Large induced forests in planar graphs with girth 4

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Abstract

We give a new lower bound on the order of a largest induced forest in planar graphs with girth 4. We prove that a triangle-free planar graph of order n admits an induced forest of order at least $\frac{6n+7}{11}$, improving the lower bound of Salavatipour [M.R. Salavatipour. Large induced forests in triangle-free planar graphs. Graphs and Combinatorics, 22:113–126, 2006].

Keywords: Planar Graphs, Triangle-free Planar Graphs, Induced Forests, Feedback Vertex Sets, Albertson-Berman Conjecture

1 Introduction

Let G be a graph. A *decycling set* or *feedback vertex set* S of G is a subset of the vertices of G such that removing the vertices of S from G yields an acyclic graph. Thus S is a decycling set of G if and only if the graph induced by $V(G)\backslash S$ in G is an induced forest of G. The FEEDBACK VERTEX SET DECISION PROBLEM (which consists of, given a graph G and an integer k, deciding whether there is a decycling set of G of size k) is known to be NP-complete, even restricted to the case of planar graphs, bipartite graphs or perfect graphs [10]. It is thus legitimate to seek bounds for the size of a decycling set or for the order of an induced forest. The smallest size of a decycling set of G is called the *decycling number* of G, and the highest order of an induced forest of G is called the *forest number* of G, denoted respectively by $\phi(G)$ and $a(G)$. Note that the sum of the decycling number and the forest number of G is equal to the order of G (i.e. $|V(G)| = a(G) + \phi(G)$).

Mainly, the community focuses on the following challenging conjecture due to Albertson and Berman [3]:

Conjecture 1 (Albertson and Berman [3])**.** *Every planar graph of order n admits an induced forest of order at least $\frac{n}{2}$.*

Conjecture 1, if true, would be tight (for $n \geq 3$ multiple of 4) because of the disjoint union of complete graphs on four vertices (Akiyama and Watanabe [1] gave examples where the conjecture differs from the optimal by at most one half for all n), and would imply that every planar graph has an independent set on at least a quarter of its vertices, the only known proof of which relies on the Four-Color Theorem.

The best known lower bound to date for the forest number of a planar graph is due to Borodin and is a consequence of the acyclic 5-colorability of planar graphs [6]. We recall that an acyclic coloring is a proper vertex coloring such that the graph induced by the vertices of any two color classes is a forest. From this result we obtain the following theorem:

Theorem 2 (Borodin [6]). *Every planar graph of order n admits an induced forest of order at least $\frac{2n}{5}$.*

As a consequence of the acyclic 3-colorability of outerplanar graphs, Hosono [9] showed the following theorem which is best possible.

Theorem 3 (Hosono [9]). *Every outerplanar graph of order n admits an induced forest of order at least $\frac{2n}{3}$.*

The tightness of Theorem 3 is shown by the example in Figure 1.

Figure 1: Example that proves the tightness of Theorem 3.

Lower and upper bounds on forest number of planar graphs with girth 5 and 7 has also been deduced from results on acyclic coloring by Fertin et al. [8].

Theorem 4 (Fertin et al. [8]).

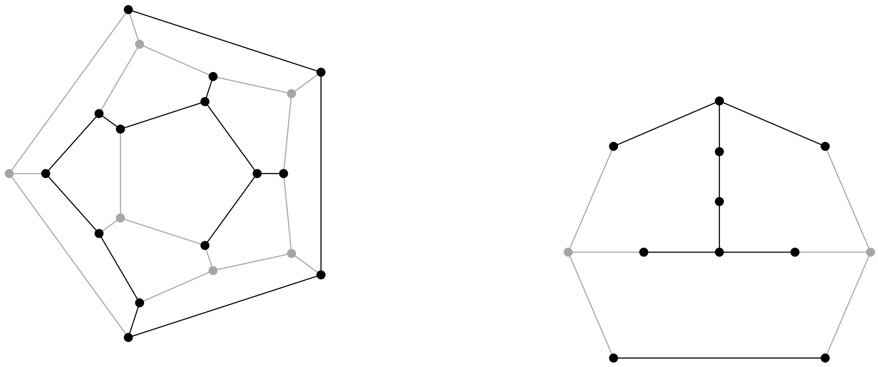
(1) Every planar graph of order n and girth at least 5 admits an induced forest of order at least $\frac{n}{2}$. Moreover, for $n \equiv 0(\mod 20)$, there exist planar graphs of order n and girth 5 having forest number $\frac{7n}{10}$ (disjoint copies of the dodecahedron, see Figure 2a).

(2) Every planar graph of order n and girth at least 7 admits an induced forest of order at least $\frac{2n}{3}$. Moreover, for $n \equiv 0(\mod 12)$, there exist planar graphs of order n and girth 7 having forest number $\frac{5n}{6}$ (disjoint copies of the graph depicted in Figure 2b).

Kowalik et al. [12] made the following conjecture on planar graph of girth at least 5:

Conjecture 5 (Kowalik et al. [12]). *Every planar graph with girth at least 5 and order n admits an induced forest of order at least $\frac{7n}{10}$.*

This conjecture, if true, would be tight due to Theorem 4. Very recently, Kelly and Liu [11], and Shi and Xu [14], independently improved Theorem 4(1). We note that Shi and Xu additionally characterize equality.



(a) The dodecahedron is a planar graph of girth 5 with forest number 14.

(b) This graph is a planar graph of girth 7 with forest number 10.

Figure 2: Examples of Theorem 4.

Theorem 6 (Kelly and Liu [11], Shi and Xu [14]). *Every connected planar graph of girth at least* 5*, order* n*, and size* m *has an induced forest of order at least* $\frac{8n-2m-2}{7}$*.*

Using Euler's formula, that implies that every connected planar graph with girth at least 5 and order n has an induced forest of order at least $\frac{(2n+2)}{3} \approx \frac{7n}{10.5}$ (recall that $\frac{7n}{10}$ is conjectured).

