting from the ring graph $R(\mathcal{F})$ and then adding the following:

- Add the vertices of a tree T whose internal vertices have all degree 3 and which has $2m + 4n$ leaves denoted by $u_{1,1}, u_{1,2} \ldots, u_{n,1}, u_{n,2}, u'_{1,1}, u'_{1,2}, \ldots, u'_{n,1}, u'_{n,2}, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, \ldots, c_{m,1}, c_{m,2}$, where the pairs $u_{i,1}, u_{i,2}$ and $u'_{i,1}, u'_{i,2}$ have the same parent in T for $i \in [n]$ and so do each of the pairs $c_{j,1}, c_{j,2}$, $j \in [m]$. Here the vertices $c_{j,1}, c_{j,2}$ correspond to the clause C_j
- Join each vertex $c_{j,1}, c_{j,2}$, $j \in [m]$ to the 3 vertices in R which correspond to the literals which correspond to C_j and add the edge $c_{j,1}c_{j,2}$.
- Add $4n$ new vertices $w_{i,1}, w_{i,2}, w'_{i,1}, w'_{i,2}$, $i \in [n]$. Add the edges $w_{i,1}w_{i,2}, w'_{i,1}w'_{i,2}$, $i \in [n]$.
- For each $i \in [n]$ add the edges $w_{i,1}a_{i,1}, w_{i,2}a_{i,2}, w'_{i,1}b_{i,1}, w'_{i,2}b_{i,2}$.
- For each $i \in [n]$ join each of $u_{i,1}$ and $u_{i,2}$ by edges to the vertices $w_{i,1}, w_{i,2}$ and add the edge $u_{i,1}u_{i,2}$.
- For each $i \in [n]$ join each of $u'_{i,1}$ and $u'_{i,2}$ by edges to the vertices $w'_{i,1}, w'_{i,2}$ and add the edge $u'_{i,1}u'_{i,2}$.

We first prove that in any $(\delta \geq 2, \delta \geq 3)$-partition (V_1, V_2) of X all the vertices of T must be in the same set V_i. Note that we can not have $c_{j,1}$ and $c_{j,2}$ in different sets of the partition because then one of their 3 neighbours in $V(R)$ must be in both sets. By a similar argument, for each $i \in [n]$ the vertices $u_{i,1}$ and $u_{i,2}$ must belong to the same set in the partition and the vertices $u'_{i,1}$ and $u'_{i,2}$ must belong to the same set in the partition. It is easy to check that this implies our claim for T.

If $\mathcal{F}$ is satisfiable, then by Theorem 2.2, we can find vertex disjoint cycles C, C' in R such that C contains a vertex corresponding to a literal of C_j for each j and $V(R) = V(C) \cup V(C')$. Now we let $V_1 = V(C')$ and $V_2 = V(X) - V_1$. It is easy to check that this is a $(\delta \geq 2, \delta \geq 3)$-partition because C must contain exactly one of the vertices $a_{i,1}, a_{i,2}$ and exactly one of the vertices $b_{i,1}, b_{i,2}$ for each $i \in [n]$.
Suppose now that (V_1, V_2) is a good partition of X. By the argument above we have $V(T) \subset V_i$ for $i = 1$ or $i = 2$. This must be $i = 2$ since in the graph $X - V(T)$ all vertices except the switch vertices have degree 2. Thus we have $V(T) \subset V_2$ and each of the vertices $c_{j,1}, c_{j,2}$, $j \in [m]$ have a neighbour in

$V(R)$ which is also in V_2. As in earlier proofs it is easy to check that if some vertex of one of the paths $P_{i,1}, P_{i,2}$ is in V_2 then all vertices of that path are in V_2. As in the proof of the previous theorem, we now conclude that for each switch $\{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}\}$, exactly one of the vertices $a_{i,1}, a_{i,2}$ and exactly one of the vertices $b_{i,1}, b_{i,2}$ is in V_2. Now we see that the vertices in V_1 and V_2 both induce a cycle in R and as the vertices $c_{j,1}, c_{j,2}$ have degree 2 outside R, the cycle in R which is in V_2 must contain a neighbour of each of $c_{j,1}, c_{j,2}$, $j \in [m]$. Hence, by Theorem 2.2, $\mathcal{F}$ is satisfiable. $\diamond$

Corollary 4.5 *For every $k \geq 2$ it is $\mathcal{NP}$-complete to decide if a given graph with minimum degree at least $k+1$ has a $(\delta \geq k, \delta \geq k+1)$-partition.*

