On the structure of Schnyder woods on orientable surfaces

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ON THE STRUCTURE OF SCHNYDER WOODS ON ORIENTABLE SURFACES*

Daniel Gonçalves,† *Kolja Knauer,*‡ *and Benjamin Lévêque,*§

Abstract. We propose a simple generalization of Schnyder woods from the plane to maps on orientable surfaces of higher genus. This is done in the language of angle labelings. Generalizing results of de Fraysseix and Ossona de Mendez, and Felsner, we establish a correspondence between these labelings and orientations and characterize the set of orientations of a map that correspond to such a Schnyder labeling. Furthermore, we study the set of these orientations of a given map and provide a natural partition into distributive lattices depending on the surface homology. This generalizes earlier results of Felsner and Ossona de Mendez. In the particular case of toroidal triangulations, this study enables us to identify a canonical lattice that lies at the core of several bijection proofs.

1 Introduction

Schnyder [27] introduced Schnyder woods for planar triangulations with the following local property:

Definition 1 (Schnyder property) *Given an embedded graph G, a vertex v and an orientation and coloring*[1] *of the edges incident to v with the colors 0, 1, 2, we say that v satisfies the* Schnyder property, *(see Figure 1) if v satisfies the following local property:*

- *Vertex v has out-degree one in each color.*
- *The edges $e_0(v)$, $e_1(v)$, $e_2(v)$ leaving v in colors 0, 1, 2, respectively, occur in counterclockwise order.*
- *Each edge entering v in color i enters v in the counterclockwise sector from $e_{i+1}(v)$ to $e_{i-1}(v)$.*

*This work was supported by grants EGOS ANR-12-JS02-002-01 and GATO ANR-16-CE40-0009-01

†*CNRS, Université de Montpellier, LIRMM, UMR 5506, CC477, 161 rue Ada, 34095 Montpellier Cedex 5, France,* daniel.goncalves@lirmm.fr

‡*Aix Marseille Université, LIF, UMR 7279, Parc Scientifique et Technologique de Luminy, 163 avenue de Luminy - Case 901, 13288 Marseille Cedex 9, France,* kolja.knauer@lif.univ-mrs.fr

§*CNRS, Université Grenoble Alpes, G-SCOP, UMR 5272, 46 Avenue Félix Viallet, 38031 Grenoble Cedex 1, France,* benjamin.leveque@grenoble-inp.fr

[1] Throughout the paper colors and some of the indices are given modulo 3.

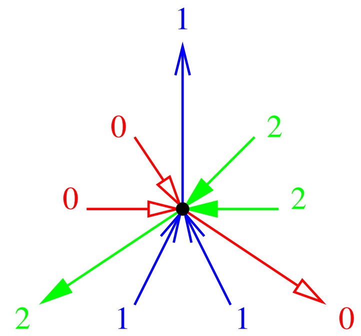


Figure 1: The Schnyder property. The depicted correspondence between red, blue, green, 0, 1, 2, and the arrow shapes will be used through the paper.

Definition 2 (Schnyder wood) *Given a planar triangulation G, a* Schnyder wood *is an orientation and coloring of the inner edges of G with the colors* 0, 1, 2 *(edges are oriented in one direction only), where each inner vertex v satisfies the* Schnyder property*.*

See Figure 2 for an example of a Schnyder wood.

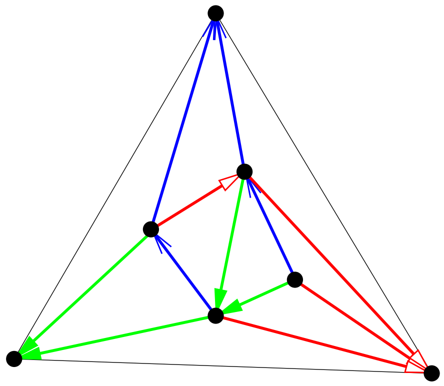


Figure 2: Example of a Schnyder wood of a planar triangulation.

Schnyder woods are today one of the main tools in the area of planar graph representations. Among their most prominent applications are the following: They provide a machinery to construct space-efficient straight-line drawings [28, 20, 9], yield a characterization of planar graphs via the dimension of their vertex-edge incidence poset [27, 9], and are used to encode triangulations [25, 3]. Further applications lie in enumeration [4], representation by geometric objects [14, 17], graph spanners [5], etc. The richness of these applications has stimulated research towards generalizing Schnyder woods to non planar graphs.

For higher genus triangulated surfaces, a generalization of Schnyder woods has been proposed by Castelli Aleardi, Fusy and Lewiner [7], with applications to encoding. Actually, they prove that every triangulation with genus g admits a so-called g-Schnyder wood, which is a partition of the edge set (where $2g$ edges have been doubled) into 3 parts, where the Schnyder property is followed except at the doubled edges. In this definition, the simplicity and the symmetry of the original definition of Schnyder woods are lost. Furthermore, this definition does not extend to non-triangular graphs (as Schnyder woods do in the planar case). Here we propose an alternative generalization of Schnyder woods for higher genus that generalizes the one proposed in [18] for the toroidal case. The existence of these objects for large classes is open (see the conjectures in Section 2.5), but this generalization of Schnyder woods is elegant and extends the case of non-triangular plane graphs. This allows us to preserve the link between the Schnyder woods of a graph, and those of its dual.

A closed curve on a surface is *contractible* if it can be continuously transformed into a single point. Given a graph embedded on a surface, a *contractible loop* is an edge forming a contractible curve. Two edges of an embedded graph are called *homotopic multiple edges* if they have the same extremities and their union encloses a region homeomorphic to an open disk. Except if stated otherwise, we consider graphs embedded on surfaces that do not have contractible loops nor homotopic multiple edges. Note that this is a weaker assumption, than the graph being *simple*, i.e. not having loops nor multiple edges. A graph embedded on a surface is called a *map* on this surface if all its faces are homeomorphic to open disks. A map is a *triangulation* if all its faces have size three.

In this paper we consider finite maps. Maps are preferred to embedded graphs because they allow the dual embedded graph to be uniquely defined. We will see also that the notion of cyclic order around faces is needed in the following section. We denote by n be the number of vertices and m the number of edges of a graph. Given a graph embedded on a surface, we use f for the number of faces. Euler's formula says that any map on an orientable surface of genus g satisfies $n - m + f = 2 - 2g$. In particular, the plane is the surface of genus 0, the torus the surface of genus 1, the double torus the surface of genus 2, etc. By Euler's formula, a triangulation of genus g has exactly $3n + 6(g - 1)$ edges. So having a generalization of Schnyder woods in mind, for all $g \geq 2$ there are too many edges to force all vertices to have out-degree exactly three. This problem can be overcome by allowing vertices to fulfill the Schnyder property "several times", i.e. such vertices have out-degree 6, 9, etc. with the color property of Figure 1 repeated several times (see Figure 3).

Figure 4 is an example of such a Schnyder wood on a triangulation of the double torus. The double torus is represented by an octagon whose sides are identified according to their labels. All the vertices of the triangulation have out-degree three except two vertices, the circled ones, that have out-degree six. Each of the latter appear twice in the representation.