Akiyama and Watanabe [1], and Albertson and Haas [2] independently raised the following conjecture:

Conjecture 7 (Akiyama and Watanabe [1], and Albertson and Haas [2]). *Every bipartite planar graph of order* n *admits an induced forest of order at least* $\frac{5n}{8}$*.*

This conjecture, if true, would be tight for n multiple of 8: for example if G is the disjoint union of k cubes, then we have $a(G) = 5k$ and G has order $8k$ (see Figure 3). Motivated by Conjecture 7, Alon [4] proved the following theorem using probabilistic methods:

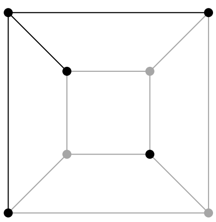


Figure 3: The cube has forest number 5.

Theorem 8 (Alon [4]). *There exist some* $b > 0$ *and* $b' > 0$ *such that:*

- *For every bipartite graph* G *with* n *vertices and average degree at most* d *(*≥ 1*),* $a(G) \geq (\frac{1}{2} + e^{-bd^2})n$*.*

- *For every $d \geq 1$ and all sufficiently large n there exists a bipartite graph with n vertices and average degree at most d such that $a(G) \leq (\frac{1}{2} + e^{-b'\sqrt{d}})n$.*

The lower bound was later improved by Colon et al. [7] to $a(G) \geq (1/2 + e^{-b''d})n$ for some constant b''.

Conjecture 7 also led to some research for lower bounds of the forest number of triangle-free planar graphs (as a superclass of bipartite planar graphs). Alon et al. [5] proved the following theorem and corollary:

Theorem 9 (Alon et al. [5]). *Every triangle-free graph of order n and size m admits an induced forest of order at least $n - \frac{m}{4}$.*

Corollary 10 (Alon et al. [5]). *Every triangle-free cubic graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

Theorem 9 is tight because of the union of cycles of length 4. In a planar graph with girth at least g, order n, and size m with at least a cycle, the number of faces is at most $\frac{2m}{g}$ (since all the faces' boundaries have length at least g). Then, by Euler's formula, $\frac{2m}{g} \geq m - n + 2$, and thus $m \leq \frac{g}{g-2}(n-2)$. In particular, triangle-free planar graphs of order $n \geq 3$ have size at most $2n - 4$.

As a consequence of Theorem 9, for every triangle-free planar graph G of order n, we have $a(G) \geq n/2$. That lower bound was improved for $n \geq 1$ by Salavatipour [13].

Theorem 11 (Salavatipour [13]). *Every triangle-free planar graph of order n and size m admits an induced forest of order at least $\frac{29n-6m}{32}$ and thus at least $\frac{17n+24}{32} \approx \frac{5n}{9.41}$.*

In 2010, Kowalik et al. [12] proposed that for every triangle-free planar graph G of order n and size m, $a(G) \geq \frac{119n-24m-24}{128} \geq \frac{71n+72}{128}$. However, the proof contains a flaw (contrarily to what the authors claim, the minimum counter-example is not necessarily connected). In Section 2, we give an infinite family of counter-examples for $a(G) \geq \frac{119n-24m-24}{128}$ and we propose an improvement of Theorem 11, which thus leads to the best known lower bound to our knowledge:

Theorem 12. *Every triangle-free planar graph of order n and size m admits an induced forest of order at least* $\max\{\frac{38n-7m}{44}, n - \frac{m}{4}\}$.

We note that Theorem 12 improves Theorem 9 when $m > \frac{3n}{2}$. Hence by Euler's formula the following corollary holds:

Corollary 13. *Every triangle-free planar graph of order $n \geq 1$ admits an induced forest of order at least $\frac{6n+7}{11} \approx \frac{5n}{9.17}$.*

2 Proof of Theorem 12

We first give a counter-example to the bound of Kowalik et al. [12]: we consider the disjoint union of k cubes. There are $8k$ vertices and $12k$ edges, hence Kowalik et al.'s lower bound tells us that there is an induced forest of size at least $\frac{119(8k)-24(12k)-24}{128} = 5k + (k-1)\frac{3}{16}$. However there cannot be an induced forest of more than 5 vertices in a cube (see Figure 3), and thus the biggest induced forest in our graph contains $5k$ vertices, which contradicts the lower bound. Furthermore, by increasing k, one can see that the biggest induced forest can be arbitrarily smaller than the supposed lower bound.

The proof of Theorem 12 consists in looking for a minimal counter-example G, proving some structural properties on G, and concluding that it cannot verify Euler's formula, which is contradictory.

Consider $G = (V, E)$. For a set $S \subset V$, let $G - S$ be the graph constructed from G by removing the vertices of S and all the edges incident to some vertex of S. If $x \in V$, then we denote $G - \{x\}$ by $G - x$. For a set S of vertices such that $S \cap V = \emptyset$, let $G + S$ be the graph constructed from G by adding the vertices of S. If $x \notin V$, then we denote $G + \{x\}$ by $G + x$. For a set F of pairs of vertices of G such that $F \cap E = \emptyset$, let $G + F$ be the graph constructed from G by adding the edges of F. If e is a pair of vertices of G and $e \notin E$, we denote $G + \{e\}$ by $G + e$. For a set $W \subset V$, we denote by $G[W]$ the subgraph of G induced by W.

We call a vertex of degree d, at least d, and at most d, a *d-vertex*, a *d^+-vertex*, and a *d^--vertex* respectively. Similarly, we call a cycle of length l, at least l, and at most l an *l-cycle*, an *l^+-cycle*, and an *l^--cycle* respectively, and by extension a face of length l, at least l, and at most l an *l-face*, an *l^+-face*, and an *l^--face* respectively.

Let $\mathcal{P}_4$ be the class of triangle-free planar graphs.We will prove of the following more general statement than Theorem 12:

$$0 \leq a \leq 1 \tag{1}$$
$$0 \leq b \tag{2}$$
$$a - 6b \leq 0 \tag{3}$$
$$3a - 10b \leq 1 \tag{4}$$
$$8a - 12b \leq 5 \tag{5}$$

Theorem 14. *If a and b are positive constants such that equations (1)–(5) are verified, then $a(G) \geq an - bm$ for all $G \in \mathcal{P}_4$.*

That series of inequalities defines a polygon represented in Figure 4, and for a triangle-free planar graph of given order n and size m, the highest lower bound will be given by maximizing $an - bm$ for a and b in this polygon. This maximum will be achieved at a vertex of the polygon. Moreover, by Euler's formula, every triangle-free planar graph of order $n \geq 3$ and size m satisfies $0 \leq m \leq 2n - 4$. Therefore for $n \geq 3$ the maximum will always be achieved at the intersection of either $3a - 10b = 1$ and $8a - 12b = 5$, or $8a - 12b = 5$ and $a = 1$. The corresponding intersections are $(b, a) = (\frac{7}{44}, \frac{38}{44})$ and $(b, a) = (\frac{1}{4}, 1)$, represented in Figure 4.

