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# On non-repetitive sequences of arithmetic progressions: the cases $k \in \{4, 5, 6, 7, 8\}$

Borut Lužar\*<sup>§</sup> Martina Mockovčiaková<sup>†</sup> Pascal Ochem<sup>‡</sup>  
Alexandre Pinlou<sup>‡</sup> Roman Soták<sup>§</sup>

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## Abstract

A  $d$ -subsequence of a sequence  $\varphi = x_1 \dots x_n$  is a subsequence  $x_i x_{i+d} x_{i+2d} \dots$ , for any positive integer  $d$  and any  $i$ ,  $1 \leq i \leq n$ . A  $k$ -Thue sequence is a sequence in which every  $d$ -subsequence, for  $1 \leq d \leq k$ , is non-repetitive, i.e. it contains no consecutive equal subsequences. In 2002, Grytczuk proposed a conjecture that for any  $k$ ,  $k+2$  symbols are enough to construct a  $k$ -Thue sequences of arbitrary lengths. So far, the conjecture has been confirmed for  $k \in \{1, 2, 3, 5\}$ . Here, we present two different proving techniques, and confirm it for all  $k$ , with  $2 \leq k \leq 8$ .

**Keywords:** non-repetitive sequence,  $k$ -Thue sequence,  $(k+2)$ -conjecture

## 1 Introduction

A *repetition* in a sequence  $\varphi$  is a subsequence  $\rho = x_1 \dots x_{2t}$  of consecutive terms of  $\varphi$  such that  $x_i = x_{t+i}$  for every  $i = 1, \dots, t$ . The length of a repetition is hence always even and comprised of two identical *repetition blocks*,  $\rho_1 = x_1 \dots x_t$  and  $\rho_2 = x_{t+1} \dots x_{2t}$ . A sequence is called *non-repetitive* or *Thue* if it does not contain any repetition. Surprisingly, as shown by Thue [12] (see [1] for a translation), having three distinct symbols suffices to construct non-repetitive sequences of arbitrary lengths. This result is a fundamental piece in the theory of combinatorics on words. After that, a number of other concepts related to repetitions has been presented (see e.g. [2] for more details).

In this paper, we continue dealing with the following generalization. A (possibly infinite) sequence  $\varphi$  is  $k$ -Thue (or *non-repetitive up to mod  $k$* ) if every  $d$ -subsequence of

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\*Faculty of Information Studies, Novo mesto, Slovenia. E-Mail: borut.luzar@gmail.com

<sup>†</sup>NTIS, University of West Bohemia, Pilsen, Czech Republic. E-Mail: mmockov@ntis.zcu.cz

<sup>‡</sup>LIRMM, Université Montpellier 2, Montpellier, France.

E-Mails: {pascal.ochem,alexandre.pinlou}@lirmm.fr

<sup>§</sup>Institute of Mathematics, Faculty of Science, Pavol J. Šafárik University, Košice, Slovakia. E-Mail: roman.sotak@upjs.sk

$\varphi$  is Thue, for  $1 \leq d \leq k$ . By a  $d$ -subsequence of  $\varphi$  we mean an arithmetic subsequence  $x_i x_{i+d} x_{i+2d} \dots$  of  $\varphi$ . Consider a sequence

$$a \underline{b} d \underline{c} b \underline{c},$$

which is Thue, but not 2-Thue, since the 2-subsequence  $b c c$  is not Thue. On the other hand,

$$\underline{a} b c \underline{a} d b$$

is 2-Thue, but not 3-Thue, due to the repetition in the 3-subsequence  $a a$ .

This generalization was introduced by Currie and Simpson [6] and has been immediately followed by an intriguing conjecture due to Grytczuk [8].

**Conjecture 1** (Grytczuk, 2002). *For any positive integer  $k$ ,  $k+2$  distinct symbols suffice to construct a  $k$ -Thue sequence of any length.*

It is easy to show that having only  $k+1$  symbols there is a repetition in any sequence of length at least  $2k+2$ , so the bound  $k+2$  is tight.

Since 1-Thue sequences are simply Thue sequences, the above mentioned result establishes the conjecture for  $k=1$ . The conjecture has also been confirmed for  $k=2$  in [6] and independently in [11], for  $k=3$  in [6], and for  $k=5$  in [4]. Although it has been considered also for the case  $k=4$  by Currie and Pierce [5] using an application of the fixing block method, it remains open for all the cases except  $k \in \{1, 2, 3, 5\}$ .

Several upper bounds have been established, first being  $e^{33k}$  due to Grytczuk [8], and then substantially improved to  $2k + O(\sqrt{k})$  in [9]. Currently the best known upper bound is due to Kranjc et al. [11].

**Theorem 1** (Kranjc et al., 2015). *For any integer  $k \geq 2$ ,  $2k$  distinct symbols suffice to construct a  $k$ -Thue sequence of any length.*

The proof of the above is constructive and provides  $k$ -Thue sequences of given lengths.

The aim of this paper is two-fold. The main contribution is answering Conjecture 1 in affirmative for several additional values of  $k$ .

**Theorem 2.** *For any  $k \in \{4, 5, 6, 7, 8\}$ ,  $k+2$  distinct symbols suffice to construct a  $k$ -Thue sequence of any length.*

Moreover, we present two different techniques of proving the above theorem. In the former, described in Section 3, we use exhaustive computer search to determine morphisms for each  $k$ ,  $k \in \{4, 5, 6, 7, 8\}$ , from which we construct  $k$ -Thue sequences. In the latter, described in Section 5, we use concatenation of special blocks given by another morphism. The purpose of the latter one is to introduce its ability to deal with larger  $k$ 's, therefore we only prove the cases  $k=4$  and  $k=6$ . We believe, in the future, it could be used for proving Conjecture 1 for infinitely many values of  $k$ .

## 2 Preliminaries

In this section, we introduce additional terminology and notation used in the paper. Throughout the paper,  $i$  and  $t$  are used to determine positive integers, unless more details are given.

An  $\mathbb{A}$ -*sequence* (or simply a *sequence* when the alphabet is known from the context or not relevant) of length  $t$  is an ordered tuple of  $t$  symbols from some alphabet  $\mathbb{A}$ . Let  $\varphi = x_1 \dots x_t$  be a sequence. A subsequence of  $\varphi$  of consecutive terms  $x_i \dots x_j$ , for some  $i, j$ ,  $1 \leq i \leq j \leq t$ , is denoted by  $\varphi(i, j)$ . A *term* indicates an element of a sequence at a specified index. A *block* is a subsequence of consecutive terms of some sequence. When we refer to a term as a term of a block, by its index we mean the index of a term in the block. We denote the term at index  $i$  in a sequence  $\varphi$  (resp. a block  $\beta$ ) by  $\varphi(i)$  (resp.  $\beta(i)$ ).

A *prefix* of a sequence  $\varphi = x_1 \dots x_r$  is a sequence  $\pi = x_1 \dots x_s$ , for some integer  $s \leq r$ . A *suffix* is defined analogously. In a sequence  $\varphi$  consider a pair of sequences  $\pi$  and  $\varepsilon$  such that  $\pi\varepsilon$  is a subsequence of  $\varphi$ ,  $\pi$  has length at least 1, and  $\varepsilon$  is a prefix of  $\pi\varepsilon$ . The *exponent* of  $\pi\varepsilon$  is

$$\exp(\pi\varepsilon) = \frac{|\pi\varepsilon|}{|\pi|}.$$

If a sequence has exponent  $p$ , we call it a  $p$ -*repetition*. A sequence is  $q^+$ -*free* if it contains no  $p$ -repetition such that  $p > q$ . For sequences over 3-letter alphabets, Dejean [7] proved the following.

**Theorem 3** (Dejean, 1972). *Over 3-letter alphabets there exist  $\frac{7}{4}^+$ -free sequences of arbitrary lengths.*

A *morphism* is a mapping  $\mu$  which assigns to each symbol of an alphabet a sequence. Applied to a sequence  $\varphi$ ,  $\mu(\varphi)$  is the sequence obtained from  $\varphi$  where every symbol is replaced by its image according to  $\mu$ . We say that a morphism is  $k$ -*uniform* if it maps every symbol from the domain to some sequence of length  $k$ .

Given a sequence  $\varphi = \beta_1 \dots \beta_t$  comprised of blocks  $\beta_i$ , for  $1 \leq i \leq t$ , the *covering subsequence*  $\hat{\sigma}$  of a subsequence  $\sigma$  in  $\varphi$  is the subsequence  $\varphi(i, j)$ , where  $i$  is the index of the first term of the block containing the first term of  $\sigma$ , and  $j$  is the index of the last term of the block containing the last term of  $\sigma$ .

An  $i$ -*shift* of  $\varphi$  is the sequence  $\varphi^i = x_{i+1} \dots x_\ell x_1 \dots x_i$ , i.e. the sequence  $\varphi$  with the subsequence of the first  $i$  elements moved to the end. Let  $\varphi$  be a sequence of length  $\ell$ . We define the *circular sequence*  $\varphi_\zeta^{(\ell, t)}$  of order  $\ell$  and length  $\ell^2 \cdot t$  as

$$\varphi_\zeta^{(\ell, t)} = \underbrace{\varphi^0 \varphi^1 \dots \varphi^{\ell-1} \dots \varphi^0 \varphi^1 \dots \varphi^{\ell-1}}_t.$$

We call each subsequence  $\varphi^i$  of  $\varphi_\zeta^{(\ell, t)}$  a  $\zeta$ -*block*.