In this paper we formalize this idea to obtain a concept of Schnyder woods applicable to general maps (not only triangulations) on arbitrary orientable surfaces. This is based on the definition of Schnyder woods via angle labelings in Section 2. We prove several basic properties of these objects that extend the properties of Schnyder woods in

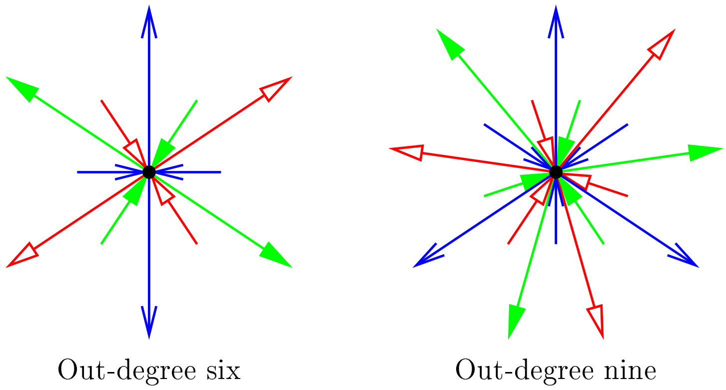


Figure 3: The Schnyder property repeated several times around a vertex.

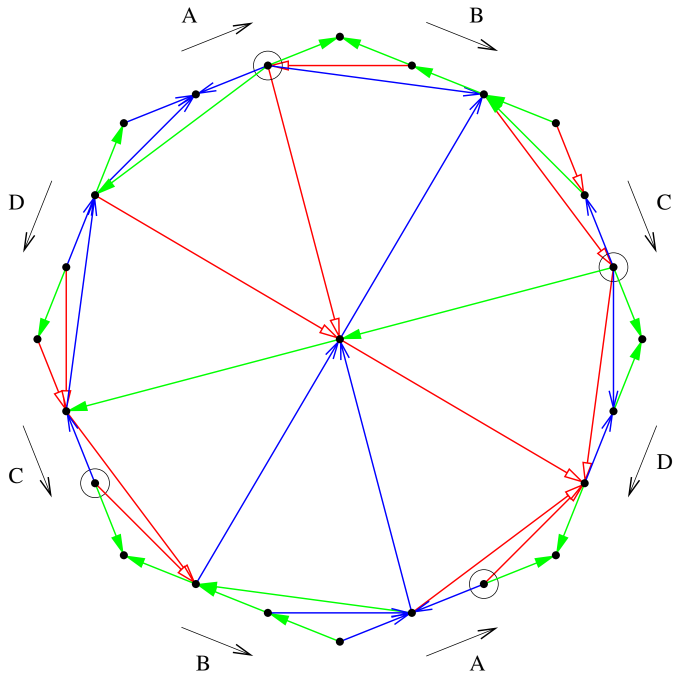


Figure 4: A Schnyder wood of a triangulation of the double torus.

the plane. These properties are generally stated in the universal cover[2] of the considered map. While every map admits a “trivial” Schnyder wood, the existence of a non-trivial one remains open but leads to interesting conjectures.

By a result of de Fraysseix and Ossona de Mendez [15], for any planar triangulation there is a bijection between its Schnyder woods and the orientations of its inner edges where every inner vertex has out-degree three. Thus, any orientation with the proper

[2]The *universal cover* of a map G is an infinite plane map that is locally isomorphic to G. A full definition is provided in Section 2.4.

out-degree corresponds to a Schnyder wood and there is a unique way, up to symmetry of the colors, to assign colors to the oriented edges in order to fulfill the Schnyder property at every inner vertex. This is not true in higher genus as already in the torus, there exist orientations that do not correspond to any Schnyder wood (see Figure 5). In Section 3, we exhibit a simple necessary and sufficient condition for an orientation to correspond to a Schnyder wood. This characterization is stated in terms of the homology class of the considered orientation. In Section 3 we thus consider the orientations of maps through the lens of homology.

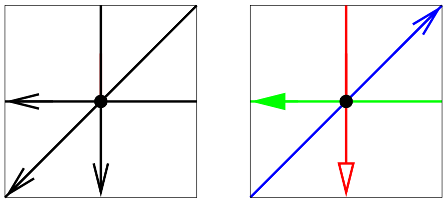


Figure 5: Two different orientations of a toroidal triangulation. Only the one on the right corresponds to a Schnyder wood.

In Section 4, we study the transformations between Schnyder orientations. We obtain a partition of the set of Schnyder woods into homology classes of orientations, each of these classes being a distributive lattice. In these lattices two orientations (of the same map G) are linked if one is obtained from the other by reversing the edges of some directed cycles (corresponding to the borders of faces of an associated map $\widetilde{G}$). This generalizes corresponding results obtained for the plane by Ossona de Mendez [24] and Felsner [11].

In Section 5, we focus on toroidal triangulations for which a particular lattice, called *canonical lattice*, is identified. It is proved that the toroidal Schnyder woods introduced in [18], that have a global property called *crossing property*, all belong to this lattice. The minimal element of the canonical lattice has recently been used to obtain bijections, and an optimal linear encoding method for toroidal triangulations by Despré, the first author, and the third author [8], generalizing previous results of Poulalhon and Schaeffer for the plane [25]. An analogous canonical property has also been found for a generalization of transversal structures to essentially 4-connected toroidal triangulations with some enumerative consequences (see [6]).

Note that the results presented here also appear in the Habilitation manuscript of the third author [19], where they are combined with [8, 18] to present these articles in a unified way.

2 Generalization of Schnyder woods

We now introduce angle labeling, and we will show that they allow an elegant characterization of an extension of Schnyder woods to planar maps. These angle labelings then allow us to define our generalization of Schnyder woods for any map, and this illustrates how this generalization naturally extends the planar case. We will then study these Schnyder woods on the universal cover of the considered map, and we will see that several properties of the plane extend to these maps. We then conclude the section with two conjectures on the existence of these Schnyder woods for large families of maps.

2.1 Angle labelings

Consider a map G on an orientable surface. An *angle labeling* of G is a labeling of the angles of G (i.e. face corners of G) in colors 0, 1, 2. More formally, we denote an angle labeling by a function $\ell : \mathcal{A} \to \mathbb{Z}_3$, where $\mathcal{A}$ is the set of angles of G. Given an angle labeling, we define several properties of vertices, faces and edges that generalize the notion of Schnyder angle labeling in the planar case [13].

Consider an angle labeling ℓ of G. A vertex or a face v is of *type* k, for $k \geq 1$, if the labels of the angles around v form, in counterclockwise order, $3k$ nonempty intervals such that in the j-th interval all the angles have color ($j \mod 3$). A vertex or a face v is of *type* 0, if the labels of the angles around v are all of color i for some i in $\{0, 1, 2\}$.

An edge e is of *type* 1 *or* 2 if the labels of the four angles incident to edge e are, in clockwise order, $i-1$, i, i, $i+1$ for some i in $\{0, 1, 2\}$. The edge e is of *type* 1 if the two angles with the same color are incident to the same extremity of e and of *type* 2 if the two angles are incident to the same side of e. An edge e is of *type* 0 if the labels of the four angles incident to edge e are all i for some i in $\{0, 1, 2\}$. The edges of type 0, 1, and 2 respectively correspond to edges with 0, 1, and 2 outgoing half-edges (See Figure 6).

If there exists a function $f : V \to \mathbb{N}$ such that every vertex v of G is of type $f(v)$, we say that ℓ is f-VERTEX. If we do not want to specify the function f, we simply say that ℓ is VERTEX. We sometimes use the notation K-VERTEX if the labeling is f-VERTEX for a function f with $f(V) \subseteq K$. When $K = \{k\}$, i.e. f is a constant function, then we use the notation k-VERTEX instead of f-VERTEX. Similarly we define FACE, K-FACE, k-FACE, EDGE, K-EDGE, k-EDGE.

The following lemma expresses that property EDGE is the central notion here. Properties K-VERTEX and K-FACE are used later on to express additional requirements on the angle labelings that are considered.