Let us show that any of the two lower bounds can be higher than the other, for graphs of arbitrarily high order.

For the disjoint union of k cubes (which is a graph of order $8k$ and size $12k$), the two lower bounds are equal to $5k$.

We consider now a graph composed of k disjoint cubes, where we remove an edge from each cube. This graph has $8k$ vertices and $11k$ edges. In this case we have $n - \frac{m}{4} = \frac{21}{4}k > \frac{38n-7m}{44} = \frac{227}{44}k$. More simply, for an independent set, $n - \frac{m}{4} = n > \frac{38n-7m}{44} = \frac{38n}{44}$.

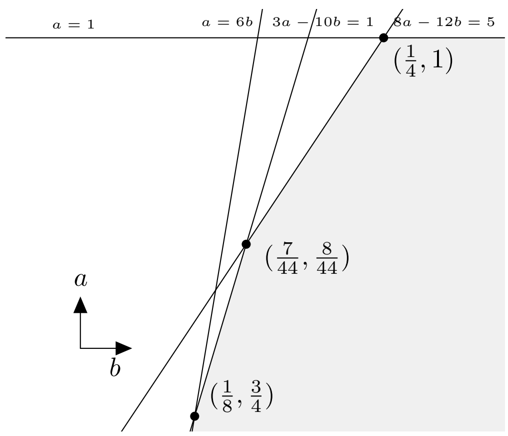


Figure 4: The top-left part of the polygon of the constraints on a and b.

We now consider a graph composed of k disjoint cubes, where we add an edge from each cube to the next one and an edge from the last one to the first one. This graph has $8k$ vertices and $13k$ edges. In this case, we have $n - \frac{m}{4} = \frac{19}{4}k < \frac{38n-7m}{44} = \frac{213}{44}k$. Also observe that for a quadrangulation on n vertices and $2n-4$ edges (i.e. a planar graph on n vertices that has only 4-faces), $n - \frac{m}{4} = \frac{n}{2} + 1 < \frac{38n-7m}{44} = \frac{6n+7}{11}$.

Let us now proceed to the proof of Theorem 14. For this proof we mainly adapt the methods of Kowalik et al. [12]. Let $G = (V, E)$ be a plane embedding of a counter-example to Theorem 14 with the minimum order. Let $n = |V|$ and $m = |E|$. We will use the scheme presented in Observation 15 for most of our lemmas.

Observation 15. *Let α, β, γ be integers satisfying $\alpha \geq 1$, $\beta \geq 0$, $\gamma \geq 0$ and $a\alpha - b\beta \leq \gamma$. Let $H^* \in \mathcal{P}_4$ be a graph with $|V(H^*)| = n - \alpha$ and $|E(H^*)| \leq m - \beta$.*

By minimality of G, H^* admits an induced forest F^* of order at least $a(n-\alpha) - b(m-\beta)$. If there is an induced forest F of G of order at least $|V(F^*)| + \gamma$, then we get a contradiction: as $a\alpha - b\beta \leq \gamma$, we have $|V(F)| \geq an - bm$.

Table 1 contains the values of (α, β, γ) that will be used throughout this section. For each one, the inequality $a\alpha - b\beta \leq \gamma$ is a consequence of the constraints (1)–(5). For instance, by adding (1) and (4), we get $a + (3a - 10b) \leq 1 + 1$, i.e. $4a - 10b \leq 2$. Simplifying by two yields the inequality $2a - 5b \leq 1$, which is the second line of Table 1.

We will now prove a series of lemmas on the structure of G.

Lemma 16. *Graph G is 2-edge-connected.*

Proof. By contradiction, suppose $V(G)$ is partitioned into two partite sets V_1 and V_2 such that there is at most one edge between vertices of V_1 and V_2. Consider graph $G[V_i]$ induced by the vertices of V_i (for $i = 1, 2$) with $n_i = |V_i|$ vertices and $m_i = |E(G[V_i])|$ edges. By minimality of G, $G[V_i]$ admits an induced forest, say F_i, with at least $an_i - bm_i$ vertices. Now the union of F_1 and F_2 (more formally, $G[V(F_1) \cup V(F_2)]$) is an induced forest of G having at

α	β	γ	proof
1	6	0	(3)
2	5	1	((1) + (4))/2
3	5	2	(3(1) + (4))/2
1	1	1	(1) + (2)
5	9	3	((1) + (3) + (5))/2
6	8	4	((1) + (5)) $*$ 2/3
4	10	2	(1) + (4)
7	13	4	((1) + 3(4) + 4(5))/6
3	10	1	(4)
8	12	5	(5)
6	14	3	((3) + (4) + (5))/2
8	19	4	((1) + (3) + 2(4) + (5))/2
9	24	4	((3) + 3(4) + (5))/2
10	23	5	((1) + 9(4) + 4(5))/6
9	19	5	(3(1) + (3) + 2(4) + (5))/2

Table 1: The various triples (α,β,γ) and the combinations of inequalities which imply $a\alpha - b\beta \leq \gamma$.

least $an_1 - bm_1 + an_2 - bm_2 = a(n_1 + n_2) - b(m_1 + m_2) \geq an - bm$ vertices as $m \geq m_1 + m_2$. A contradiction. $\square$

In particular, Lemma 16 implies that there is no 1^--vertex in G.