Apart from concatenation of sequences, we define another sequence combining operation. Let  $\varphi_1$  and  $\varphi_2$  be sequences of lengths  $\ell \cdot t$ . A *sequence wreathing of order  $\ell$*  of  $\varphi_1$  and  $\varphi_2$ , denoted by  $\varphi_1 \wr_\ell \varphi_2$ , is consecutive concatenation of  $k$  subsequent elements of  $\varphi_1$  and  $\varphi_2$ , i.e.

$$\varphi_1 \wr_\ell \varphi_2 = \varphi_1(1, \ell) \varphi_2(1, \ell) \dots \varphi_1((t-1)\ell + 1, t \cdot \ell) \varphi_2((t-1)\ell + 1, t \cdot \ell).$$

We call the sequences  $\varphi_1$  and  $\varphi_2$  the *base* and the *wrap* of sequence wrapping  $\varphi_1 \wr_\ell \varphi_2$ , respectively. Additionally, the blocks  $\varphi_1(i\ell + 1, (i+1)\ell)$  and  $\varphi_2(i\ell + 1, (i+1)\ell)$  are respectively called a *base-block* and a *wrap-block*.

We conclude this section with two lemmas we will use in the forthcoming sections. The former, due to Currie [3], states that insertion of non-repetitive subsequences (over distinct alphabets) into a non-repetitive sequence preserves non-repetitiveness.

**Lemma 4** (Currie, 1991). *Let  $\varphi_0 = x_1 \dots x_t$  be a non-repetitive  $\mathbb{A}$ -sequence, and  $\varphi_1, \dots, \varphi_{t+1}$  be non-repetitive  $\mathbb{B}$ -sequences, where  $\mathbb{A}, \mathbb{B}$  are disjoint alphabets. Additionally, the length of any  $\varphi_i$ ,  $1 \leq i \leq t+1$ , may be 0. Then, the sequence  $\varphi_1 x_1 \dots \varphi_t x_t \varphi_{t+1}$  is non-repetitive.*

Proving that a non-repetitive sequence  $\varphi$  is  $k$ -Thue for some integer  $k > 1$ , one needs to show that every  $\ell$ -subsequence of  $\varphi$  is non-repetitive for every integer  $\ell$ ,  $1 \leq \ell \leq k$ . To prove that an  $\ell$ -subsequence is non-repetitive, it suffices to have enough information about  $\varphi$  as we show in the next lemma. Let  $\varphi = \beta_1 \dots \beta_t$  be a sequence comprised of blocks  $\beta_i$ ,  $1 \leq i \leq t$ . We say that a block  $\beta_i$  is *uniquely determined by a subset of terms* if there is no block  $\beta_j$ ,  $\beta_i \neq \beta_j$ , having the same terms at the same positions. E.g., from the construction of circular sequences, we have the following.

**Observation 1.** *A  $\zeta$ -block  $\varphi^i$ ,  $0 \leq i \leq \ell-1$ , is uniquely determined by one term, i.e., given at least one term of a  $\varphi^i$ , one can determine  $i$ .*

We use the following lemma as a tool for proving that some  $d$ -subsequence of a Thue sequence does not contain a repetition.

**Lemma 5.** *Let  $\sigma$  be an  $\ell$ -subsequence of a sequence  $\varphi = \beta_1 \beta_2 \dots \beta_t$ , for some positive integers  $\ell$  and  $t$ . Let  $\rho_1 \rho_2$  be a repetition in  $\sigma$ , and let, for some  $j$ ,  $\gamma_1 = \beta_{j+1} \dots \beta_{j+r}$ ,  $\gamma_2 = \beta_{j+r+1} \dots \beta_{j+2r}$  be the covering sequences of  $\rho_1$  and  $\rho_2$ , respectively. If it holds that*

- *the terms of  $\rho_1$  uniquely determine the blocks  $\beta_i$ , for  $i \in \{j+1, j+r\}$ ;*
- *the terms of  $\rho_2$  uniquely determine the blocks  $\beta_i$ , for  $i \in \{j+r+1, j+2r\}$ ;*
- *all the terms of  $\rho_1, \rho_2$  appear in  $\gamma_1, \gamma_2$  at the same indices within their blocks, respectively;*

*then  $\gamma_1 \gamma_2$  is a repetition in  $\varphi$ .*

*Proof.* Since all the blocks are uniquely determined and  $r > 0$ , it follows that  $\beta_{j+i} = \beta_{j+r+i}$  for every  $j$ . □

### 3 Technique #1: Exhaustive Search for Morphisms

The aim of this section is to present a compact proof of Theorem 2. For completeness, in the proof, we provide constructions of  $k$ -Thue sequences also for  $k \in \{2, 3\}$ . We used an exhaustive computer search to determine appropriate morphisms which are then applied to appropriate sequences.

*Proof of Theorem 2.* Let  $\mathbb{A}_3$  and  $\mathbb{A}_{k+2}$  be alphabets on 3 and  $k+2$  letters, respectively. For every  $k$ ,  $2 \leq k \leq 8$ , let a morphism  $\mu_k : \mathbb{A}_3^* \rightarrow \mathbb{A}_{k+2}^*$  be defined as given below:

- $k = 2$ : a 7-uniform morphism

$$\begin{aligned} \mu_2(0) &= 0310213 \\ \mu_2(1) &= 0230132 \\ \mu_2(2) &= 0120321 \end{aligned}$$

- $k = 3$ : a 14-uniform morphism

$$\begin{aligned}\mu_3(0) &= 10231402310243 \\ \mu_3(1) &= 01243024130243 \\ \mu_3(2) &= 01240312401234\end{aligned}$$

- $k = 4$ : a 12-uniform morphism

$$\begin{aligned}\mu_4(0) &= 012350412534 \\ \mu_4(1) &= 012345103245 \\ \mu_4(2) &= 012340521345\end{aligned}$$

- $k = 5$ : a 27-uniform morphism

$$\begin{aligned}\mu_5(0) &= 012345601235460235146023546 \\ \mu_5(1) &= 012345601234650134625013465 \\ \mu_5(2) &= 012345061234065123460152346\end{aligned}$$

- $k = 6$ : a 23-uniform morphism

$$\begin{aligned}\mu_6(0) &= 01234560172436501243756 \\ \mu_6(1) &= 01234560127354061235476 \\ \mu_6(2) &= 01234560123746510324657\end{aligned}$$

- $k = 7$ : a 36-uniform morphism

$$\begin{aligned}\mu_7(0) &= 012345670812345608721345687201345678 \\ \mu_7(1) &= 012345670182345601872345618702345687 \\ \mu_7(2) &= 012345670128345670281345762801345768\end{aligned}$$

- $k = 8$ : a 30-uniform morphism

$$\begin{aligned}\mu_8(0) &= 012345678902315647890312645789 \\ \mu_8(1) &= 012345678902143675982014365789 \\ \mu_8(2) &= 012345678019324568079123548679\end{aligned}$$

In what follows, we show that  $\mu_k(\varphi')$  is  $k$ -Thue for every  $(\frac{7}{4})^+$ -free sequence  $\varphi' \in \mathbb{A}_3^*$ . Using a computer, we have verified the following.

**Claim 1.** *Let  $\varphi$  be any non-repetitive sequence over  $\mathbb{A}_3$  of length at most 40. For each morphism  $\mu_k$ ,  $\mu_k(\varphi)$  is  $k$ -Thue.*

Next, for every  $k$  and  $d$  such that  $2 \leq k \leq 8$  and  $1 \leq d \leq k$ , we consider every sequence  $\delta = x_1x_2x_3x_4$  of length 4 over  $\mathbb{A}_3$  and every  $d$ -subsequence  $\sigma$  of  $\mu_k(\delta)$  such that  $\sigma$  intersects both the prefix  $\mu_k(x_1)$  and the suffix  $\mu_k(x_4)$  of  $\mu_k(\delta)$ . We again used a computer to check that if such a  $d$ -subsequence appears in two sequences  $\mu(\delta)$  and  $\mu(\delta')$ , where  $\delta = x_1x_2x_3x_4$  and  $\delta' = x'_1x'_2x'_3x'_4$ , then  $x_2x_3 = x'_2x'_3$ .