Lemma 1 *An* EDGE *angle labeling is* VERTEX *and* FACE.

Proof. Let ℓ be an EDGE angle labeling. Consider two counterclockwise consecutive angles a, a' around a vertex (or a face). Property EDGE implies that $\ell(a') = \ell(a)$ or $\ell(a') = \ell(a) + 1$ (see Figure 6). Thus by considering all the angles around a vertex or a face, it is clear that ℓ is also VERTEX and FACE. □

Thus we define a Schnyder labeling as follows:

Definition 3 (Schnyder labeling) *Given a map G on an orientable surface, a* Schnyder labeling *of G is an* EDGE *angle labeling of G.*

Figure 6 shows how an EDGE angle labeling defines an "orientation" and coloring of the edges of the graph with edges oriented in one direction or in two opposite directions.

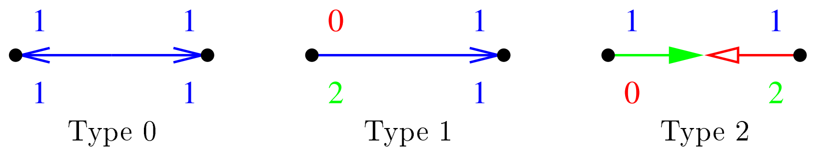


Figure 6: Correspondence between EDGE angle labelings and some orientations and colorings of the edges.

In the next two sections, the correspondence from Figure 6 is used to show that Schnyder labelings correspond to or generalize previously defined Schnyder woods in the plane and in the torus. Hence, they are a natural generalization of Schnyder woods for higher genus.

2.2 Planar Schnyder woods

Originally, Schnyder woods were defined only for planar triangulations [27]. Felsner [9, 10] extended this definition to planar maps. To do so he allowed edges to be oriented in one direction or in two opposite directions (originally only one direction was possible). The formal definition is the following:

Definition 4 (Planar Schnyder wood) *Given a planar map G. Let x_0, x_1, x_2 be three vertices occurring in counterclockwise order on the outer face of G. The* suspension *G^σ is obtained by attaching a half-edge that reaches into the outer face to each of these special vertices. A* planar Schnyder wood *rooted at x_0, x_1, x_2 is an orientation and coloring of the edges of G^σ with the colors 0, 1, 2, where every edge e is oriented in one direction or in two opposite directions (each direction having a distinct color and being outgoing), satisfying the following conditions:*

- *Every vertex satisfies the Schnyder property, and the half-edge at x_i is directed outward and colored i.*
- *There is no interior face whose boundary is a monochromatic cycle.*

See Figure 7 for two examples of planar Schnyder woods.

The correspondence of Figure 6 gives the following bijection, as proved by Felsner [10]:

Proposition 1 ([10]) *If G is a planar map and x_0, x_1, x_2 are three vertices occurring in counterclockwise order on the outer face of G, then the planar Schnyder woods of G^σ*

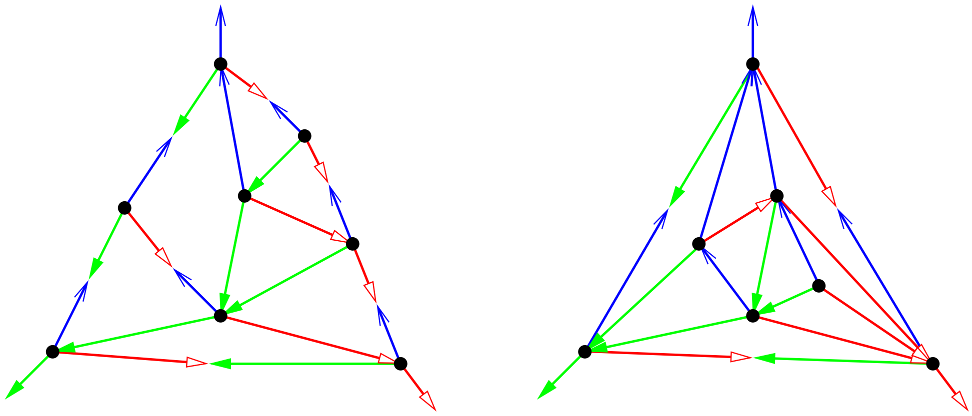


Figure 7: A planar Schnyder wood of a planar map and of a planar triangulation.

*are in bijection with the {1,2}-*EDGE*, 1-*VERTEX*, 1-*FACE *angle labelings of* G^σ *(with the outer face being 1-*FACE *but in clockwise order).*

Felsner [9] and Miller [22] characterized the planar maps that admit a planar Schnyder wood. Namely, they are the internally 3-connected maps (i.e. those with three vertices on the outer face such that the graph obtained from G by adding a vertex adjacent to the three vertices is 3-connected).

2.3 Generalized Schnyder woods

Any map (on any orientable surface) admits a trivial EDGE angle labeling: the one with all angles labeled i (and thus all edges, vertices, and faces are of type 0). A natural non-trivial case, that is also symmetric for the duality, is to consider EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labelings of general maps (where $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$). In planar Schnyder woods only type 1 and type 2 edges are used. Here we allow type 0 edges because they seem unavoidable for some maps (see discussion below). This suggests the following definition of Schnyder woods in higher genus.

First, the generalization of the Schnyder property is the following:

Definition 5 (Generalized Schnyder property) *Given a map G on a genus $g \geq 1$ orientable surface, a vertex v and an orientation and coloring of the edges incident to v with the colors 0, 1, 2, we say that v satisfies the* generalized Schnyder property *(see Figure 3), if v satisfies the following local property for $k \geq 1$:*

- *Vertex v has out-degree $3k$.*
- *The edges $e_0(v), \ldots, e_{3k-1}(v)$ leaving v in counterclockwise order are such that $e_j(v)$ has color j mod 3.*

- *Each edge entering v in color i enters v in a counterclockwise sector from $e_j(v)$ to $e_{j+1}(v)$ with $i \equiv j - 1 \pmod 3$.*

Then, the generalization of Schnyder woods is the following (where the three types of edges depicted on Figure 6 are allowed):

Definition 6 (Generalized Schnyder wood) *Given a map G on a genus $g \geq 1$ orientable surface, a* generalized Schnyder wood *of G is an orientation and coloring of the edges of G with the colors* 0, 1, 2*, where every edge is oriented in one direction or in two opposite directions (each direction having a distinct color and being outgoing, or each direction having the same color and being incoming), satisfying the following conditions:*

- *Every vertex satisfies the generalized Schnyder property.*
- *There is no face whose boundary is a monochromatic cycle.*

When there is no ambiguity we call "generalized Schnyder woods" just "Schnyder woods". See Figure 8 for two examples of Schnyder woods in the torus.

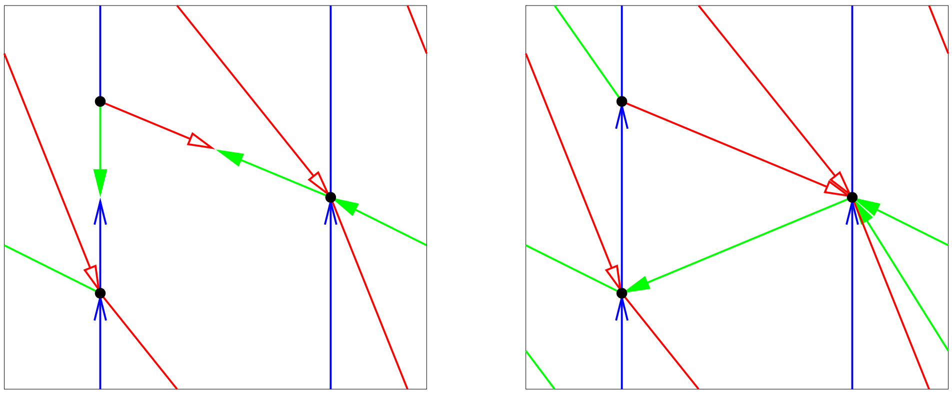


Figure 8: A Schnyder wood of a toroidal map and of a toroidal triangulation.