Proof: This follows by induction on k with Theorem 4.4 as the base case in the same way as we proved the last part of Theorem 4.3. $\diamond$

Proposition 4.6 *There is a polynomial algorithm for deciding whether a given graph of minimum degree 5 has a $(\delta \geq 2, \delta \geq 3)$-partition.*

Proof: Let G have $\delta(G) \geq 5$. By Theorem 1.1 and the algorithmic version of this result from [7] we may assume that G has a 3-cycle C. Denote its vertex set by $\{a, b, c\}$. If $\delta(G[V - V(C)] \geq 3)$ we are done as we can take $V_1 = V(C)$, so assume there is a vertex d which adjacent to all vertices of C. Then $\{a, b, c, d\}$ induce a K_4. If $\delta(G[V - \{a, b, c, d\}) \geq 2$ we can take $V_2 = \{a, b, c, d\}$, so we can assume that there is a vertex e which is adjacent to all 4 vertices in $\{a, b, c, d\}$ and now $\{a, b, c, d, e\}$ induce a K_5. Now if $G[V - \{a, b, c, d, e\}]$ contains a cycle C' we can conclude by starting with $V_2 = \{a, b, c, d, e\}$ and adding vertices of $V - V(C') - \{a, b, c, d, e\}$ to V_2 as long as there is one with at least 3 neighbours in V_2. When the process stops $(V - V_2, V_2)$ is a good partition. Hence we can assume that $G[V - \{a, b, c, d, e\}]$ is acyclic. If one connected component of $G[V - \{a, b, c, d, e\}]$ is non trivial with a spanning tree T, then two leaves u, v of T will share a neighbour in $\{a, b, c, d, e\}$. Without loss of generality this is e and now the K_4 induced by a, b, c, d and the cycle formed by e, u, v and the path between u and v in T are disjoint and we can find a good partition as we did above. Hence if we have not found the partition yet we must have that $G[V - \{a, b, c, d, e\}]$ is an independent set I, all of whose vertices are joined to all vertices in $\{a, b, c, d, e\}$. If $|I| \geq 2$ it is easy to find a good partition consisting of a 3-cycle on a, b and one vertex from I as V_1 and the remaining vertices as V_2. Finally if $|I| = 1$ there is no solution. $\diamond$

5 Further 2-partition problems

In [31] Thomassen proved that every graph G of connectivity at least $k_1 + k_2 - 1$ and minimum degree at least $4k_1 + 4k_2 + 1$ has a 2-partition (V_1, V_2) so that $G[V_i]$ is k_i-connected for $i = 1, 2$.

It is natural to ask about the complexity of deciding whether a graph has a 2-partition (V_1, V_2) with prescribed lower bounds on the (edge-)connectivity of $G[V_i]$, $i \in [2]$.

We start with a simple observation.

Proposition 5.1 *There exits a polynomial algorithm for deciding whether a given graph has a 2-partition (V_1, V_2) such that $G[V_1]$ is connected and $G[V_2]$ is 2-edge-connected.*

Proof: Suppose first that G is not 2-edge-connected. If G has more than two connected components it is a 'no'-instance. If it has two components, it is a 'yes'-instance if and only if one of these is 2-edge-connected. Hence we can assume that G is connected but not 2-edge-connected. Now it is easy to see that there is a good partition if and only if the block-cutvertex tree of G has a nontrivial block which is a leaf in the block-cutvertex tree. Thus assume below that G is 2-edge-connected. Now consider an ear-decomposition (sometimes called a handle-decomposition) of G where we start from an arbitrary cycle C. Let P be the last non-trivial ear that we add and let u, v be the end vertices of P. Then $V_1 = V(P) - \{u, v\}$ and $V_2 = V - V_1$ is a good partition. $\diamond$

Perhaps a bit surprisingly, if we require just a bit more for the connected part, the problem becomes $\mathcal{NP}$-complete.

Both $\mathcal{NP}$-completeness proofs below use reductions from a given 3-SAT formula $\mathcal{F}$ so we only describe the necessary modifications of $R(\mathcal{F})$.