Thus, long enough  $d$ -subsequences of  $\mu_k(\varphi)$  allow to determine  $\varphi$ , except maybe for the first and the last term of  $\varphi$ . So, if a large repetition  $\rho$  occurs in some  $d$ -subsequence of  $\mu_k(\varphi)$ , then  $\varphi$  contains a factor  $uvu$  such that  $u$  is large and  $|v| \leq 2$ . For  $|u| \geq 7$ , such a factor  $uvu$  cannot appear in a  $(\frac{7}{4})^+$ -free sequence. On the other hand, if  $|u| \leq 6$ , then the length of  $\varphi$  is at most 18 (including possible first and last term). For such sequences,  $\mu_k(\varphi)$  are  $k$ -Thue by Claim 1. This completes the proof.  $\square$

## 4 Construction of Thue sequences using Hexagonal Morphism

Recently, in his master thesis, Kočiško [10] introduced a uniform morphism  $\kappa$ , which maps a term  $x$  of a sequence to a block of three symbols regarding the mapping of the predecessor of  $x$ . In particular, instead of using an alphabet  $\mathbb{A} = \{1, 2, 3\}$  an auxiliary alphabet

$$\overline{\mathbb{A}} = \{\overline{1}, \underline{1}, \overline{2}, \underline{2}, \overline{3}, \underline{3}\}$$

is used. The morphism  $\kappa$  is then defined as

$$\begin{aligned} \kappa(\overline{1}) &= \overline{1} \underline{2} \overline{3}, & \kappa(\overline{2}) &= \overline{2} \underline{3} \overline{1}, & \kappa(\overline{3}) &= \overline{3} \underline{1} \overline{2}, \\ \kappa(\underline{1}) &= \underline{3} \overline{2} \underline{1}, & \kappa(\underline{2}) &= \underline{1} \overline{3} \underline{2}, & \kappa(\underline{3}) &= \underline{2} \overline{1} \underline{3}. \end{aligned}$$

For a positive integer  $t$ , we recursively define the sequence

$$\overline{\varphi}_\kappa^t = \kappa(\overline{\varphi}_\kappa^{t-1}),$$

where  $\overline{\varphi}_\kappa^0 = \overline{1}$ . Notice that for every  $t$ , every symbol from  $\overline{\mathbb{A}}$  is a neighbor of at most two symbols of  $\overline{\mathbb{A}}$  (if  $t > 3$ , then precisely two); we say that neighboring symbols are *adjacent*. The adjacency is also preserved between the blocks of three symbols to which the symbols from  $\overline{\mathbb{A}}$  are mapped by  $\kappa$ ; we denote these blocks  $\overline{\kappa}$ -triples. Due to its structure, we refer to  $\kappa$  as the *hexagonal morphism*. In Fig. 1, the adjacencies between the symbols and the  $\overline{\kappa}$ -triples, and the mappings of  $\kappa$  are depicted.

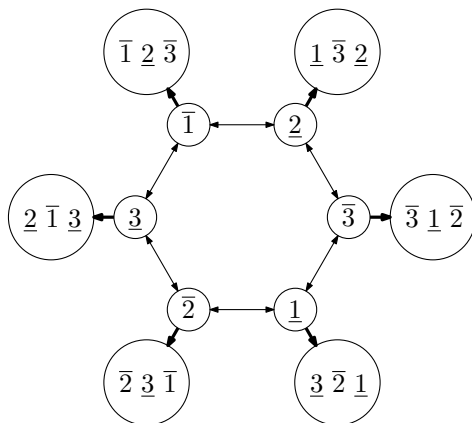


Figure 1: The graph of adjacencies between the symbols of  $\overline{\mathbb{A}}$  and  $\overline{\kappa}$ -triples, and the mappings defined by  $\kappa$ .

Let  $\pi : \overline{\mathbb{A}} \rightarrow \mathbb{A}$  be a projection of symbols from the auxiliary alphabet  $\overline{\mathbb{A}}$  to  $\mathbb{A}$  defined as  $\pi(\overline{a}) = a$  and  $\pi(\underline{a}) = a$ , for every  $a \in \{1, 2, 3\}$ . By  $\varphi_\kappa^t$ , we denote the projected sequence  $\overline{\varphi}_\kappa^t$ , i.e.  $\varphi_\kappa^t = \pi(\overline{\varphi}_\kappa^t)$ ; similarly a projected  $\overline{\kappa}$ -triple  $\tau$ ,  $\pi(\tau)$ , is referred to as a  $\kappa$ -block.

By the definition of  $\varphi_\kappa^t = \{x_i\}_{i=1}^{3^t}$  and the mapping  $\kappa$ , one can easily derive the following basic properties:

- (K<sub>1</sub>) For every pair of adjacent  $\kappa$ -blocks  $\tau$  and  $\sigma$ , the sequence  $\tau\sigma$  is Thue.
- (K<sub>2</sub>) The length of  $\varphi_\kappa^t$  is  $3^t$ , and  $x_{3i+1}x_{3i+2}x_{3i+3}$  is a  $\kappa$ -block for every  $i$ ,  $0 \leq i < 3^{t-1}$ .

- (K<sub>3</sub>)  $\{x_{3i+1}, x_{3i+2}, x_{3i+3}\} = \{1, 2, 3\}$  for every  $i$ ,  $0 \leq i < 3^{t-1}$ .
- (K<sub>4</sub>)  $x_{3i+2} \neq x_{3(i+1)+2}$  for every  $i$ ,  $0 \leq i < 3^{t-1} - 1$ .
- (K<sub>5</sub>) Any three consecutive terms  $x_{j+1}x_{j+2}x_{j+3}$  of  $\varphi_\kappa^t$ , which do not belong to the same  $\kappa$ -block, uniquely determine the two  $\kappa$ -blocks they belong to.
- (K<sub>6</sub>) For a pair  $\tau_1, \tau_2$  of adjacent  $\kappa$ -blocks it holds that the first term of  $\tau_1$  is distinct from the third term of  $\tau_2$ .
- (K<sub>7</sub>) If a pair of distinct  $\kappa$ -blocks has the same first or last term, then they are adjacent.
- (K<sub>8</sub>) A pair of adjacent  $\kappa$ -blocks is not adjacent to any other common  $\kappa$ -block.
- (K<sub>9</sub>) The middle term of the  $\kappa$ -block  $\pi(\kappa(i))$ ,  $i \in \overline{\mathbb{A}}$ , equals  $\pi(i) + 1$  (modulo 3).
- (K<sub>10</sub>) A pair of distinct  $\kappa$ -symbols  $x_1$  and  $x_2$ , where  $x_1$  and  $x_2$  are the first (last) terms of adjacent  $\kappa$ -blocks  $\tau_1$  and  $\tau_2$ , uniquely determines  $\tau_1$  and  $\tau_2$ .
- (K<sub>11</sub>) A  $\kappa$ -block  $\tau_1$  and at least one term of a  $\kappa$ -block  $\tau_2$  adjacent to  $\tau_1$  uniquely determine  $\tau_2$ .
- (K<sub>12</sub>) A pair of adjacent  $\kappa$ -blocks is in  $\varphi_\kappa^t$  always separated by an even number of  $\kappa$ -blocks, since the graph of adjacencies is bipartite.

We use (some of) the properties above, to prove the following theorem.

**Theorem 6** (Kočiško, 2013). *The sequence  $\varphi_\kappa^t$  is Thue, for every non-negative integer  $t$ .*

For the sake of completeness, we present a short proof of Theorem 6 here also.

*Proof.* We prove the theorem by induction. Clearly,  $\varphi_\kappa^0$  is Thue. Consider the sequence  $\varphi_\kappa^t = \{x_i\}_{i=1}^{3^t}$  and suppose that  $\varphi_\kappa^j$  is Thue for every  $j < t$ . Suppose for a contradiction that there is a repetition in  $\varphi_\kappa^t$  and let  $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$  be a repetition with the minimum length (for later purposes we distinguish two repetition factors, although  $\rho_1 = \rho_2$ ). By (K<sub>1</sub>), we have that  $r \geq 3$ . We consider two subcases regarding the length  $r$  of  $\rho_1 (= \rho_2)$ .

Suppose first that  $r$  is divisible by 3. Then, as we show in the following claim, we may assume that the term  $y_1$  is the first term of some  $\kappa$ -block.

**Claim 2.** *Let  $r$  be divisible by 3. If  $y_1 = x_{3i+2}$  (resp.  $y_1 = x_{3i+3}$ ) for some  $i$ ,  $0 \leq i < 3^{t-1}$ , then  $x_{3i+1}x_{3i+2} \dots x_{3i+2r}$  (resp.  $x_{3(i+1)+1}x_{3(i+1)+2} \dots x_{3(i+1)+2r}$ ) is also a repetition.*

*Proof.* Suppose that  $y_1 = x_{3i+2}$ . By (K<sub>3</sub>), every  $\kappa$ -block is uniquely determined by two symbols. So  $x_{3i+1} = y_r$  and hence  $x_{3i+1} \dots x_{3i+2r} = y_r y_1 \dots y_{2r-1}$  is a repetition. A proof for the case  $y_1 = x_{3i+3}$  is analogous. ♦

Hence, we have that  $\rho = \tau_1 \dots \tau_{\frac{r}{3}}$ , where  $\tau_j$  are  $\kappa$ -blocks for every  $j$ ,  $1 \leq j \leq \frac{r}{3}$ . But in this case, there is a repetition already in  $\varphi_\kappa^{t-1}$ , contradicting the induction hypothesis.

Therefore, we may assume that  $r$  is not divisible by 3. This means that the first terms  $y_1$  and  $y_{r+1}$  of the two repetition factors  $\rho_1$  and  $\rho_2$ , respectively, are at different positions within the  $\kappa$ -blocks they belong to. For example, if  $r = 3k + 1$ , and  $y_1$  is the first term of the  $\kappa$ -block  $y_1 y_2 y_3$ , then  $y_{r+1}$  is the second term of the  $\kappa$ -block  $y_r y_{2r+1} y_{2r+2}$ . There are hence six possible cases regarding the position of  $y_1$  and  $y_{r+1}$  in their  $\kappa$ -blocks.