The first and third author already defined Schnyder woods for toroidal maps in [18]. The above definition is broader than the one in [18] where an additional (global) condition is required (see Section 5).

Figure 4 is an example of a Schnyder wood on a triangulation of the double torus. The correspondence from Figure 6 immediately gives the following bijection whose proof is omitted.

Proposition 2 *If G is a map on a genus $g \geq 1$ orientable surface, then the generalized Schnyder woods of G are in bijection with the* EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE *angle labelings of G.*

The examples in Figures 8 and 4 do not have type 0 edges. However, for all $g \geq 2$, there are genus g maps, with vertex degrees and face degrees at most five. Figure 9 depicts how to construct such maps, for all $g \geq 2$. For these maps, type 0 edges are unavoidable. Indeed, take such a map with an angle labeling that has only type 1 and type 2 edges. Around a type 1 or type 2 edge there are exactly three changes of labels, so in total there are exactly $3m$ such changes. As vertices and faces have degree at most five, they are either of type 0 or 1, hence the number of label changes should be at most $3n + 3f$. Thus, $3m \leq 3n + 3f$, which contradicts Euler's formula for $g \geq 2$. Furthermore, note that the maps described in Figure 9, as well as their dual maps, are 3-connected.

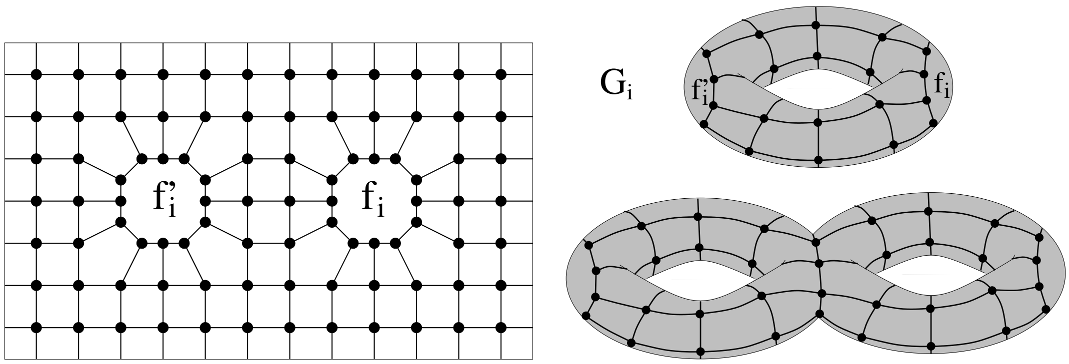


Figure 9: A toroidal map G_i with two distinguished faces, f_i and f'_i. Take g copies G_i with $1 \leq i \leq g$ and glue them by identifying f_i and f'_{i+1} for all $1 \leq i < g$. Faces f_1 and f'_g are filled to have only vertices and faces of degree at most five.

An orientation and coloring of the edges corresponding to an EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labelings is given for the double-toroidal map of Figure 10. It contains two edges of type 0 and it is 1-VERTEX and 1-FACE. Similarly, one can obtain EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labelings for any map in Figure 9.

2.4 Schnyder woods in the universal cover

In this section we prove some properties of Schnyder woods in the universal cover. We refer to [21] for the general theory of universal covers. The *universal cover* of the torus (resp. an orientable surface of genus $g \geq 2$) is a surjective mapping p from the plane (resp. the open unit disk) to the surface that is locally a homeomorphism. The universal cover of the torus is obtained by replicating a flat representation of the torus to tile the plane (see Figure 11). Figure 12 shows how to obtain the universal cover of the double torus. The key property is that a closed curve $\mathcal{C}$ on the surface corresponds to a set of closed curves in the universal cover if and only if $\mathcal{C}$ is contractible.

Universal covers can be used to represent a map on an orientable surface as an infinite planar map. Any property of the map can be lifted to its universal cover, as long as it is defined locally. Thus universal covers are an interesting tool for the study of Schnyder labelings since all the definitions we have given so far are purely local.

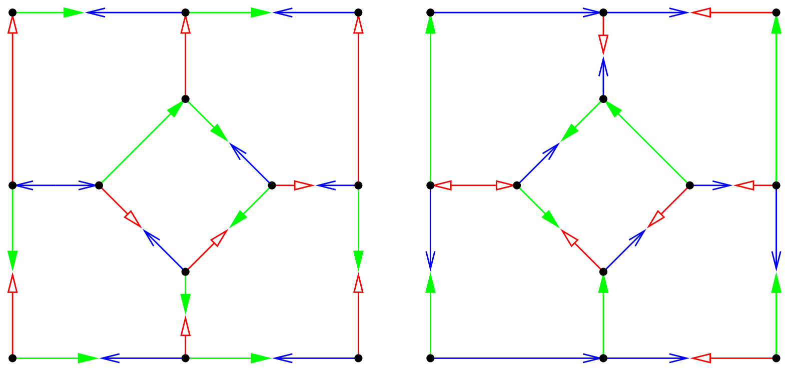


Figure 10: An orientation and coloring of the edges of a double-toroidal map that correspond to an EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labeling. Here, the two parts are toroidal and the two central faces are identified (by preserving the colors) to obtain a double-toroidal map.

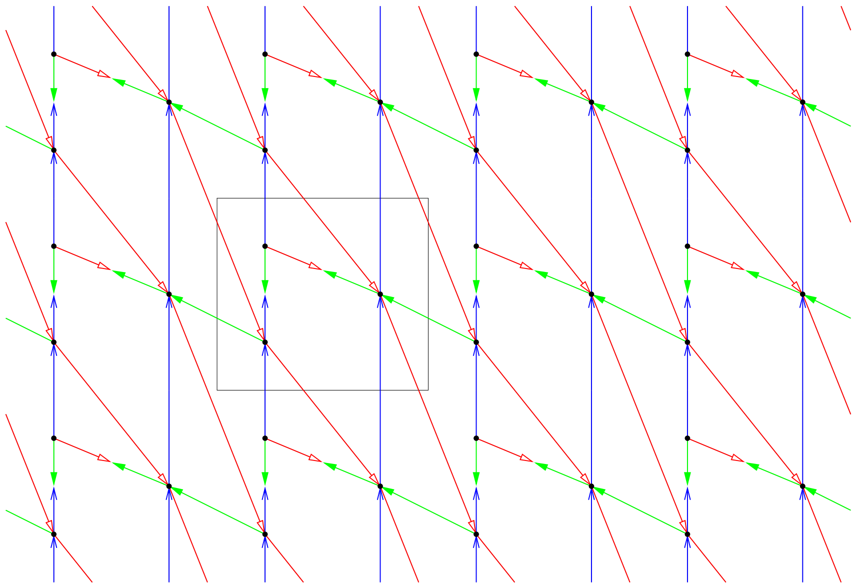
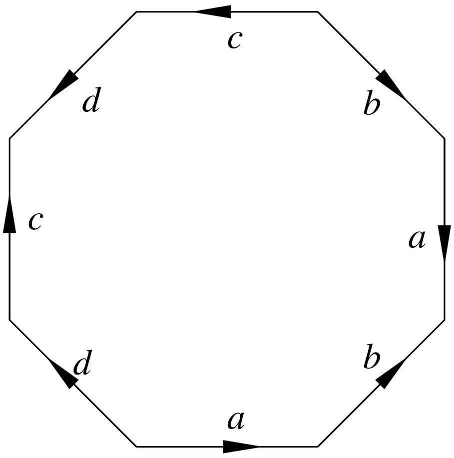


Figure 11: Universal cover of the toroidal map in the left of Figure 8.