Lemma 17. *Every vertex in G has degree at most 5.*

Proof. By contradiction, suppose $v \in V(G)$ is a 6^+-vertex. Observation 15 applied to $H^* = G - v$ with $(\alpha, \beta, \gamma) = (1, 6, 0)$ and $F = F^*$ completes the proof. $\square$

Lemma 18. *If v is a 3-vertex adjacent to a 4^+-vertex w in G, then the two other neighbors of v have a common neighbor different from v.*

Proof. Let x and y be the two neighbors of v different from w. Suppose that they do not have a common neighbor different from v. Let $H^* = G + xy - \{w, v\}$. Graph H^* has $n - 2$ vertices and $m' \leq m - 5$ edges. As x and y do not have a common neighbor in G other than v, the addition of the edge xy does not create any triangle in H^*, thus $H^* \in \mathcal{P}_4$. Let F' be any induced forest of H^*. Adding v to F' (more formally, consider $G[V(F') \cup \{v\}]$) leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. $\square$

Lemma 19. *There is no 2-vertex adjacent to a 4^+-vertex in G.*

Proof. Let v be a 2-vertex adjacent to a 4^+-vertex w and $H^* = G - \{v, w\}$. Graph H^* has $n - 2$ vertices and $m' \leq m - 5$ edges. Let F' be any induced forest of H^*. Adding v to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. $\square$

Lemma 20. *There is no 3-vertex adjacent to two 2-vertices in G.*

Proof. Let v be a 3-vertex adjacent to two 2-vertices u and w and $H^* = G - \{u, v, w\}$. Graph H^* has $n-3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^*. Adding u and w to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ completes the proof. □

Lemma 21. *Every vertex in G has degree at least* 3.

Proof. Let v be a 2-vertex. Suppose that v has a neighbor u of degree 2 and a neighbor w of degree 3. Let $H^* = G - \{u, v, w\}$. Graph H^* has $n-3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^*. Adding u and v to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction.

Suppose that v has two neighbors of degree 3, say u and w. Consider three cases according to the number of neighbors u and w have in common.

- Suppose u and w have only v in common. Let $H^* = G + uw - v$. Graph H^* has $n-1$ vertices and $m' = m - 1$ edges. Observe that $H^* \in \mathcal{P}_4$. Let F' be any induced forest of H^*. Adding v to F' (more formally, consider $G[V(F') \cup \{v\}]$) does not create any cycle (the edge uw is just subdivided in uv, vw). Observation 15 applied to $(\alpha, \beta, \gamma) = (1, 1, 1)$ leads to a contradiction.

- Suppose u and w have two neighbors in common, say v and x. Let y be the last neighbor of u. By Lemma 20, both x and y have degree at least 3. Note that x and y are not adjacent because G has girth at least 4. Let $H^* = G - \{u, v, w, x, y\}$. Graph H^* has $n-5$ vertices and, since y and w are not adjacent (otherwise u and w have three common neighbors), $m' \leq m - 9$ edges. Let F' be any induced forest of H^*. Adding u, v and w to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction.

- Suppose u and w have three neighbors in common. Let x and y be the ones that are not v. Suppose x is a 4^+-vertex and let $H^* = G - \{u, v, w, x, y\}$. Graph H^* has $n-5$ vertices and $m' \leq m - 9$ edges (recall that y is a 3^+-vertex by Lemma 20). Let F' be any induced forest of H^*. Adding u, v and w to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction.

  W.l.o.g. we assume that x and y are 3-vertices. Let z be the third neighbor of x. Let $H^* = G - \{u, v, w, x, y, z\}$. Graph H^* has $n-6$ vertices and $m' \leq m - 8$ edges. Let F' be any induced forest of H^*. Adding u, v, x and y to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (6, 8, 4)$ leads to a contradiction.

Therefore, by Lemmas 16 and 19, every 2-vertex has only neighbors of degree 2. As G is connected (Lemma 16), either G does not have any 2-vertex or it is 2-regular. If G is 2-regular, then G is a n-cycle and thus $m = n$. Since $G \in \mathcal{P}_4$, we have $n \geq 4$. It is clear that G has an induced forest of size $n - 1$. Recall that $8a - 12b \leq 5$ and $a \leq 1$; this gives that $4(a - b) \leq 3$. Since $n \geq 4$, we can deduce that $an - bm = (a - b)n \leq n - 1$. This contradicts the fact that G is a counter-example. Therefore, G has minimum degree at least 3. This completes the proof. □

Lemma 22. *There is no 4-cycle in G with*

- *at least one 4^+-vertex and two opposite 3-vertices, or*
- *one 3-vertex opposite to a 4-vertex that has an edge going to the interior of the cycle and one going to the exterior of it.*

In particular there is no 4-cycle with exactly three 3-vertices in G.

Proof.

- Let $C = v_0v_1v_2v_3$ be a cycle such that v_0 and v_2 have degree 3 and v_3 is a 4^+-vertex. Suppose v_1 is a 4^+-vertex. Let $H^* = G - C$. Graph H^* has $n - 4$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^*. Adding v_0 and v_2 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ leads to a contradiction. Therefore v_1 has degree 3.

  Let u_0, u_1 and u_2 be the third neighbors of v_0, v_1 and v_2, respectively. Suppose $u_0 = u_2$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0\}$. Graph H^* has $n - 5$ vertices and $m' \leq m - 9$ edges. Let F' be any induced forest of H^*. Adding v_0, v_1 and v_2 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction. So u_0 and u_2 are distinct.

  By Lemma 18, $u_0u_1 \in E$ and $u_1u_2 \in E$. Assume u_0 (or u_2) has at most one neighbor $w \notin \{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 13$ edges. Let F' be any induced forest of H^*. Adding v_0, v_1, v_2 and u_0 to H^* leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$ leads to a contradiction. Thus both of the vertices u_0 and u_2 have at least two neighbors that are not in $\{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1 and v_2 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction.

- Let $C = v_0v_1v_2v_3$ be a cycle such that v_0 is a 3-vertex and v_2 is a 4-vertex with an edge going to the interior of the cycle and one going to the exterior of it. If v_1 and v_3 have degree 3, then we fall into the previous case. Therefore w.l.o.g. v_1 is a 4^+-vertex. Let $H^* = G - C$. Graph H^* has $n - 4$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^*. Adding v_0 and v_2 to F' leads to an induced forest of G. Indeed, if adding v_2 creates a cycle, then there is a path from the interior to the exterior of C in H^*, which is impossible. Observation 15 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof.