Suppose first that  $y_1$  is the first term of the  $\kappa$ -block  $x_1 x_2 x_3$ . By (K<sub>3</sub>),  $x_1, x_2$ , and  $x_3$  are pairwise distinct. Since  $\rho_1 = \rho_2$ , we thus know three consecutive elements of two  $\kappa$ -blocks



(the one of  $y_{r+1}$  and the subsequent one). By  $(K_5)$ , we can determine both  $\kappa$ -blocks, which gives us information about the term  $y_4$ . Using  $(K_5)$  again, we can determine the  $\kappa$ -block  $y_4y_5y_6$ , namely  $y_4y_5y_6 = x_2x_1x_3$  in the case when  $r \equiv 1 \pmod 3$ , and  $y_4y_5y_6 = x_1x_3x_2$  in the case when  $r \equiv 2 \pmod 3$ . Using the information obtained by determining  $\kappa$ -blocks using  $(K_5)$ , we infer that every  $\kappa$ -block of  $\rho_1$  ends with  $x_3$  in the former case, or starts with  $x_1$  in the latter case. As  $\rho_1$  and  $\rho_2$  are concatenated, this leads us to contradiction on the existence of a repetition. With a similar argument, we obtain a contradiction in the case when  $y_{r+1}$  is the first term of its  $\kappa$ -block.

Suppose now that  $r = 3k + 1$ , for some positive integer  $k$ , and  $y_1$  is the second term of its  $\kappa$ -block, say  $x_3x_1x_2$ . Then  $y_{r+1} = x_1$  and  $y_{r+2} = x_2$ , where  $y_{r+1}$  and  $y_{r+2}$  belong to distinct  $\kappa$ -blocks. Notice that there are two possibilities for the value of  $y_{r+3}$ , namely  $x_1$  and  $x_3$ . However, regardless the choice, after determining the  $\kappa$ -block  $y_{r+2}y_{r+3}y_{r+4}$  by  $(K_5)$ , and continue by alternately determining  $\kappa$ -blocks in  $\rho_1$  and  $\rho_2$ , as described above, we infer that in both cases, every  $\kappa$ -block in  $\rho_1$  ends with  $x_2$ , a contradiction. An analogous analysis may be performed in the last case, when  $r = 3k + 2$  and  $y_1$  being the third term of its  $\kappa$ -block.  $\square$

## 5 Technique #2: Transposition & Cyclic Blocks

In this section, we present alternative proofs to answer Conjecture 1 in affirmative for the cases  $k = 4$  and  $k = 6$ . For each of the two cases we present a special morphism and apply it on a non-repetitive sequence. Then, we use sequence wreathing to extend the sequence by circular blocks.

### 5.1 The case $k = 4$

In this part, to prove the case  $k = 4$  in Theorem 2, we combine the sequence  $\varphi_\kappa^t$  obtained by the hexagonal morphism and the circular sequence  $\varphi_\zeta^{(3,3^t)}$  by wreathing. We construct  $\varphi_\kappa^t$  over the alphabet  $\{1, 2, 3\}$ , and  $\varphi_\zeta^{(3,3^{t-1})}$  over the alphabet  $\{4, 5, 6\}$ . We define

$$\varphi_4^t = \varphi_\kappa^t \wr_3 \varphi_\zeta^{(3,3^{t-1})}.$$

For clarity, we refer to the base-blocks of  $\varphi_4^t$  as  $\kappa$ -blocks (recall that the wrap-blocks are called  $\zeta$ -blocks). Additionally, the terms from  $\kappa$ -blocks (resp.  $\zeta$ -blocks) are called  $\kappa$ -terms (resp.  $\zeta$ -terms).

**Lemma 7.** *The sequence  $\varphi_4^t$  is 4-Thue for every non-negative integer  $t$ .*

*Proof.* Since  $\varphi_\kappa^t$  is Thue by Theorem 6,  $\varphi_4^t$  is also Thue by Lemma 4. Thus, it remains to prove that every  $d$ -subsequence of  $\varphi_4^t$  is Thue, for every  $d \in \{2, 3, 4\}$ . Observe first that by  $(K_1)$ ,  $(K_6)$ , and the definition of circular sequences, every five consecutive terms of  $\varphi_4^t$  are distinct. This in particular means that

(P<sub>1</sub>) *there are no repetitions of length 2 or 4 in any  $d$ -subsequence of  $\varphi_4^t$ .*

Moreover,

(P<sub>2</sub>) *in every  $d$ -subsequence of  $\varphi_4^t$  there are at most two consecutive  $\kappa$ -terms or  $\zeta$ -terms;*

- (P<sub>3</sub>) *in every  $d$ -subsequence of  $\varphi_4^t$  any repetition contains  $\kappa$ -terms and  $\zeta$ -terms;*
- (P<sub>4</sub>) *if a  $\kappa$ -term (resp.  $\zeta$ -term) in a  $d$ -subsequence  $\sigma$  of  $\varphi_4^t$ , whose predecessor and successor in  $\sigma$  are  $\zeta$ -terms (resp.  $\kappa$ -terms), is at index  $i$  within its  $\kappa$ -block (resp.  $\zeta$ -block), then every  $\kappa$ -term (resp.  $\zeta$ -term) in  $\sigma$  is at index  $i$  within its  $\kappa$ -block (resp.  $\zeta$ -block).*

All the latter three properties are direct corollaries of (P<sub>1</sub>) and the fact that every  $\kappa$ -block and  $\zeta$ -block is of length 3.

Now, we prove that every  $d$ -subsequence of  $\varphi_4^t$  is non-repetitive, considering three cases with regard to  $d$ . In each case, we assume there is a repetition  $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$  in some  $d$ -subsequence  $\sigma$  and eventually reach a contradiction on its existence.

By (P<sub>3</sub>), there is at least one  $\zeta$ -term in  $\rho_1$ . Moreover, by the definition of circular sequences and  $\varphi_4^t$ , every three consecutive  $\zeta$ -terms in  $\rho_1$  (ignoring the  $\kappa$ -terms) are distinct, unless  $d = 4$  and the  $\zeta$ -terms of  $\rho_1$  are at indices 1 and 3 in  $\zeta$ -blocks. However, in such a case, by construction of circular sequences, without loss of generality, consecutive  $\zeta$ -terms of  $\rho_1$  are 4 4 5 5 6 6  $\dots$ , which means that  $r$  must be divisible by 6, to have the same sequence of  $\zeta$ -terms in  $\rho_2$ . This implies that

- (P<sub>5</sub>) *the number of  $\zeta$ -terms in  $\rho_1$  is divisible by 3,*

and consequently, since in  $\zeta$ -blocks the symbols repeat at the same indices in every third block:

- (P<sub>6</sub>) *the number of  $\zeta$ -blocks to which the  $\zeta$ -terms of  $\rho_1$  belong to in  $\varphi_4^t$  is divisible by 3.*

Observe that, by the above properties,

- (P<sub>7</sub>) *the first terms of  $\rho_1$  and  $\rho_2$  are either both  $\kappa$ -terms or  $\zeta$ -terms, and moreover, they appear at the same index within their blocks in  $\varphi_4^t$ .*

Now, we start the analysis regarding  $d$ :

- $d = 2$ .

Suppose first that  $y_1$  is the first term of some  $\kappa$ -block. Then,  $\rho_1$  is comprised alternately of two  $\kappa$ -terms (the first and the third terms of a  $\kappa$ -block in  $\varphi_4^t$ ) and one  $\zeta$ -term (the second term of its  $\zeta$ -block in  $\varphi_4^t$ ). Consequently,  $y_{r+1}$  is the first term of a  $\kappa$ -block also, and the last term of  $\rho_1$  must be a  $\zeta$ -term. By (K<sub>3</sub>), every  $\kappa$ -block is uniquely determined by two of its terms, hence one can determine all  $\kappa$ -blocks to which the  $\kappa$ -terms of  $\rho_1$  and  $\rho_2$  belong to in  $\varphi_4^t$ . Similarly, all the  $\zeta$ -blocks, to which  $\zeta$ -terms of  $\rho_1$  and  $\rho_2$  belong, are uniquely determined by Observation 1. Moreover, since the terms  $y_r$  and  $y_{r+1}$  belong to different blocks, we can apply Lemma 5 obtaining a contradiction on the existence of  $\rho_1\rho_2$ .

Suppose now that  $y_1$  is the third term of some  $\kappa$ -block. A similar argument as in the paragraph above shows that  $y_r$  is the first term of some  $\kappa$ -block  $\gamma_r$  of  $\varphi_4^t$ , while  $y_{r+1}$  is the third term of  $\beta_r$ . Note that the terms  $y_2, y_3$  and  $y_{r+2}, y_{r+3}$  uniquely determine the same block  $\gamma_2$ . Consider now the  $\kappa$ -block  $\gamma_1$  to which  $y_1$  belongs. By (K<sub>7</sub>), it is one of the two possible  $\kappa$ -blocks that end with  $y_1$ , and since  $\gamma_1$  and  $\gamma_r$  are adjacent to  $\gamma_2$ , by (K<sub>8</sub>), we infer that  $\gamma_1 = \gamma_r$ . Thus, taking the first term  $z$  of  $\gamma_1$ , we have a repetition  $zy_1 \dots y_{r-1} y_r \dots y_{2r-1}$  in  $\sigma$ , which satisfies the assumptions of Lemma 5. Hence, there is a repetition in  $\varphi_4^t$ , a contradiction.

Next, suppose that  $y_1$  is the second term of some  $\kappa$ -block. Then, by  $(P_4)$ , all the  $\kappa$ -terms in  $\rho_1$  and  $\rho_2$  are the second terms of  $\kappa$ -blocks in  $\varphi_4^t$ . By  $(K_9)$ , we have that the second terms of  $\kappa$ -blocks in  $\varphi_4^t$  are exactly the terms of  $\varphi_\kappa^{t-1}$  shifted by 1, and thus form a non-repetitive sequence. Using Lemma 4, we infer that the sequence  $\sigma$  is also non-repetitive, a contradiction.