Theorem 5.2 *It is $\mathcal{NP}$-complete to decide whether an undirected graph $G = (V, E)$ has a vertex partition (V_1, V_2) so that $G[V_1]$ is 2-edge-connected and $G[V_2]$ is connected and non-acyclic.*

Proof: We add vertices and edges to $R = R(\mathcal{F})$ as follows:

- For each clause C_j, $j \in [m]$ we add a vertex c_j and join it by three edges to the three literal vertices of R corresponding to C_j (as we did in several proofs above).
- Add new vertices $c'_1, c'_2, \ldots, c'_m$ and edges $c_j c'_j$, $j \in [m]$.
- Add new vertices $\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \alpha'_n$ and the edges $\alpha_i a_{i,1}, \alpha_i a_{i,2}, \alpha_i \alpha'_i$, $i \in [n]$
- Add new vertices $\beta_1, \ldots, \beta_n$, $\beta'_1, \ldots, \beta'_n$ and the edges $\beta_i b_{i,1}, \beta_i b_{i,2}, \beta_i \beta'_i$, $i \in [n]$.

We claim that the resulting graph G has a vertex partition (V_1, V_2) such that $G[V_1]$ is 2-edge-connected and $G[V_2]$ is connected and non-acyclic if and only if $\mathcal{F}$ is satisfiable. Note that, by construction, for every good partition every vertex of G which is not in R must belong to V_2. In particular if a path $P_{i,j}$ contains a vertex of V_2 then all the vertices of $P_{i,j}$ are in V_2. Since we want $G[V_2]$ to be connected, the edges between α_i, β_i, $i \in [n]$ and R imply that for every $i \in [n]$ at most one of the vertices $a_{i,1}, a_{i,2}$ and a most one of the vertices $b_{i,1}, b_{i,2}$ can belong to V_1. This implies that exactly one of $a_{i,1}, a_{i,2}$ and exactly one of the vertices $b_{i,1}, b_{i,2}$ belong to V_1 as otherwise V_1 would be empty. Now it is easy to check that the desired partition exists if and only if R contains a cycle C' which uses precisely one of the paths $P_{i,1}, P_{i,2}$ for $i \in [n]$ and avoids at least one literal vertex for every clause of $\mathcal{F}$. Thus, by Theorem 2.2, $\mathcal{F}$ is satisfiable if and only if G has a good partition. $\diamond$

Since we can decide whether a graph has two vertex disjoint cycles in polynomial time [9, 25] the following result, whose easy proof we leave to the interested reader, implies that it is polynomial to decide whether a graph has a 2-partition into two connected and non-acyclic graphs.

Proposition 5.3 *A graph G has a 2-partition (V_1, V_2) such that $G[V_i]$ is connected and has a cycle for $i = 1, 2$ if and only if G has a pair of disjoint cycles and either G is connected or it has exactly two connected components, each of which contain a cycle.*

Theorem 5.4 *It is $\mathcal{NP}$-complete to decide whether a graph $G = (V, E)$ has a vertex partition (V_1, V_2) so that each of $G[V_1]$ and $G[V_2]$ are 2-edge-connected.*

Proof: Let $\mathcal{F}$ be a 3-SAT formula and let $G = G(\mathcal{F})$ be the graph we constructed in the proof above. Let G_1 be the graph obtained by adding the following vertices and edges to G:

- add new vertices c''_j, $j \in [m]$, q_j, $j \in \{0\} \cup [m]$ and γ
- add the edges $c''_j c'_j$, $j \in [m]$
- add the edges of the path $q_0 c''_1 q_1 c''_2 q_2 \ldots c''_m q_m$
- complete this path into a cycle W by adding the edges $\gamma q_0, \gamma q_m$
- add an edge from γ to all vertices in $\{\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n\}$.

We claim that G' has a vertex-partition into two 2-edge-connected graphs if and only if $\mathcal{F}$ is satisfiable. First observe that in any good partition (V_1, V_2) we must have all vertices of W inside V_1 or V_2. This follows from the fact that each q_i has degree 2 so it needs both its neighbours in the same set. Without loss of generality, W is a cycle in V_2. After deleting the vertices of W we have exactly the graph G in the proof of Theorem 5.2 above and it is easy to see that all vertices not in $V(R)$ must belong to V_2 in any good partition. This implies that G' has the desired vertex-partition if and only if the graph G has a partition (V_1, V'_2) so that $G[V_1]$ is 2-edge-connected and $G[V'_2]$ is connected and non-acyclic. This problem is $\mathcal{NP}$-complete by Theorem 5.2 so the proof is complete. $\diamond$

By inspecting the proof above it is not difficult to see that the following holds.

Theorem 5.5 *It $\mathcal{NP}$-complete to decide whether a graph has a 2-partition (V_1, V_2) such that each of the graphs $G[V_i]$, $i = 1, 2$ are 2-connected.*

It may be worth while to try and extend the results of this section to higher (edge)-connectivities.

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