□

Lemma 23. *There is no 4-face with four 3-vertices in G.*

Proof. Suppose that there is such a 4-face $C = v_0v_1v_2v_3$, and let u_i be the third neighbor of v_i for $i = 0..3$. In the following, we consider the indices of the u_i and v_i modulo 4. If for some $i_0 \in \{0, 1, 2, 3\}$, $u_{i_0} = u_{i_0+1}$, then we have a triangle. Suppose now that $u_{i_0} = u_{i_0+2}$ for some $i_0 \in \{0, 1, 2, 3\}$, w.l.o.g. say $i_0 = 0$. In the cycle $v_0v_1v_2u_0$, the vertices v_0 and v_2 are two opposite 3-vertices. By Lemma 22, u_0 is a 3-vertex. Observe that u_1v_1 and u_3v_3 are separated by the cycle $v_0v_1v_2u_0$. Hence one of them is a bridge, contradicting Lemma 16.

Therefore all the u_i are distinct. We now consider the question of the presence or not of the edges u_iu_{i+1}. Consider the case $u_iu_{i+1} \notin E$ and $u_{i+1}u_{i+2} \notin E$ for some $i \in \{0, 1, 2, 3\}$, w.l.o.g. say $i = 0$. If $u_0u_2 \in E$, then either $u_2u_3 \notin E$ or $u_0u_3 \notin E$ (otherwise G has a triangle), and $u_1u_3 \notin E$ by planarity of G. Therefore up to the permutation of the indices, $u_0u_1 \notin E$, $u_1u_2 \notin E$ and $u_0u_2 \notin E$. We then define $H^* = G + x + \{xu_0, xu_1, xu_2\} - \{v_0, v_1, v_2, v_3\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges and belongs to $\mathcal{P}_4$ as u_0u_1, u_0u_2 and u_1u_2 are not in E. Let F' be any induced forest of H^*. Let F be the subgraph of G induced by $V(F')\backslash\{x\}$ plus v_0, v_1 and v_2 if $x \in F'$ or plus v_0 and v_2 if $x \notin F'$. Subgraph F is an induced forest of G. Hence, Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction. Therefore there must be an i such that $u_iu_{i+1} \in E$ and $u_{i+2}u_{i+3} \in E$, w.l.o.g. $u_0u_1 \in E$ and $u_2u_3 \in E$.

Let $G' = G - C$. Graph G' has $n - 4$ vertices and $m - 8$ edges. Let us now count, for each of the u_i's, the number of the neighbors of u_i that are not in $A = \{v_0, v_1, v_2, v_3, u_0, u_1, u_2, u_3\}$. The edges that are known in $G[A]$ are represented in Figure 5.

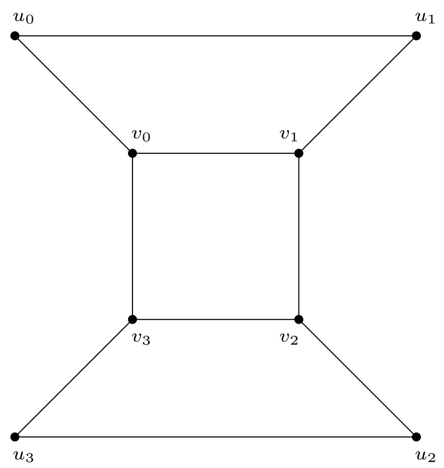


Figure 5: The graph $G[A]$ (only the edges that are known to be there are represented).

- Suppose w.l.o.g. u_0 has only neighbors in A, and another $u_{i'}$ has at most one neighbor not in A. Let $H^* = G' - \{u_0, u_1, u_2, u_3\}$. Graph H^* has $n - 8$ vertices. By Lemma 21, each of the u_i has degree at least 3. Graph H^* has $m' \leq m - 12$ edges. Let F' be any induced forest of H^*. Adding the vertices u_0, $u_{i'}$, v_1, v_2 and v_3 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (8, 12, 5)$ leads to a contradiction.
- Suppose w.l.o.g. u_0 has at most one neighbor not in A, and all the other u_i have each at least one neighbor not in A. Vertex u_0 is not adjacent both to u_2 and u_3 since G has girth at least 4. Let i_0 be such that $i_0 \neq 0$ and $u_0u_{i_0} \notin E$ (either $i_0 = 2$ or $i_0 = 3$). Let $H^* = G' - \{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$ (we remove all the vertices of A except u_{i_0}). Graph H^* has $n - 7$ vertices. Let us count the number of edges in G' that have an endvertex in

$\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. If $i_0 = 2$, then there are at least two edges for the neighbors of u_1 and u_3 that are not in A, plus the edges u_0u_1 and u_2u_3, plus one edge since u_0 has degree at least 3, thus at least 5 edges of H^* have an endvertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. If $i_0 = 3$, then there are at least two edges for the neighbors of u_1 and u_2 that are not in A, plus the edges u_0u_1 and u_2u_3, plus one edge since u_0 has degree at least 3, thus at least 5 edges of H^* have an endvertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. In both cases, H^* has $m' \leq m - 13$ edges. Let F' be any induced forest of H^*. Adding the vertices u_0, v_1, v_2 and v_3 to F' leads to an induced forest of G, since there is no path between u_0 and u_{i_0} in $G[\{v_1, v_2, v_3, u_0, u_{i_0}\}]$. Observation 15 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$ leads to a contradiction.

- So all the u_i have at least two neighbors not in A. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges, and if F' is any induced forest in H^*, then adding the vertices v_0, v_1 and v_2 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction and completes the proof.

□

Lemma 24. *There is no separating* 4*-cycle with four* 3*-vertices in* G.

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. We will consider the indices of the v_i modulo 4 in what follows. Since G is 2-edge-connected (Lemma 16), two of the v_i have their third neighbor in the interior of C, and the two other have theirs outside of it. There is a v_i such that the third neighbors of v_{i+1} and v_{i+2} are separated by C, w.l.o.g. for $i = 0$. Then let u be the third neighbor of v_0. Let $H^* = G - C - u$. Graph H^* has $n - 5$ vertices, and $m' \leq m - 9$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1 and v_2 to F' leads to a forest of G, thus Observation 15 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction. □

Lemma 25. *There is no* 3*-vertex adjacent to a* 5*-vertex in* G.