Finally, suppose that  $y_1$  is a  $\zeta$ -term. Let  $\gamma$  be the last  $\zeta$ -block in  $\varphi_4^t$  to which some term of  $\rho_2$  belongs. Since a  $\zeta$ -block is uniquely determined by at least one of its terms, using  $(P_5)$ , we infer that the  $\zeta$ -block of  $\varphi_4^t$  following  $\gamma$  is equal to the  $\zeta$ -block uniquely determined by  $y_1$ . Let  $y \in \{y_2, y_3\}$  be the first  $\kappa$ -term of  $\rho_1$ . The observation above implies that there exists a repetition in  $\sigma$  starting with  $y$  and ending with the  $\zeta$ -terms before  $y$  in  $\rho_1$ . Such a repetition cannot exist due to the analysis of the cases above.

- $d = 3$ .

Suppose that  $y_1$  is the first term of some  $\kappa$ -block. Clearly, the first term of  $\rho_2$  is also a  $\kappa$ -term, and thus the number of  $\kappa$ -terms in  $\rho_1$  is divisible by 3, by  $(P_5)$ . Therefore, there are at least six  $\kappa$ -terms in  $\rho_1\rho_2$ , meaning there are two distinct consecutive  $\kappa$ -terms. Using  $(K_{10})$  and  $(K_{11})$ , we can uniquely determine all  $\kappa$ -blocks to which the  $\kappa$ -terms of  $\rho_1$  and  $\rho_2$  belong. So, by Lemma 5, we obtain a contradiction.

If  $y_1$  is the third term of some  $\kappa$ -block, we use the same argument as in the paragraph above.

The argument when  $y_1$  is the second term of some  $\kappa$ -block is analogous to the subcase in the case  $d = 2$ , where  $y_1$  is the second term of some  $\kappa$ -block.

In the case when  $y_1$  is a  $\zeta$ -term, we can again translate the analysis to the one of the above cases, since the  $\zeta$ -triples have period 3.

- $d = 4$ .

Suppose that  $y_1$  is the first term of some  $\kappa$ -block. By  $(P_1)$ , the length of  $\rho_1$  is at least 3. Furthermore,  $y_2$  and  $y_3$  are a  $\zeta$ -terms (the second terms of some  $\zeta$ -block) and a  $\kappa$ -term (the third term of some  $\kappa$ -block), respectively. By  $(P_4)$ , all  $\zeta$ -terms in  $\rho_1$  are the second terms of  $\zeta$ -blocks. Thus, by  $(P_5)$  and the fact that for every  $\zeta$ -term in  $\rho_1$  there are two  $\zeta$ -blocks in  $\hat{\rho}_1$ , we have that the number of  $\zeta$ -blocks in  $\hat{\rho}_1$  is divisible by 6. By  $(P_7)$ ,  $y_{r+1}$  is also the first term of some  $\kappa$ -block in  $\varphi_4^t$ , meaning that the number of  $\kappa$ -blocks in  $\hat{\rho}_1$  is also divisible by 6 and that the number of  $\kappa$ -blocks between the blocks of  $y_1$  and  $y_{r+1}$  is odd. Hence, by  $(K_{12})$ , the  $\kappa$ -blocks of  $y_1$  and  $y_{r+1}$  are the same. Analogously, all the blocks of the  $\kappa$ -terms  $y_i$  in  $\rho_1$  are the same as the  $\kappa$ -blocks of  $y_{i+r}$  in  $\rho_2$ . Thus, there is a repetition in  $\varphi_4^t$  also, a contradiction.

Suppose now that  $y_1$  is the second term of some  $\kappa$ -block. Similarly as in the case above, we notice that all  $\kappa$ -terms of  $\rho_1$  are the second terms in their  $\kappa$ -blocks in  $\hat{\rho}_1$ , and that the number of  $\kappa$ -blocks in  $\hat{\rho}_1$  is divisible by 6. Again, we deduce that for every two  $\kappa$ -terms  $y_i$  and  $y_j$  in  $\sigma$ , there are even number of  $\kappa$ -blocks between the  $\kappa$ -blocks of  $y_i$  and  $y_j$  in  $\hat{\sigma}$ . It follows that every pair of equal  $\kappa$ -symbols in  $\sigma$  belongs to the same  $\kappa$ -block, and hence  $(\hat{\rho}_1) = \hat{\rho}_2$ , a contradiction.

The cases, when  $y_1$  is the third term of some  $\kappa$ -block, or the second term of some  $\zeta$ -block are analogous to the first case. The cases, when  $y_1$  is the first or the third term of some  $\zeta$ -block are analogous to the second case.

□

## 5.2 The case $k = 6$

In this part, we present a construction of a 6-Thuë sequence using 8 symbols, in a similar way as for the case  $k = 4$ . Again, we wreath a Thuë sequence with a circular sequence, but now, the base sequence is formed by blocks of four symbols, where in each block we only permute symbols in fixed pairs.

Similarly as in Section 4, we start by constructing a Thuë sequence over an alphabet

$$\mathbb{B} = \{1, 2, 3, 4\}$$

of 4 symbols. Let a morphism  $\lambda$ , mapping a symbol from the sequence to a block of four distinct symbols, be defined as

$$\lambda(1) = 1\ 2\ 3\ 4, \quad \lambda(2) = 2\ 1\ 4\ 3, \quad \lambda(3) = 1\ 2\ 4\ 3, \quad \lambda(4) = 2\ 1\ 3\ 4.$$

For a positive integer  $t$ , we recursively define the sequence

$$\varphi_\lambda^t = \lambda(\varphi_\lambda^{t-1}),$$

where  $\varphi_\lambda^0 = 1$ . Notice that for every positive integer  $t$ , every symbol from  $\mathbb{B}$  is a neighbor of all symbols of  $\mathbb{B}$ . The blocks of four symbols to which the symbols from  $\mathbb{B}$  are mapped by  $\lambda$ , are referred to as  $\lambda$ -blocks. In Fig. 2, the mappings of  $\lambda$  are depicted.

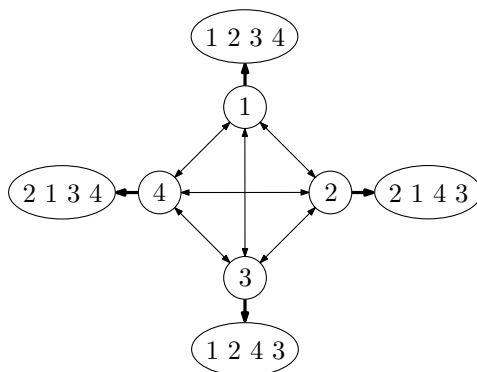


Figure 2: The graph of adjacencies between the symbols of  $\mathbb{B}$  and  $\lambda$ -blocks, and the mappings defined by  $\lambda$ .

We first observe some basic properties of the sequence  $\varphi_\lambda^t$ , for any positive integer  $t$ .

- ( $L_1$ ) For any pair of adjacent  $\lambda$ -blocks  $\gamma_1$  and  $\gamma_2$ , the sequence  $\gamma_1\gamma_2$  is Thuë.
- ( $L_2$ ) The length of  $\varphi_\lambda^t$  is  $4^t$ , and  $x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4}$  is a  $\lambda$ -block for every  $i$ ,  $0 \leq i \leq 4^{t-1} - 1$ .
- ( $L_3$ )  $\{x_{4i+1}, x_{4i+2}\} = \{1, 2\}$  and  $\{x_{4i+3}, x_{4i+4}\} = \{3, 4\}$  for every  $i$ ,  $0 \leq i \leq 4^{t-1} - 1$ . Consequently, by knowing at least one term at index 1 or 2, and at least one term at index 3 or 4, the  $\lambda$ -block is uniquely determined.
- ( $L_4$ ) For every  $i$ ,  $0 \leq i \leq 4^{t-1} - 3$ , it holds:  $x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4} \neq x_{4i+9}x_{4i+10}x_{4i+11}x_{4i+12}$  (this is in fact a consequence of ( $L_3$ )).
- ( $L_5$ ) Two consecutive  $\lambda$ -blocks with the same first two terms are mapped from  $\{1, 3\}$  or  $\{2, 4\}$ . Similarly, two consecutive  $\lambda$ -blocks with the same last two terms are mapped from  $\{1, 4\}$  or  $\{2, 3\}$ .

- (L<sub>6</sub>) Let  $\gamma_1$  and  $\gamma_2$  be distinct  $\lambda$ -blocks with equal terms at indices 1 and 2 or at indices 3 and 4. For  $\lambda$ -blocks  $\gamma_3, \gamma_4$ , and  $\gamma_5$ , in  $\varphi_\lambda^t$ , there is at most one of the subsequences  $\gamma_1\gamma_3\gamma_5$  and  $\gamma_2\gamma_4\gamma_5$ , since otherwise the property (L<sub>3</sub>) would be violated in  $\varphi_\lambda^{t-1}$ .
- (L<sub>7</sub>) If for two  $\lambda$ -blocks  $\gamma_1 = x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4}$  and  $\gamma_2 = x_{4j+1}x_{4j+2}x_{4j+3}x_{4j+4}$  there is such  $\ell \in \{1, 2, 3, 4\}$  that  $x_{4i+\ell} = x_{4j+\ell}$  and 4 divides  $|j - i|$ , then  $\gamma_1 = \gamma_2$ . On the other hand, if 4 does not divide  $|j - i|$ , but  $|j - i|$  is even, then  $\gamma_1 \neq \gamma_2$ .
- (L<sub>8</sub>) If for a  $\lambda$ -block  $\gamma$  one term is known, then it is one of two possible  $\lambda$ -blocks. In particular, if the known term is at index 1 or 2 in  $\gamma$ , then either  $\lambda^{-1}(\gamma) \in \{1, 3\}$  or  $\lambda^{-1}(\gamma) \in \{2, 4\}$ . If the known term is at index 3 or 4 in  $\gamma$ , then either  $\lambda^{-1}(\gamma) \in \{1, 4\}$  or  $\lambda^{-1}(\gamma) \in \{2, 3\}$ .