In a map, a *walk* is a (possibly infinite) sequence of edges traversed in a given direction, such that the head of an edge in the sequence coincides with the tail of the next edge in the sequence (possibly two successive edges in the sequence are the same edge traversed in opposite directions). A *path* is a walk with no repeated vertices. A *closed* walk is a finite walk such that the tail of the first edge in the sequence coincides with the head of the last edge. A *cycle* is a closed walk with no repeated vertices.

Consider a map G on a genus $g \geq 1$ orientable surface. Let G^∞ be the infinite planar map drawn on the universal cover and defined by $p^{-1}(G)$ (where p is a surjective



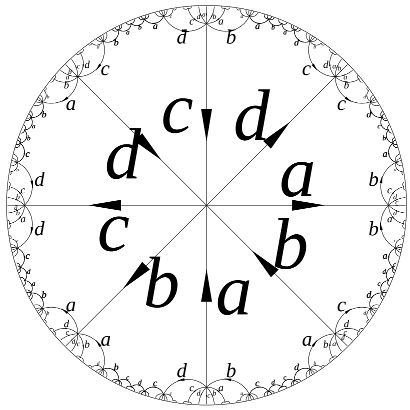


Figure 12: Canonical representation and universal cover of the double torus (source : Yann Ollivier `http://www.yann-ollivier.org/maths/primer.php`).

mapping that is locally a homeomorphism from the plane, or from the open unit disk, to that surface).

We need the following general lemma concerning universal covers:

Lemma 2 *Suppose that for a finite set of vertices X of G^∞, the graph $G^\infty \setminus X$ is not connected. Then $G^\infty \setminus X$ has a finite connected component.*

Proof. Suppose the lemma is false and $G^\infty \setminus X$ is not connected and has no finite component. Then it has a face bounded by an infinite number of vertices. As G is finite, the vertices of G^∞ have bounded degree. Putting back the vertices of X, a face bounded by an infinite number of vertices would remain. The corresponding face in G is not homeomorphic to an open disk, a contradiction with G being a map. □

Suppose now that G is given with a Schnyder wood (i.e. an EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labeling by Proposition 2). Consider the orientation and coloring of the edges of G^∞ corresponding to the Schnyder wood of G.

Let G_i^∞ be the directed graph of edges of color i of G^∞, i.e. G_i^∞ is the graph whose vertex set is the vertex set of G^∞, whose edge set is the set of edges of G^∞ that are colored (or half-colored) i, and whose edges are oriented in one direction only, corresponding precisely to the direction of color i in G^∞. The graph $(G_i^\infty)^{-1}$ is the graph obtained from G_i^∞ by reversing the direction of all its edges. The graph $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$ is obtained from the graph G by orienting edges in one or two directions depending on whether this orientation is present in G_i^∞, $(G_{i-1}^\infty)^{-1}$ or $(G_{i+1}^\infty)^{-1}$. Similarly to what happens for planar Schnyder woods (see [9]), we have the following:

Lemma 3 *The graph $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$ does not contain directed cycles.*

Proof. Suppose there is a directed cycle C in $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$. By Lemma 2, there is a finite number of vertices and of faces inside C. Assume now that C is a cycle containing the minimum number of faces inside. Let D be the finite submap of G^∞ whose border is C. Suppose by symmetry that C turns around D counterclockwisely. Then, by the Schnyder property (see Definition 5), every vertex of D has at least one outgoing edge of color $i+1$ in D. This is clear for the vertices inside D, but note that for every vertex v on C, the edge of C leaving v (resp. incoming in v) is an edge in the counterclockwise sector from $e_{i-1}(v)$ to $e_{i+1}(v)$ (resp. from $e_{i+1}(v)$ to $e_{i-1}(v)$), excluding (resp. including) $e_{i-1}(v)$ and $e_{i+1}(v)$. This implies that $e_{i+1}(v)$ belongs to D. Thus there is a cycle of color $(i+1)$ in D, and its edges coincide with C by minimality of C. So C is a cycle of $(G_{i+1}^\infty)^{-1}$. Then, again by the Schnyder property, every vertex of D has at least one outgoing edge of color i in D. Thus, there is a cycle of color i in D and its edges coincide with C by minimality of C. So C is a cycle of (G_i^∞). Thus all the edges of C are oriented in G^∞ in color i counterclockwisely and in color $i+1$ clockwisely.

By the definition of Schnyder woods, there is no face the boundary of which is a monochromatic cycle, so D is not a face. Let vx be an edge in the interior of D that is outgoing for v. The vertex v can be either in the interior of D or in C (if v has more than three outgoing edges). In both cases, v has necessarily an edge e_i of color i and an edge e_{i+1} of color $i+1$, leaving v and in the interior of D. Consider $W_i(v)$ (resp. $W_{i+1}(v)$) a monochromatic walk starting from e_i (resp. e_{i+1}), obtained by following outgoing edges of color i (resp. $i+1$). By minimality of C those walks are not self-intersecting in D. We hence have that $W_i(v) \setminus v$ and $W_{i+1}(v) \setminus v$ intersect C. Thus each of these walks contains a non-empty subpath from v to C. If these two subpaths intersect, let u be an intersection such that the subpaths of $W_i(v)$ and $W_{i+1}(v)$ going from u to C are not intersecting. The union of these two paths, plus a part of C contradicts the minimality of C. □

Let v be a vertex of G^∞. For each color i, vertex v is the starting vertex of some walks of color i, we denote the union of these walks by $P_i(v)$. Every vertex has at least one outgoing edge of color i and the set $P_i(v)$ is obtained by following all these edges of color i starting from v. Note that for some vertices v, the set $P_i(v)$ may consist of a single walk. This is the case when v cannot reach a vertex of out-degree six or more. Note also that by Lemma 3 the graph $P_i(v)$ is acyclic, and as each of its vertices has positive out-degree, it is an infinite graph.

Lemma 4 *For every vertex v and color i, the two graphs $P_{i-1}(v)$ and $P_{i+1}(v)$ intersect only on v.*

Proof. If $P_{i-1}(v)$ and $P_{i+1}(v)$ intersect on two vertices, then $G_{i-1}^\infty \cup (G_{i+1}^\infty)^{-1}$ contains a cycle, contradicting Lemma 3. □

Recall that a graph is *k-connected* if it has at least $k+1$ vertices and if it remains connected after removing any $k-1$ vertices. Extending the notion of essentially 2-connectedness defined in [23] for the toroidal case, we say that G is *essentially k-connected* if G^∞ is k-connected. Note that the notion of being essentially k-connected is different from G being k-connected. There are no implications in any direction [19], but note that since G is a map, it is essentially 1-connected.