Proof. Let v be a 3-vertex adjacent to a 5-vertex u. Let w and x be the two other neighbors of v.

We first assume that w or x, w without loss of generality, is a 4^+-vertex. Let $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^*. Adding v to F' leads to an induced forest of G. Thus Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction.

Therefore w and x are 3-vertices. By Lemma 18, w and x have a common neighbor (distinct from v), which has degree 3 by Lemma 22. Finally Lemmas 23 and 24 lead to a contradiction, completing the proof. □

Lemma 26. *There is no separating* 4*-cycle with at least two* 3*-vertices in* G.

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. By Lemmas 22 and 24, C has exactly two 3-vertices. By Lemmas 21, 22 and 25, the two 3-vertices are adjacent, the two other vertices have degree 4 and none of the 4-vertices has a neighbor inside C and the other one outside C. W.l.o.g. the 3-vertices are v_0 and v_1. Let u_0 and u_1 be the third neighbors of v_0 and v_1 respectively.

If $u_0v_2 \in E$ or $u_1v_3 \in E$, say $u_0v_2 \in E$ w.l.o.g., then either $v_0v_1v_2u_0$ or $v_0v_3v_2u_0$ has a 3-vertex (v_0) opposite to a 4-vertex (v_2) with an edge going inside and one going outside of it, contradicting Lemma 22. Therefore $u_0v_2 \notin E$ and $u_1v_3 \notin E$.

By Lemma 18, $u_0u_1 \in E$; thus C does not separate u_0 and u_1, say u_0 and u_1 are in the exterior of C up to changing the plane embedding. By Lemmas 21–25, u_0 and u_1 are 4-vertices. At least one of v_2 or v_3, say v_2, has two neighbors inside of C (otherwise the cycle is not separating). Let $H^* = G - \{v_0, v_1, v_3, u_1\}$. Graph H^* has $n-4$ vertices and $m' \leq m-10$ edges, and if F' is any induced forest of H^*, then adding v_0 and v_1 to F' leads to an induced forest of G (since v_2 is only connected to the interior and u_0 to the exterior of C). Observation 15 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof. □

Lemma 27. *There is no* 4*-face with exactly two* 3*-vertices in* G*.*

Proof. Let $C = v_0v_1v_2v_3$ be such a face. By Lemmas 21 and 22 the two 3-vertices are adjacent. W.l.o.g. v_0 and v_1 have degree 3, and v_2 and v_3 have degree 4 (by Lemmas 21 and 25). Let u_0 and u_1 be the third neighbors of v_0 and v_1 respectively. By Lemma 18 applied to v_0 and v_3, and v_1 and v_2, $u_0u_1 \in E$. Then by Lemma 26, $v_0v_1u_1u_0$ cannot be a separating cycle, and so it is the boundary of some 4-face. If both u_0 and u_1 have degree 3, we have a contradiction to Lemma 23. If one has degree 3 and the other has degree at least 4, we have a contradiction to Lemma 22. Finally, by Lemma 25, u_0 and u_1 are 4-vertices.

If v_2 is adjacent to u_0, then $u_0v_0v_1v_2$ is a separating 4-cycle, with two 3-vertices, contradicting Lemma 26. Hence v_2u_0 is not in E. Similarly, v_3u_1 is not in E. Since $G \in \mathcal{P}_4$, either u_0 and v_2 do not have a common neighbor, or u_1 and v_3 do not have a common neighbor. By symmetry assume that u_0 and v_2 do not have a common neighbor. Let $H^* = G + u_0v_2 - \{u_1, v_0, v_1, v_3\}$. Graph H^* has $n-4$ vertices, $m' \leq m-10$ edges and belongs to $\mathcal{P}_4$. Let F' be any induced forest of H^*. Adding v_0 and v_1 to F' leads to an induced forest of G (intuitively the edge u_0v_2 is just subdivided). Observation 15 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof. □

Lemma 28. *There is no* 4*-cycle with at least two* 3*-vertices in* G*.*

Proof. It follows from Lemmas 22, 23, 26 and 27. □

Lemma 29. *There is no* 4*-face with exactly one* 3*-vertex in* G*.*

Proof. Let $C = v_0v_1v_2v_3$ be such a face. W.l.o.g. v_0 is the 3-vertex and v_1, v_2 and v_3 are 4^+-vertices. By Lemma 25, v_1 and v_3 are 4-vertices. Let u_0 be the third neighbor of v_0. Vertex u_0 is different from v_2 and non-adjacent to v_1 and v_3 (G is triangle-free).

Let us first assume that $u_0v_2 \in E$. By Lemmas 21, 25 and 28, u_0 is a 4-vertex. Assume v_2 has degree 5. Let $H^* = G - \{u_0, v_0, v_2\}$. Graph H^* has $n-3$ vertices and $m-10$ edges. Let F' be any induced forest of H^*. Adding the vertex v_0 to F' leads to an induced forest of G (since $u_0v_0v_1v_2$ separates the neighbours of v_1 that are not in C and the neighbours of v_3 that are not in C). Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction. Hence v_2 has degree 4. Then either $v_0v_1v_2u_0$ or $v_0v_3v_2u_0$ has a 3-vertex opposite to a 4-vertex with a neighbor in the interior and one in the exterior of it, contradicting Lemma 22.

Thus u_0 is non-adjacent to v_2. By Lemma 18, v_1 and u_0 have a common neighbor other than v_0, say u_1. It is distinct from all the vertices we defined previously. By Lemma 28 applied to $v_0v_1u_1u_0$, u_0 and u_1 have degree at least 4. By Lemma 25, u_0 has degree exactly 4.