We leave the above properties to the reader to verify and proceed by proving that  $\varphi_\lambda^t$  is Thue.

**Lemma 8.** *The sequence  $\varphi_\lambda^t$  is Thue for every non-negative integer  $t$ .*

*Proof.* Suppose the contrary, and let  $t$  be the minimum such that there is a repetition in  $\varphi_\lambda^t$ . Denote the  $i$ -th term of  $\varphi_\lambda^t$  by  $x_i$ . Let  $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$  be a repetition of minimum length. We first show that  $r > 4$ . The cases with  $r \leq 3$  are trivial, so suppose  $r = 4$ . By (L<sub>1</sub>), we have that  $y_1$  is not at index  $4i + 1$  in  $\varphi_\lambda^t$  (for any  $i, 0 \leq i \leq 4^{t-1} - 1$ ), and by (L<sub>3</sub>), it is not at index  $4i + 2$  nor  $4i + 4$ . Hence, assume  $y_1$  is at index  $4i + 3$ . Denote the  $\lambda$ -block  $x_{4i+5}x_{4i+6}x_{4i+7}x_{4i+8} (= y_3y_4y_5y_6)$  by  $\gamma_1$ . By (L<sub>1</sub>), we have that  $\gamma_0 = x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4} = y_4y_3y_5y_6$  and similarly,  $\gamma_2 = x_{4i+9}x_{4i+10}x_{4i+11}x_{4i+12} = y_3y_4y_6y_5$ . By (L<sub>5</sub>), this means that if  $\gamma_1 \in \{1, 2\}$ , then  $\gamma_0, \gamma_2 \in \{3, 4\}$ , and analogously, if  $\gamma_1 \in \{3, 4\}$ , then  $\gamma_0, \gamma_2 \in \{1, 2\}$ , a contradiction to (L<sub>3</sub>). Hence,  $r > 4$ .

Let  $j$  be the index of  $y_1$  in  $\varphi_\lambda^t$ , i.e.  $y_1 = x_j$ . If  $j$  is odd, then by (L<sub>3</sub>), either  $x_j x_{j+1} = \{1, 2\}$  or  $x_j x_{j+1} = \{3, 4\}$ , and without loss of generality, we may assume the former. Thus, also  $x_{j+r} x_{j+r+1} = \{1, 2\}$ , which implies that  $r$  must be even. In the case when  $j$  is even, (L<sub>3</sub>) similarly implies that  $x_j \in \{1, 2\}$  and  $x_{j+1} \in \{3, 4\}$ , and hence  $x_{j+r} \in \{1, 2\}$  and  $x_{j+r+1} \in \{3, 4\}$ . Consequently,  $r$  is again even. Finally observe that by (L<sub>2</sub>), from  $r$  being even and  $x_j = x_{j+r}$  it follows that  $r$  is divisible by 4.

Suppose now that  $j = 4i + 1$ , for some  $i$ . Then, since  $r$  is divisible by 4,  $\rho_1$  and  $\rho_2$  are comprised of  $\frac{r}{4}$   $\lambda$ -blocks each, the first starting with  $x_j$ . This in turn means that there is a repetition in  $\varphi_\lambda^{t-1}$  as every  $\lambda$ -block represents one term in  $\varphi_\lambda^{t-1}$ , a contradiction to the minimality of  $t$ .

Next, suppose  $j = 4i + 2$ . By (L<sub>3</sub>), we have that  $x_{j-1} = x_{j+r-1}$ , and hence  $\rho'_1\rho'_2 = x_{j-1}x_j \dots x_{j+r-1}x_{j+r} \dots x_{j+2r-2}$  is also a repetition in  $\varphi_\lambda^t$ , where  $j - 1 = 4i + 1$ , and hence the reasoning in the above paragraph applies.

Suppose  $j = 4i + 4$ . Then, analogous to the previous case, we infer  $x_{j+r} = x_{j+2r}$ , and hence  $\rho'_1\rho'_2 = x_{j+1} \dots x_{j+r}x_{j+r+1} \dots x_{j+2r}$  is also a repetition in  $\varphi_\lambda^t$ , where  $j + 1 = 4(i + 1) + 1$ , so the reasoning for  $j = 4i + 1$  applies again.

Finally, consider the case with  $j = 4i + 3$ . If  $r = 8$ , from  $x_{j+2}x_{j+3}x_{j+4}x_{j+5} = x_{j+10}x_{j+11}x_{j+12}x_{j+13}$  it follows that the  $\lambda$ -block  $x_{j+6}x_{j+7}x_{j+8}x_{j+9}$  is surrounded by the same  $\lambda$ -blocks, which contradicts (L<sub>4</sub>). Hence, we may assume  $r \geq 12$ . Since the  $\lambda$ -blocks  $x_{j+6}x_{j+7}x_{j+8}x_{j+9}$  and  $x_{j+r+6}x_{j+r+7}x_{j+r+8}x_{j+r+9}$  are equal, and  $r$  is divisible by 4, it follows that also  $x_{j-2}x_{j-1}x_j x_{j+1} = x_{j+r-2}x_{j+r-1}x_{j+r}x_{j+r+1}$  and we may apply the reasoning for the case with  $j = 4i + 1$  on the repetition  $x_{j-2} \dots x_{j+2r-3}$ . Hence,  $\varphi_\lambda^t$  is Thue.  $\square$

Now, take the circular sequence  $\varphi_\zeta^{(4,4^{t-1})}$ , with  $\varphi = 5\ 6\ 7\ 8$ , and use sequence wreathing on  $\varphi_\lambda^t$  and  $\varphi_\zeta^{(4,4^{t-1})}$  to obtain the sequence

$$\varphi_6^t = \varphi_\lambda^t \wr_4 \varphi_\zeta^{(4,4^{t-1})}.$$

Similarly as above, we refer to the base-blocks of  $\varphi_6^t$  as  $\lambda$ -blocks, and to the wrap-blocks as  $\zeta$ -blocks. The terms of  $\lambda$ -blocks (resp.  $\zeta$ -blocks) are referred to as  $\lambda$ -terms (resp.  $\zeta$ -terms). The sequence  $\varphi_6^2$  is hence:

$$\underbrace{1\ 2\ 3\ 4}_{\lambda(1)}\ 5\ 6\ 7\ 8\ \underbrace{2\ 1\ 4\ 3}_{\lambda(2)}\ 6\ 7\ 8\ 5\ \underbrace{1\ 2\ 4\ 3}_{\lambda(3)}\ 7\ 8\ 5\ 6\ \underbrace{2\ 1\ 3\ 4}_{\lambda(4)}\ 8\ 5\ 6\ 7$$

It remains to prove that  $\varphi_6^t$  is also 6-*Thue*.

**Lemma 9.** *The sequence  $\varphi_6^t$  is 6-*Thue* for every non-negative integer  $t$ .*

*Proof.* By Lemmas 4 and 8, we have that  $\varphi_6^t$  is *Thue*. Thus, we only need to prove that every  $d$ -subsequence of  $\varphi_6^t$  is also *Thue*, for every  $d \in \{2, 3, 4, 5, 6\}$ . First, we list some general properties and then consider  $d$ -subsequences separately regarding the values of  $d$ .

By  $(L_3)$  and the definition of circular sequences, every seven consecutive terms of  $\varphi_6^t$  are distinct. Hence,

$(R_1)$  there are no repetitions of length 2 or 4 in any  $d$ -subsequence  $\varphi_6^t$ .

Furthermore, since the length of any  $\lambda$ -block and  $\zeta$ -block in  $\varphi_6^t$  is 4, one can deduce that:

$(R_2)$  in every  $d$ -subsequence of  $\varphi_6^t$  there are at most two consecutive  $\lambda$ -terms or  $\zeta$ -terms;

$(R_3)$  in every  $d$ -subsequence of  $\varphi_6^t$  any repetition contains  $\lambda$ -terms and  $\zeta$ -terms;

Given a  $d$ -subsequence  $\sigma = z_1 z_2 \dots z_n$  of  $\varphi_6^t$  consisting of  $n$  elements, we define a mapping  $\vartheta: \Sigma \rightarrow \{N, C\}^n$ , where  $\Sigma$  represents the set of all  $d$ -subsequences of  $\varphi_6^t$ , mapping  $\sigma$  to an  $n$ -component vector,  $i$ -th component being  $N$  if  $z_i$  belongs to a  $\lambda$ -block and  $C$  otherwise ( $N$  and  $C$  standing for a **n**on-cyclic and **c**yclic element, respectively). We call  $\vartheta(\sigma)$  the *type vector* of  $\sigma$ .

$(T_1)$  The type vector of any 2-subsequence contains  $CCNN$  or  $NNCC$  in the first five components (depending on the position of the first term in the sequence).