Theorem 1 *If a map G on a genus $g \geq 1$ orientable surface admits an* EDGE*,* $\mathbb{N}^*$-VERTEX*,* $\mathbb{N}^*$-FACE *angle labeling, then G is essentially 3-connected.*

Proof. Towards a contradiction, suppose that there exist two vertices x, y of G^∞ such that $G' = G^\infty \setminus \{x, y\}$ is not connected. Then, by Lemma 2, the graph G' has a finite connected component R. Let v be a vertex of R. By Lemma 3, for $0 \leq i \leq 2$, the infinite graph $P_i(v)$ does not lie in R so it intersects either x or y. So for two distinct colors i, j, we have $P_i(v)$ and $P_j(v)$ intersect in a vertex distinct from v, a contradiction to Lemma 4. □

2.5 Conjectures on the existence of Schnyder woods

Proving that every triangulation on a genus $g \geq 1$ orientable surface admits a 1-EDGE angle labeling would imply the following theorem of Barát and Thomassen [2]:

Theorem 2 ([2]) *Every simple triangulation on a genus $g \geq 1$ orientable surface admits an orientation of its edges such that every vertex has out-degree divisible by three.*

Recently, Theorem 2 has been improved by Albar, the first author, and the second author [1]:

Theorem 3 ([1]) *Every simple triangulation on a genus $g \geq 1$ orientable surface admits an orientation of its edges such that every vertex has out-degree at least three, and divisible by three.*

Note that Theorems 2 and 3 are proved only in the case of simple triangulations (i.e. no loops and no multiple edges). We believe them to be true also for non-simple triangulations without contractible loops nor homotopic multiple edges.

Theorem 3 suggests the existence of 1-EDGE angle labelings with no sinks, i.e. 1-EDGE, $\mathbb{N}^*$-VERTEX angle labelings. One can easily check that in a triangulation, a 1-EDGE angle labeling is also 1-FACE. Thus we can hope that a triangulation on a genus $g \geq 1$ orientable surface admits a 1-EDGE, $\mathbb{N}^*$-VERTEX, 1-FACE angle labeling. Note that a 1-EDGE, 1-FACE angle labeling of a map implies that faces have size three. So we propose the following conjecture, whose "only if" part follows from the previous sentence:

Conjecture 1 *A map on a genus $g \geq 1$ orientable surface admits a 1-*EDGE*,* $\mathbb{N}^*$-VERTEX*, 1-*FACE *angle labeling if and only if it is a triangulation.*

If true, Conjecture 1 would strengthen Theorem 3 in two ways. First, it considers more triangulations (not only simple ones). Second, it requires the coloring property around vertices.

How about general maps? We propose the following conjecture, whose "only if" part is Theorem 1:

Conjecture 2 *A map on a genus $g \geq 1$ orientable surface admits an* EDGE*,* $\mathbb{N}^*$-VERTEX*,* $\mathbb{N}^*$-FACE *angle labeling if and only if it is essentially 3-connected.*

Conjecture 2 implies Conjecture 1 since for a triangulation every face would be of type 1, and thus every edge would be of type 1. Conjecture 2 is proved in [18] for $g = 1$ whereas both conjectures are open for $g \geq 2$.

3 Characterization of Schnyder orientations

As already mentioned, contrarily to the plane, the out-degrees in an orientation do not characterize Schnyder woods. In this section we study the orientations of maps through the lens of homology, and this eventually leads to a characterization of the orientations corresponding to Schnyder woods.

3.1 A bit of homology

In the next sections, we need a bit of surface homology of general maps, which we will discuss now. For a deeper introduction to homology we refer to [16].

For the sake of generality, in this subsection we consider that maps may have loops or multiple edges. Consider a map $G = (V, E)$, on an orientable surface of genus g, given with an arbitrary orientation of its edges. This fixed arbitrary orientation is implicit in all the paper and is used to handle flows. A *flow* ϕ on G is a vector in $\mathbb{Z}^E$. For any $e \in E$, we denote by ϕ_e the coordinate e of ϕ.

A walk W has a *characteristic flow* $\phi(W)$ defined by:

$$\phi(W)_e := \#\text{times } W \text{ traverses } e \text{ forward} - \#\text{times } W \text{ traverses } e \text{ backward}$$

This definition naturally extends to a set of walk $\mathcal{W}$ by summing the characteristic flows (i.e. $\phi(\mathcal{W})_e := \sum_{W \in \mathcal{W}} \phi(W)_e$). From now on we consider that a set of walks and its characteristic flow are the same object and by abuse of notation we can write W instead of $\phi(W)$. As an *oriented edge set* X (i.e. an *oriented subgraph* H) can be seen as a set of one-edge walks, it has a characteristic flow $\phi(X)$ (resp. $\phi(H)$) which we may refer to by simply X (resp. H).

A *facial walk* is a closed walk bounding a face. Let $\mathcal{F}$ be the set of counterclockwise facial walks and let $\mathbb{F} = <\phi(\mathcal{F})>$ be the subgroup of $(\mathbb{Z}^E, +)$ generated by $\mathcal{F}$. Two flows ϕ, ϕ' are *homologous* if $\phi - \phi' \in \mathbb{F}$. They are *weakly homologous* if $\phi - \phi' \in \mathbb{F}$ or $\phi + \phi' \in \mathbb{F}$. We say that a flow ϕ is 0-homologous if it is homologous to the zero flow, i.e. $\phi \in \mathbb{F}$.

Let $\mathcal{W}$ be the set of *closed* walks and let $\mathbb{W} = <\phi(\mathcal{W})>$ be the subgroup of $(\mathbb{Z}^E, +)$ generated by $\mathcal{W}$. The group $H(G) = \mathbb{W}/\mathbb{F}$ is the *first homology group* of G. It is well-known that $H(G)$ only depends on the genus of the map, and actually it is isomorphic to $\mathbb{Z}^{2g}$.

A set $\{B_1, \ldots, B_{2g}\}$ of (closed) walks of G is said to be a *basis for the homology* if the equivalence classes of their characteristic vectors $([\phi(B_1)], \ldots, [\phi(B_{2g})])$ generate $H(G)$. Then for any closed walk W of G, we have $W = \sum_{F \in \mathcal{F}} \lambda_F F + \sum_{1 \leq i \leq 2g} \mu_i B_i$ for some $\lambda \in \mathbb{Z}^{\mathcal{F}}, \mu \in \mathbb{Z}^{2g}$. Moreover one of the λ_F can be set to zero (and then all the other coefficients are unique). Indeed, for any map, there exists a set of cycles that forms a basis for the homology and it is computationally easy to build. A possible way is by considering a spanning tree T of G, and a spanning tree T^* of G^* that contains no edges dual to T. By Euler's formula, there are exactly $2g$ edges in G that are not in T nor dual to edges of T^*. Each of these $2g$ edges forms a unique cycle with T. It is not hard to see that this set of cycles forms a basis for the homology.

The edges of the dual G^* of G are oriented such that the dual e^* of an edge e of G goes from the face on the right of e to the face on the left of e. Let $\mathcal{F}^*$ be the set of counterclockwise facial walks of G^*. Consider $\{B_1^*, \ldots, B_{2g}^*\}$ a set of closed walks of G^* that form a basis for the homology. We now introduce the following function β, defined for two flows p and d of G and G^*, respectively, as follows:

$$\beta(p, d) = \sum_{e \in G} p_e d_{e^*}$$

Note that β is a bilinear function. This function is needed to define the parameter δ in the following subsection, and to study the transformations between generalized Schnyder woods in Section 4.1. For this last purpose we also prove the following new lemma.

Lemma 5 *Given two flows ϕ, ϕ' of G, the following properties are equivalent to each other:*

1. *The two flows ϕ, ϕ' are homologous.*
2. *For each closed walk W of G^* we have $\beta(\phi, W) = \beta(\phi', W)$.*
3. *For each $F \in \mathcal{F}^*$, we have $\beta(\phi, F) = \beta(\phi', F)$, and, for each $1 \leq i \leq 2g$, we have $\beta(\phi, B_i^*) = \beta(\phi', B_i^*)$.*

Proof. (1. $\Longrightarrow$ 3.) Suppose that ϕ, ϕ' are homologous. Then we have $\phi - \phi' = \sum_{F \in \mathcal{F}} \lambda_F F$ for some $\lambda \in \mathbb{Z}^{\mathcal{F}}$. It is easy to see that, for each closed walk W of G^*, a facial walk $F \in \mathcal{F}$ satisfies $\beta(F, W) = 0$, so $\beta(\phi, W) = \beta(\phi', W)$ by linearity of β.