Suppose $u_1v_3 \in E$. As C is a face, the last neighbor of v_1 ($\neq v_0, v_2, u_1$), say w_1, is not in the interior of C. The cycle $v_0v_1u_1v_3$ separates u_0 and v_2. Suppose first that $v_0v_1u_1v_3$ does not separate u_0 and w_1. Then $v_0v_1u_1u_0$ separates v_3 and w_1. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1\}$. Graph G^* has $n-6$ vertices and $m' \leq m-14$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1 and v_3 to F' leads to an induced forest of G. Hence Observation 15 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction. Therefore $v_0v_1u_1v_3$ separates u_0 and w_1. Assume u_1 has degree 5. Let $H^* = G - \{u_1, v_0, v_3\}$. Graph H^* has $n-3$ vertices and $m-10$ edges. Let F' be any induced forest of H^*. Adding the vertex v_0 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction. Hence u_1 has degree 4. Then $v_0v_1u_1v_3$, $v_0u_0u_1v_3$ or $v_0v_1u_1u_0$ has a 3-vertex opposite to a 4-vertex with a neighbor in the interior and one in the exterior of it, contradicting Lemma 22.

So u_1 cannot be adjacent to v_3. As $u_1v_3 \notin E$ and $u_0v_2 \notin E$, by Lemma 18 v_3 and u_0 have a common neighbor distinct from v_0, say u_3. By what precedes and by symmetry, it is of degree at least 4 and non-adjacent to v_0, v_1, v_2 and u_1 (it has a role similar to that of u_1, and is non-adjacent to u_1 because of the girth assumption). See Figure 6 for a reminder of the structure of $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$. Vertex v_0 has degree 3, v_1, v_3 and u_0 are 4-vertices, and v_2, u_1 and u_3 are 4^+-vertices. Recall that $u_1v_3 \notin E$, $u_3v_1 \notin E$ and $u_0v_2 \notin E$.

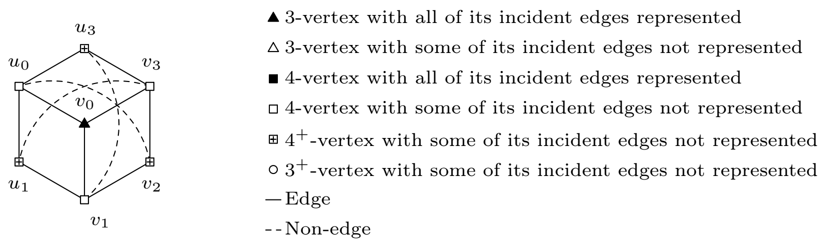


Figure 6: Graph $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$.

Let w_0, w_1 and w_3 be the fourth neighbors of u_0, v_1 and v_3 respectively. In the following we will no longer use the fact that C is a face. By the girth assumption, w_0 is not adjacent to u_1 or u_3. Suppose w_0 is adjacent to v_1 or to v_3, say $w_0v_1 \in E$. Then by the girth assumption, $w_0v_2 \notin E$. By Lemma 28 applied to $v_0v_1w_0u_0$, w_0 is a 4^+-vertex. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. Graph H^* has $n-8$ vertices and $m' \leq m-19$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G. Hence Observation 15 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. So w_0 is not adjacent to v_1 or v_3. By symmetry, w_0, w_1 and w_3 are distinct.

Suppose $w_0v_2 \in E$. Assume that C separates w_1 and w_3, or that it does not separate w_1 and w_3 nor w_0 and w_1. Then either C or $v_0v_1v_2w_0u_0$ separates w_1 and w_3. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. Graph H^* has $n-8$

vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G. Hence Observation 15 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. Thus C does not separate w_1 and w_3 but separates w_1 and w_0. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_3\}$. Graph H^* has $n - 8$ vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^*. Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G. Hence Observation 15 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. So $w_0 v_2 \notin E$, and similarly $w_1 u_3 \notin E$ and $w_3 u_1 \notin E$.

Thus the only edges that may or may not exist between the vertices we defined are $w_0 w_1$, $w_0 w_3$ and $w_1 w_3$. See Figure 7 for a reminder of the edges and vertices we know to this point. Vertex v_0 has degree 3, v_1, v_3 and u_0 are 4-vertices and v_2, u_1 and u_3 are 4^+-vertices. Vertices v_0, v_1, v_3 and u_0 have all their incident edges represented in Figure 7.

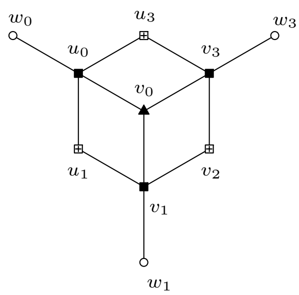


Figure 7: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1 and w_3.

Suppose $w_0 w_1 \notin E$, $w_0 w_3 \notin E$, and $w_1 w_3 \notin E$. Let $H^* = G + x + \{x w_0, x w_1, x w_3\} - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges, and is in $\mathcal{P}_4$. Let F' be any induced forest of H^*. Either $x \in F'$, then the graph induced by $V(F') \cup \{v_0, v_1, v_3, u_0\} \backslash \{x\}$ in G is a forest, or $x \notin F'$, then adding v_1, v_3 and u_0 to F' leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction. Thus there is at least one edge among $w_0 w_1$, $w_0 w_3$ and $w_1 w_3$. Moreover, since there is no triangle in G, there are no more than two of these edges. W.l.o.g. let us assume that $w_0 w_1 \notin E$ and $w_0 w_3 \in E$.

Let us now prove some claims that we will use later :

(a) Suppose that w_0 and w_1 are 4^+-vertices, or that one is a 3-vertex, the other a 4^+-vertex, and v_2, u_1 or u_3 has degree 5. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1\}$. Graph H^* has $n - 9$ vertices and $m' \leq m - 24$ edges, and adding v_0, v_1, v_3 and u_0 to any induced forest of H^* leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (9, 24, 4)$ leads to a contradiction.

(b) Suppose w_0 or w_3, say w_{i_0}, is a 3-vertex and either one of the w_i is a 4^+-vertex, or $w_1 w_3 \notin E$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. Graph H^* has $n - 10$ vertices and $m' \leq m - 23$ edges, and adding v_0, v_1, v_3, u_0 and w_{i_0} to any induced forest of H^* leads to an induced forest of G. Observation 15 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$ leads to a contradiction.