$(T_2)$  The type vector of any 4-subsequence equals  $NCNC$  or  $CNCN$  in the first four components.

Now, suppose the contrary, and let  $\rho = \rho_1 \rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$  be a repetition in some  $d$ -subsequence of  $\varphi_6^t$ . We start by analyzing possible values of  $r$ .

By  $(R_1)$ ,  $r \geq 3$ , so suppose first that  $r = 3$ . We will consider the cases regarding the type vectors of  $\rho$ . By  $(R_2)$  and  $(R_3)$ , there are six possible type vectors for  $\rho$ , namely:  $CCN\ CCN$ ,  $CNC\ CNC$ ,  $CNN\ CNN$ ,  $NCC\ NCC$ ,  $NCN\ NCN$ , and  $NNC\ NNC$ . By  $(T_1)$  and  $(T_2)$ , such a sequence does not appear in any  $\ell$ -sequence for  $\ell \in \{2, 4\}$ . Hence, it remains to consider  $\ell \in \{3, 5, 6\}$ . Let  $j$ ,  $j \in \{1, 2, 3, 4\}$ , be the index of  $y_1$  in the  $\lambda$ - or  $\zeta$ -block it belongs to.

In Table 1, we present type vectors regarding  $j$ 's and  $\ell$ 's. The only two type vectors matching the possibilities for the type vectors of  $\rho$  are in the cases  $(j, \ell) \in \{(3, 3), (2, 5)\}$ .

$j / \ell$	3	5	6
1	NNC NCC	NCN CCN	NCC NNC
2	NCC NCN	<b>NCN NCN</b>	NCC NNC
3	<b>NCN NCN</b>	NCC NCN	NNC CNN
4	NCN CCN	NNC NCC	NNC CNN

Table 1: The type vectors of  $\rho$  regarding  $j$ 's and  $\ell$ 's in the case  $r = 3$ , assuming the first term  $y_1$  lies in a  $\lambda$ -block. In the symmetric case, when  $y_1$  is in a  $\zeta$ -block, the type vector values are simply interchanged.

In the case (3,3), the indices of  $y_1, \dots, y_6$  within their blocks are respectively 3, 2, 1, 4, 3, 2. When  $y_1$  belongs to a  $\lambda$ -block,  $y_2$  and  $y_5$  must belong to consecutive  $\zeta$ -blocks. But, then  $y_2 \neq y_5$ , due to the construction of circular sequences. On the other hand, if  $y_1$  belongs to a  $\zeta$ -block, then  $y_2$  is at index 2 in a  $\lambda$ -block and  $y_5$  is at index 3 in a  $\lambda$ -block, so again  $y_2 \neq y_5$ , due to  $(L_3)$ .

In the case (2,5), the indices of  $y_1, \dots, y_6$  within their blocks are respectively 2, 3, 4, 1, 2, 3. Suppose first that  $y_1$  belongs to a  $\zeta$ -block. Then,  $y_2$  is at index 3 in a  $\lambda$ -block and  $y_5$  is at index 2 in a  $\lambda$ -block, and hence  $y_2 \neq y_5$ , due to  $(L_3)$ . Finally, suppose  $y_1$  belongs to a  $\lambda$ -block. Then,  $y_2$  is at index 3 in a  $\zeta$ -block and  $y_5$  is at index 2 in a  $\zeta$ -block, however, the two  $\zeta$ -blocks are not consecutive, and hence  $y_2 \neq y_5$ . It follows that  $r \geq 4$ .

Using the construction properties of circular sequences, we can obtain additional properties of  $r$  regarding the structure of type vectors.

**Claim 3.** *If there are two consecutive  $\zeta$ -terms in  $\rho_1$ , then 32 divides  $d \cdot r$ .*

Note that we do not require the two terms being in the same  $\zeta$ -block. *Proof.* We prove the claim by showing that having two consecutive  $\zeta$ -terms,  $x_i$  and  $x_{i+d}$ , in  $\rho_1$  imply that the corresponding two  $\zeta$ -terms,  $x_j$  and  $x_{j+d}$ , in  $\rho_2$  must appear at the same indices in their  $\zeta$ -blocks. This fact further implies that the difference between  $i$  and  $j$  is  $(8 \cdot 4)t$  (8 since each pair of  $\lambda$ - and  $\zeta$ -blocks has 8 terms, and 4, since  $\zeta$ -blocks have period 4 in  $\varphi_{\zeta}^{(4,4^t)}$ ), for some positive integer  $t$ . On the other hand, there are  $d \cdot r$  terms between  $x_i$  and  $x_j$ , and hence 32 divides  $d \cdot r$ .

We consider the cases regarding  $d$ . For  $d = 1$ , the claim is trivial. For  $d = 2$ , the pair of terms  $x_i$  and  $x_{i+2}$  can appear twice in four distinct  $\zeta$ -blocks. However, since the parity of the indices  $i$  and  $j$  must be the same in this case, they must appear in the same  $\zeta$ -block in  $\rho_2$ .

In the case  $d = 3$  a pair of two symbols appear only once in four distinct  $\zeta$ -blocks, hence there is nothing to prove. In the case  $d = 4$ , it is not possible to have two consecutive  $\zeta$ -terms.

In the cases  $d = 5$  and  $d = 6$ , the two terms belong to two consecutive  $\zeta$ -blocks. In the former, there is again only one appearance of each pair per four blocks, so it remains to consider the case  $d = 6$ . There are two possible appearances of a pair, but since the indices must have the same parity, the pair must appear in the same two  $\zeta$ -blocks. This completes the proof of the claim.  $\blacklozenge$

We continue by considering the cases regarding  $d$ .

- $d = 2$ .

If there are no two consecutive terms of  $\rho_1$  that belong to the same  $\zeta$ -block, then  $r = 4$  and  $y_1$  is a part of a  $\zeta$ -block. But in this case, there are two consecutive  $\lambda$ -blocks, uniquely determined by  $y_2, y_3$  and  $y_6, y_7$ , which must be equal as  $y_2 = y_6$  and  $y_3 = y_7$ , a contradiction to Lemma 8.

So, there is at least one  $\zeta$ -block which contains two terms of  $\rho_1$ . By Claim 3,  $r$  is divisible by  $32/2 = 16$ . Suppose  $y_1$  is the first (resp. the second) term of some  $\lambda$ -block. Then,  $y_2, y_{r+1}$ , and  $y_{r+2}$  are also  $\lambda$ -terms, and hence every  $\lambda$ -block of  $\rho$  is uniquely determined. By Lemma 5, it follows there is a repetition in  $\varphi_\lambda^t$ , a contradiction to Lemma 8.

Now, suppose  $y_1$  is the third (resp. the fourth) term of some  $\lambda$ -block  $\gamma_1$ . Let  $\gamma_2$  be the  $\lambda$ -block determined by  $y_r$  and  $y_r + 1$ . Clearly,  $\gamma_1 \neq \gamma_2$ , otherwise there is a repetition in  $\varphi_\lambda^t$ , by Lemma 5. However, since the third and the fourth terms of  $\gamma_1$  and  $\gamma_2$  are equal, they are either mapped by  $\lambda$  from  $\{1, 4\}$  or  $\{2, 3\}$ . As the  $\lambda$ -block in  $\sigma_1$  following  $\gamma_1$  is the same as the  $\lambda$ -block in  $\sigma_2$  following  $\gamma_2$ , we obtain a contradiction due to  $(L_6)$ .

Finally, suppose  $y_1$  is a part of a  $\zeta$ -block. Since  $r$  is divisible by 16, the number of  $\lambda$ -blocks in each of  $\rho_1$  and  $\rho_2$  is divisible by 4, and since all of them are uniquely determined, we have a repetition in  $\varphi_\lambda^t$  (in fact already in  $\varphi_\lambda^{t-1}$ ), a contradiction.

- $d = 3$ .

We first show that there are two consecutive  $\zeta$ -terms in  $\rho_1$ . Suppose the contrary. Then, since  $r > 3$  and the fact that the type vectors of  $\rho_1$  and  $\rho_2$  must match, there are two consecutive  $\lambda$ -terms in  $\rho_1$ . But, in the type vector, between two pairs of two consecutive  $\lambda$ -terms, for  $d = 3$ , there are two consecutive  $\zeta$ -terms, a contradiction.

Hence, we may assume there are two consecutive  $\zeta$ -terms in  $\rho_1$  and by Claim 3, 32 divides  $3r$ . Observe also that for  $d = 3$ , there is at least one term from  $\rho$  in every  $\lambda$ -block of the covering sequence of  $\rho$ . Then, by  $(L_7)$ , we infer that all  $\lambda$ -blocks in the covering sequences of  $\rho_1$  are equal to the corresponding  $\lambda$ -blocks in the covering sequences of  $\rho_2$ , and hence there is a repetition in  $\varphi_\lambda^{t-1}$ , a contradiction.

- $d = 4$ .

In this case, all the terms of  $\rho$  are at the same indices in their  $\lambda$ - and  $\zeta$ -blocks. As there is at least one  $\zeta$ -term, by construction of circular sequences, we have that 32 divides  $4r$ , and hence 8 divides  $r$ . Thus, by  $(L_7)$ , all  $\lambda$ -blocks in the covering sequences of  $\rho_1$  are equal to the corresponding ones in the covering sequences of  $\rho_2$ , and hence there is a repetition in  $\varphi_\lambda^{t-1}$ , a contradiction.

- $d = 5$ .