(3. $\implies$ 2.) Suppose that for each $F \in \mathcal{F}^*$, we have $\beta(\phi, F) = \beta(\phi', F)$, and, for each $1 \leq i \leq 2g$, we have $\beta(\phi, B_i^*) = \beta(\phi', B_i^*)$. Let W be any closed walk of G^*. We have $W = \sum_{F\in\mathcal{F}^*} \lambda_F F + \sum_{1\leq i\leq 2g} \mu_i B_i^*$ for some $\lambda \in \mathbb{Z}^{\mathcal{F}}, \mu \in \mathbb{Z}^{2g}$. Then by linearity of β we have $\beta(\phi, W) = \beta(\phi', W)$.

(2. $\implies$ 1.) Suppose $\beta(\phi, W) = \beta(\phi', W)$ for each closed walk W of G^*. Let $z = \phi - \phi'$. Thus $\beta(z, W) = 0$ for each closed walk W of G^*. We label the faces of G with elements of $\mathbb{Z}$ as follows. Choose an arbitrary face F_0 and label it 0. Then, consider any face F of G and a path P_F of G^* from F_0 to F. Label F with $\ell_F = \beta(z, P_F)$. Note that the label of F is independent from the choice of P_F. Indeed, for each two paths P_1, P_2 from F_0 to F, we have $P_1 - P_2$ is a closed walk, so $\beta(z, P_1 - P_2) = 0$ and thus $\beta(z, P_1) = \beta(z, P_2)$. Let us show that $z = \sum_{F\in\mathcal{F}} \ell_F \phi(F)$.

$$
\begin{aligned}
\sum_{F\in\mathcal{F}} \ell_F \phi(F) &= \sum_{e\in G} (\ell_{F_2} - \ell_{F_1})\, \phi(e) && \text{(face } F_2 \text{ is on the left of } e \text{ and } F_1 \text{ on the right)}\\
&= \sum_{e\in G} (\beta(z, P_{F_2}) - \beta(z, P_{F_1}))\, \phi(e) && \text{(definition of } \ell_F)\\
&= \sum_{e\in G} \beta(z, P_{F_2} - P_{F_1}) \phi(e) && \text{(linearity of } \beta)\\
&= \sum_{e\in G} \beta(z, e^*) \phi(e) && (P_{F_1} + e^* - P_{F_2} \text{ is a closed walk)}\\
&= \sum_{e\in G} \left(\sum_{e'\in G} z_{e'} \phi(e^*)_{e'^*} \right) \phi(e) && \text{(definition of } \beta)\\
&= \sum_{e\in G} z_e \phi(e)\\
&= z
\end{aligned}
$$

So $z \in \mathbb{F}$ and thus ϕ, ϕ' are homologous. □

As $\beta(p, d) = 0$ when p or d is an everywhere zero flow, Lemma 5 leads to the following lemma.

Lemma 6 *For any flow ϕ of G, the following properties are equivalent to each other:*

1. *The flow ϕ is 0-homologous.*
2. *For each closed walk W of G^* we have $\beta(\phi, W) = 0$.*
3. *For each $F \in \mathcal{F}^*$, we have $\beta(\phi, F) = 0$, and, for each $1 \leq i \leq 2g$, we have $\beta(\phi, B_i^*) = 0$.*

3.2 General characterization

Consider a map G on an orientable surface of genus g. The mapping of Figure 6 shows how an EDGE angle labeling of G can be mapped to an orientation of the edges with edges oriented in one direction or in two opposite directions. These edges can be defined more naturally in the primal-dual-completion of G.

The *primal-dual-completion* $\hat{G}$ is the map obtained from simultaneously embedding G and G^* such that vertices of G^* are embedded inside faces of G and vice-versa. Moreover, each edge crosses its dual edge in exactly one point in its interior, which also becomes a vertex of $\hat{G}$. Hence, $\hat{G}$ is a bipartite graph with one part consisting of *primal-vertices* and *dual-vertices* and the other part consisting of *edge-vertices* (of degree four). Each face of $\hat{G}$ is a quadrangle incident to one primal-vertex, one dual-vertex and two edge-vertices. Actually, the faces of $\hat{G}$ are in correspondence with the angles of G. This means that angle labelings of G correspond to face labelings of $\hat{G}$.

Given $\alpha : V \to \mathbb{N}$, an orientation of G is an *α-orientation* [11] if for every vertex $v \in V$ its out-degree $d^+(v)$ equals $\alpha(v)$. We call an orientation of $\hat{G}$ a mod_3-*orientation* if it is an α-orientation for a function α satisfying :

$$\alpha(v) \equiv \begin{cases} 0 \pmod 3 & \text{if } v \text{ is a primal- or dual-vertex,} \\ 1 \pmod 3 & \text{if } v \text{ is an edge-vertex.} \end{cases}$$

Note that an EDGE angle labeling of G corresponds to a mod_3-orientation of $\hat{G}$, by the mapping of Figure 13, where the three types of edges are represented. Type 0 corresponds to an edge-vertex of out-degree four. Type 1 and type 2 both correspond to an edge-vertex of out-degree 1; in type 1 (resp. type 2) the outgoing edge goes to a primal-vertex (resp. dual-vertex). In all cases we have $d^+(v) \equiv 1 \pmod 3$ if v is an edge-vertex. By Lemma 1, the labeling is also VERTEX and FACE. Thus, $d^+(v) \equiv 0 \pmod 3$ if v is a primal- or dual-vertex.

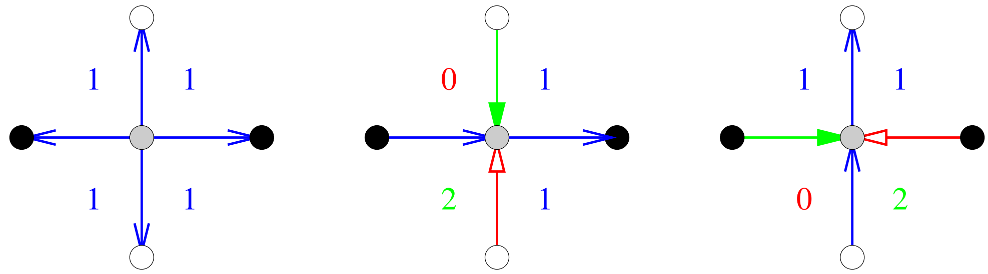


Figure 13: How to map an EDGE angle labeling to a mod_3-orientation of the primal-dual completion. Primal-vertices are black, dual-vertices are white and edge-vertices are gray. This serves as a convention for the other figures.

As mentioned earlier, de Fraysseix and Ossona de Mendez [15] give a bijection between internal 3-orientations and Schnyder woods of planar triangulations. Felsner [11] generalizes this result for planar Schnyder woods and orientations of the primal-dual completion having prescribed out-degrees. The situation is more complicated in higher genus (see Figure 5). It is not enough to prescribe out-degrees in order to characterize orientations corresponding to EDGE angle labelings. Indeed, some mod_3-orientations of $\hat{G}$ cannot be obtained from an EDGE angle labeling of G through the mapping of Figure 13. We call an orientation of $\hat{G}$ corresponding to an EDGE angle labeling of G a *Schnyder orientation*. Note that such an orientation is necessarily a mod_3-orientation. In this section we characterize which orientations of $\hat{G}$ are Schnyder orientations.