(c) Suppose w_0 and w_3 are 3-vertices and w_1 and w_3 are adjacent. Let $H^* = G - \{v_0, v_1, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. Graph H^* has $n-9$ vertices and $m' \leq m - 19$ edges, and adding v_0, v_1, u_0, w_0 and w_3 to any induced forest of H^* leads to an induced forest of G (by planarity, since $w_1w_3 \in E$ and $w_0w_3 \in E$, the cycle $v_0v_1w_1w_3v_3$ separates v_2 from w_0 in G). Observation 15 applied to $(\alpha, \beta, \gamma) = (9, 19, 5)$ leads to a contradiction.

If $w_1w_3 \in E$, then both w_0 and w_3 are 4^+-vertices (by (b) and (c)), and by symmetry w_1 is also a 4^+-vertex, which is impossible (by (a)). Hence $w_1w_3 \notin E$.

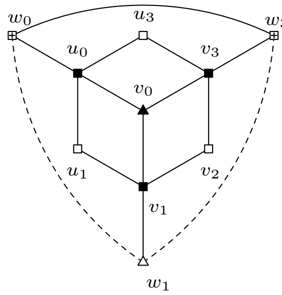


Figure 8: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1 and w_3.

Therefore w_0 and w_3 are 4^+-vertices (by (b)), thus w_1 has degree 3 (by (a)), and v_2, u_1 and u_3 have degree 4 (by (a)) (see Figure 8). Let y_0 and y_1 the two neighbors of w_1 other than v_1. By Lemma 18 they have a common neighbor other than w_1, say t. So by Lemmas 25 and 28 in $w_1y_0ty_1$, y_0 and y_1 have degree 4, and by Lemma 18 each one is adjacent either to v_2 or to u_1. If they are both adjacent to the same one, say v_2 w.l.o.g., then either $v_2v_1w_1y_0$ or $v_2v_1w_1y_1$ is a 4-cycle with a 3-vertex (w_1) opposite to a 4-vertex (v_2) that has both an edge going outside and one going inside of it, which is impossible by Lemma 22. W.l.o.g., say y_0 is adjacent to v_2 and y_1 is adjacent to u_1. At this point we know that v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0 and y_1 are distinct and do not share an edge that we do not already know. See Figure 9 for a reminder of the edges and vertices we know to this point.

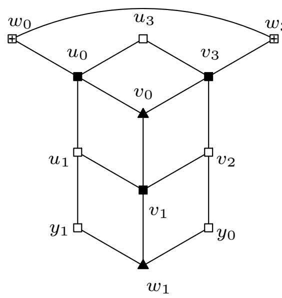


Figure 9: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3, y_0 and y_1.

Let z be the neighbor of v_2 different from v_1, v_3 and y_0. The only edges that may or not be among v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1 and z are zy_1 and zu_1, and as G is triangle-free, there is at most one of those edges. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1, z\}$. Graph H^* has $n - 10$ vertices and $m' \leq m - 23$ edges (recall that u_1 cannot be adjacent both to y_0 and y_1, and thus is not adjacent to y_0). Adding to any induced forest of H^* the vertices v_0, v_1, v_2, u_1 and w_1 leads to an induced forest of G, so Observation 15 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$ leads to a contradiction, completing the proof. □

Lemma 30. *There is no* 5*-face with only* 3*-vertices in* G*.*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a face, and u_0, u_1, u_2, u_3, and u_4 be the third neighbors of v_0, v_1, v_2, v_3 and v_4 respectively. The u_i are all distinct due to the girth assumption and Lemma 26. We will consider the indices of the u_i and v_i modulo 5. There is no edge u_iu_{i+1} for any i due to Lemma 28. Let $H^* = G + x + y + \{xu_0, xu_1, yu_2, yu_3, xy\} - C$. Graph H^* has $n - 3$ vertices and $m - 5$ edges. Let F' be any induced forest of H^*. Let F be the subgraph of G induced by the vertices of $V(F')\backslash\{x, y\}$, plus the vertices v_0 and v_3, plus v_1 if $x \in V(F')$, and plus v_2 if $y \in V(F')$. Subgraph F is an induced forest of G. Thus Observation 15 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction completing the proof. □

Lemma 31. *There is no 3-vertex adjacent to a* 3*-vertex and to a* 4*-vertex in* G*.*

Proof. Let v be a 3-vertex adjacent to a 3-vertex u and to a 4-vertex w. Let x be the third neighbor of v. By Lemma 18, x and u have a common neighbor distinct from v which contradicts Lemma 28. □

For every face f of G, let $l(f)$ be the length of f, and let $c_{4^+}(f)$ be the number of 4^+-vertices in f. For every vertex v, let $d(v)$ be the degree of v. Let k be the number of faces of G, and for every $3 \leq d \leq 5$ and every $4 \leq l$, let k_l be the number of faces of length l and n_d the number of d-vertices in G.

Each 4-vertex is in the boundary of at most four faces, and each 5-vertex is in the boundary of at most five faces. Therefore the sum of the $c_{4^+}(f)$ over all the 4-faces and 5-faces is $\sum_{f, 4 \leq l(f) \leq 5} c_{4^+}(f) \leq 4n_4 + 5n_5$. From Lemmas 25, 30 and 31 we can deduce that for each 5-face f we have $c_{4^+}(f) \geq 2$. Moreover, by Lemmas 28 and 29, for each 4-face f, $c_{4^+}(f) \geq 4$. Thus $\sum_{f, l(f)=4} c_{4^+}(f) + \sum_{f, l(f)=5} c_{4^+}(f) \geq 4k_4 + 2k_5$. Thus we have the following:

$$4n_4 + 5n_5 \geq 4k_4 + 2k_5$$

By Euler's formula, we have:

$$\begin{aligned}
-12 &= 6m - 6n - 6k \\
&= 2 \sum_{v \in V(G)} d(v) + \sum_{f \in F(G)} l(f) - 6n - 6k \\
&= \sum_{d \geq 3} (2d - 6)n_d + \sum_{l \geq 4} (l - 6)k_l \\
&\geq 2n_4 + 4n_5 - 2k_4 - k_5 \\
&\geq 0
\end{aligned}$$

This is a contradiction, which ends the proof of Theorem 14.

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