In this case,  $\lambda$ - and  $\zeta$ -blocks of the covering sequence of  $\rho_1$  contain precisely one term from  $\rho_1$  with an exception of every fifth block, which is being skipped. Hence, there are two consecutive  $\lambda$ - or  $\zeta$ -terms in  $\rho_1$  as soon as  $r > 4$ . As  $r > 3$ , the only possible  $\rho_1$  with no consecutive terms of the same type has length 4. However, in such a case, the terms  $y_1$  and  $y_{r+1}$  are not of the same type, so  $r > 4$ .

Suppose first there are no consecutive  $\zeta$ -terms in  $\rho_1$ . In that case, there are two consecutive  $\lambda$ -terms in  $\rho_1$ , and hence also in  $\rho_2$ . Moreover, since between every pair of consecutive  $\lambda$ -terms there are two consecutive  $\zeta$ -terms, the only possible  $r$  for such



$\rho$ , satisfying also that the type vectors of  $\rho_1$  and  $\rho_2$  are the same, is 8. However, then  $y_1$  and  $y_{r+1}$  are both  $\zeta$ -terms but the difference between their indices in the covering sequence is  $5 \cdot 8 = 40$ , meaning that  $y_1 \neq y_{r+1}$ .

So, we may assume there are two consecutive  $\zeta$ -terms in  $\rho_1$  and, by Claim 3, 32 divides  $5r$  (hence 32 divides  $r$  also). By  $(L_7)$ , all  $\lambda$ -blocks in the covering sequence of  $\rho_1$  that contain one term from  $\rho_1$  are equal to the corresponding  $\lambda$ -blocks in the covering sequence of  $\rho_2$ . Furthermore, since in the covering sequence of  $\rho$  three out of every four  $\lambda$ -blocks contain one term from  $\rho$ , also the  $\lambda$ -block  $\gamma_0$  without a term is uniquely determined, unless it is the first  $\lambda$ -block of  $\hat{\rho}_1$  or the last  $\lambda$ -block of  $\hat{\rho}_2$ . In the case when  $\gamma_0$  is determined, the covering sequence of  $\rho_1$  contains the same sequence of  $\lambda$ -blocks as the covering sequence of  $\rho_2$ , and so there is a repetition in  $\varphi_\lambda^{t-1}$ , a contradiction.

Hence, we may assume  $\gamma_0$  is not uniquely determined, and without loss of generality, suppose it is the first  $\lambda$ -block of  $\hat{\rho}_1$ . Since  $\gamma_0$  is not uniquely determined, it is mapped from either the third or the fourth symbol of some  $\lambda$ -block  $\xi_0$  of  $\varphi_\lambda^{t-1}$ . In the former case,  $\xi_0$  is completely determined, since  $r \geq 32$  and one can determine the  $\lambda$ -block following  $\xi_0$  in  $\varphi_\lambda^{t-1}$ , and hence also  $\gamma_0$  is completely determined. In the latter case, observe that,  $y_2 \dots y_{r+1} y_{r+2} \dots y_{2r} x_{j+5}$  (with  $y_{2r} = x_j$ ) is also a repetition, and considering it, we have all  $\lambda$ -blocks in  $\hat{\rho}$  determined, a contradiction.

- $d = 6$ .

In this case,  $\rho_1$  alternately contains two consecutive  $\lambda$ - and two consecutive  $\zeta$ -terms, with a possible shift in the beginning depending on the index of first term in the covering sequence of  $\rho$ . Hence, as  $r > 3$ , there are always two consecutive  $\zeta$ -terms in  $\rho_1$  unless  $r = 4$  and the type vector of  $\rho_1$  is  $CNNC$ . However, in that case  $y_1 \neq y_{r+1}$  by the construction of circular sequences.

Thus, we may assume there are two consecutive  $\zeta$ -terms in  $\rho_1$  and, by Claim 3, 32 divides  $6r$ , hence 16 divides  $r$ . Let  $r = 16t$ ; then the length of the covering sequence of  $\rho_1$  is  $6 \cdot 16t = 96t$  and therefore there are  $12t$   $\lambda$ -blocks, where every two out of three consecutive  $\lambda$ -blocks contain a term from  $\rho_1$ . By  $(L_7)$ , all  $\lambda$ -blocks in the covering sequence of  $\rho_1$  that contain one term from  $\rho_1$  are equal to the corresponding  $\lambda$ -blocks in the covering sequence of  $\rho_2$ . Recall that a  $\lambda$ -block is not uniquely determined by one term; it can be one of two possible (see  $(L_8)$ ).

Let  $\sigma^t$  be the covering sequence of  $\rho$  with all  $\zeta$ -blocks removed and let  $\sigma^{t-1} = \lambda^{-1}(\sigma)$ . Clearly,  $\sigma^{t-1}$  is a subsequence of  $\varphi_\lambda^{t-1}$ . Let  $\sigma_1^{t-1}$  and  $\sigma_2^{t-1}$  be the sequences defined analogously for  $\rho_1$  and  $\rho_2$ , respectively. As we deduced above,  $\sigma_1^{t-1} = z_1 z_2 \dots z_{12t}$  has  $12t$  elements. We consider four subcases regarding the index of  $z_1$  in  $\varphi_\lambda^{t-1}$ . Note that in each of the four cases, for every  $z_i$  that is a preimage of some  $\lambda$ -block with one term from  $\rho$ , we can uniquely determine which symbol  $z_i$  represents simply by  $(L_8)$  and the position of  $z_i$  in the  $\lambda$ -block of  $\varphi_\lambda^{t-1}$ . Consequently, every ‘‘complete’’  $\lambda$ -block of  $\sigma_1^{t-1}$  is uniquely determined by  $(L_3)$ , since we know at least two of its terms, and in the case, when two terms are known, they are at indices 2 and 3.

Suppose first  $z_1$  is at index  $4i + 1$  in  $\lambda^{t-1}$  for some  $i$ . Then, there are  $3t$  complete uniquely determined  $\lambda$ -blocks in  $\sigma_1^{t-1}$ , and hence by Lemma 5, there is a repetition also in  $\varphi_\lambda^{t-1}$ , a contradiction.

Next, suppose  $z_1$  is at index  $4i + 4$  in  $\lambda^{t-1}$ . There are  $3t - 1$  complete uniquely

determined  $\lambda$ -blocks in  $\sigma_1^{t-1}$  and one  $\lambda$ -block, with 3 terms  $z_{12t-2}z_{12t-1}z_{12t}$ . However, as argued above, the latter is also uniquely determined, which means that  $z_2 \dots z_{24t}w$ , where  $w$  is the element at index  $4i + 4 + 24t + 1$  in  $\varphi_\lambda^{t-1}$ , is also a repetition, and hence we may use the argumentation for  $z_1$  being at index  $4i + 1$ .

Suppose  $z_1$  is at index  $4i+3$  in  $\lambda^{t-1}$ . In this case, there are  $3t-2$  complete uniquely determined  $\lambda$ -blocks in  $\sigma_1^{t-1}$ , and two  $\lambda$ -blocks having two terms in  $\sigma_1^{t-1}$ . The second one,  $z_{12t-1}z_{12t}z_{12t+1}z_{12t+2}$  has the other two terms in  $\sigma_2^{t-1}$ . Now, if  $3t$  is divisible by 4, then the  $\lambda$ -block  $z_{-1}z_0z_1z_2$  equals  $z_{12t-1}z_{12t}z_{12t+1}z_{12t+2}$  and we again can shift the sequence to the left as above, obtaining a repetition. Hence  $3t$  is not divisible by 4. Consider the  $\lambda$ -blocks  $\alpha_1 = z_3z_4z_5z_6$  and  $\alpha_1 = z_7z_8z_9z_{10}$ . They are uniquely determined and they must be equal to the  $\lambda$ -blocks  $z_{3+12t}z_{4+12t}z_{5+12t}z_{6+12t}$  and  $z_{7+12t}z_{8+12t}z_{9+12t}z_{10+12t}$ , which is not possible due to the parity condition and  $(L_6)$ .

Finally, suppose  $z_1$  is at index  $4i+2$  in  $\lambda^{t-1}$ . Again, if  $3t$  is divisible by 4, then we shift the sequence by one to the right (I will write this nicer), as the first (incomplete)  $\lambda$ -block in  $\sigma_1^{t-1}$  must match the first (incomplete)  $\lambda$ -block in  $\sigma_2^{t-1}$ , and we obtain a repetition in  $\varphi_\lambda^{t-1}$ . Otherwise,  $3t$  is not divisible by 4, and we obtain a contradiction on the equality of first two complete  $\lambda$ -blocks in  $\sigma_1^{t-1}$  and  $\sigma_2^{t-1}$ . □

## 6 Discussion

In this paper, we improve the current state of Conjecture 1 by showing that it is true for every integer  $k$  between 1 and 8. In particular, we present two proving techniques, which are in their essence similar, but very much different in practice. Namely, the proving technique presented in Section 3 is (provided there are available computing resources) efficient for confirming Conjecture 1 for small  $k$ 's, since one can employ computing resources to verify small instances, while the statement of the conjecture then holds almost trivially for larger instances. However, to be able to prove Conjecture 1 in general or at least for an infinite number of integers, it will fail.

On the other hand, the method described in Section 5 is more promising. We are using a special construction of Thue sequences with properties allowing to prove that they are also  $k$ -Thue. This technique needs more argumentation for proving that the generated sequences are indeed  $k$ -Thue, but allows for establishing properties for a bigger set of  $k$ 's, possibly infinite.

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