Consider an orientation of the primal-dual completion $\hat{G}$. Let $Out = \{(u, v) \in E(\hat{G}) \mid v \text{ is an edge-vertex}\}$, i.e. the set of edges of $\hat{G}$ which are going from a primal- or dual-vertex to an edge-vertex. We call these edges *out-edges*. Let $\hat{G}^*$ denote the dual of $\hat{G}$. For a flow ϕ of $\hat{G}^*$, we define $\delta(\phi) := \beta(Out, \phi)$, where Out is interpreted as the characteristic flow of the edge set Out. More intuitively, if W is a walk of $\hat{G}^*$, then:

$$\begin{aligned}\delta(W) = \quad & \#\text{out-edges crossing } W \text{ from left to right} \\ & -\#\text{out-edges crossing } W \text{ from right to left.}\end{aligned}$$

The bilinearity of β implies the linearity of δ. The following lemma gives a necessary and sufficient condition for an orientation to be a Schnyder orientation.

Lemma 7 *An orientation of $\hat{G}$ is a Schnyder orientation if and only if each closed walk W of $\hat{G}^*$ satisfies $\delta(W) \equiv 0 \pmod 3$.*

Proof. ($\Longrightarrow$) Consider an EDGE angle labeling ℓ of G and the corresponding Schnyder orientation (see Figure 13). Figure 14 illustrates how δ counts the variation of the label when going from one face of $\hat{G}$ to another face of $\hat{G}$. The represented cases correspond to a walk W of $\hat{G}^*$ consisting of just one edge. If the edge of $\hat{G}$ crossed by W is not an out-edge, then the two labels in the face are the same and $\delta(W) = 0$. If the edge crossed by W is an out-edge, then the labels differ by one. If W is going counterclockwise around a primal- or dual-vertex, then the label increases by 1 (mod 3) and $\delta(W) = 1$. If W is going clockwise around a primal- or dual-vertex then the label decreases by 1 (mod 3) and $\delta(W) = -1$. One can check that this is consistent with all the edges depicted in Figure 13. Thus for each walk W of $\hat{G}^*$ from a face F to a face F', the value of $\delta(W)$ (mod 3) is equal to $\ell(F') - \ell(F)$ (mod 3). Thus if W is a closed walk then $\delta(W) \equiv 0 \pmod 3$.

($\Longleftarrow$) Consider an orientation of $\hat{G}$ such that each closed walk W of $\hat{G}^*$ satisfies $\delta(W) \equiv 0 \pmod 3$. Pick any face F_0 of $\hat{G}$ and label it 0. Consider any face F of $\hat{G}$ and a path P of $\hat{G}^*$ from F_0 to F. Label F with the value $\delta(P) \bmod 3$. Note that the label of F is independent from the choice of P as for any two paths P_1, P_2 going from F_0 to F, we have $\delta(P_1) \equiv \delta(P_2) \pmod 3$ since $\delta(P_1 - P_2) \equiv 0 \pmod 3$ as $P_1 - P_2$ is a closed walk.

Consider an edge-vertex v of $\hat{G}$ and a walk W of $\hat{G}^*$ going clockwise around v. By assumption $\delta(W) \equiv 0 \pmod 3$ and $d(v) = 4$ so $d^+(v) \equiv 1 \pmod 3$. One can check

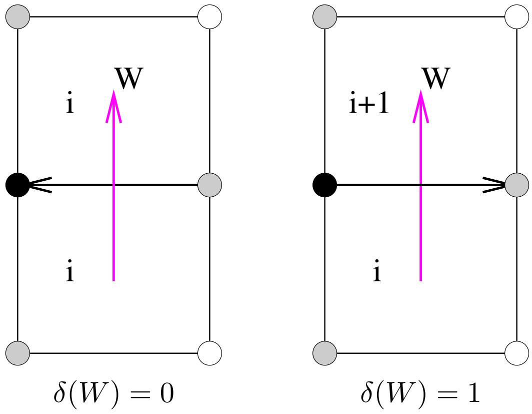


Figure 14: How δ counts the variation of the labels.

(see Figure 13) that around an edge-vertex v of out-degree four, all the labels are the same and thus v corresponds to an edge of G of type 0. One can also check that around an edge-vertex v of out-degree 1, the labels are in clockwise order, $i-1$, i, i, $i+1$ for some i in $\{0,1,2\}$ where the two faces with the same label are incident to the outgoing edge of v. Thus, v corresponds to an edge of G of type 1 or 2 depending on the outgoing edge reaching a primal- or a dual-vertex. So the obtained labeling of the faces of $\hat{G}$ corresponds to an EDGE angle labeling of G and the considered orientation is a Schnyder orientation. □

We now study properties of δ w.r.t homology in order to simplify the condition of Lemma 7. Let $\hat{\mathcal{F}}^*$ be the set of counterclockwise facial walks of $\hat{G}^*$.

Lemma 8 *In a* mod_3*-orientation of* $\hat{G}$*, each* $F \in \hat{\mathcal{F}}^*$ *satisfies* $\delta(F) \equiv 0 \pmod 3$.

Proof. If F corresponds to an edge-vertex v of $\hat{G}$, then v has degree exactly four and out-degree one or four by definition of mod_3-orientations. So there are exactly zero or three out-edges crossing F from right to left, and $\delta(F) \equiv 0 \pmod 3$.

If F corresponds to a primal- or dual-vertex v, then v has out-degree $0 \pmod 3$ by definition of mod_3-orientations. So there are exactly $0 \pmod 3$ out-edges crossing F from left to right, and $\delta(F) \equiv 0 \pmod 3$. □

Lemma 9 *In a* mod_3*-orientation of* $\hat{G}$*, if* $\{B_1, \ldots, B_{2g}\}$ *is a set of cycles of* $\hat{G}^*$ *that forms a basis for the homology, then for each closed walk* W *of* $\hat{G}^*$ *homologous to* $\mu_1 B_1 + \cdots + \mu_{2g} B_{2g}$*, we have* $\delta(W) \equiv \mu_1 \delta(B_1) + \cdots + \mu_{2g}\delta(B_{2g}) \pmod 3$.

Proof. We have $W = \sum_{F\in\hat{\mathcal{F}}^*} \lambda_F F + \sum_{1\le i\le 2g} \mu_i B_i$ for some $\lambda \in \mathbb{Z}^{\mathcal{F}}$. Then by linearity of δ and Lemma 8, the claim follows. □

Lemma 9 can be used to simplify the condition of Lemma 7 and show that if $\{B_1, \ldots, B_{2g}\}$ is a set of cycles of $\hat{G}^*$ that forms a basis for the homology, then an orientation of $\hat{G}$ is a Schnyder orientation if and only if it is a mod_3-orientation such that $\delta(B_i) \equiv 0 \pmod 3$, for all $1 \leq i \leq 2g$. Now, we define a new function γ that is used to formulate a similar characterization theorem (see Theorem 4).

Consider a (not necessarily directed) cycle C of G together with a direction of traversal. We associate to C its corresponding cycle in $\hat{G}$ denoted by $\hat{C}$. We define $\gamma(C)$ by:

$$\gamma(C) = \#\text{ edges of } \hat{G} \text{ leaving } \hat{C} \text{ on its right} - \#\text{ edges of } \hat{G} \text{ leaving } \hat{C} \text{ on its left}$$

Since it considers cycles of $\hat{G}$ instead of walks of $\hat{G}^*$, it is easier to deal with parameter γ rather than parameter δ. However γ does not enjoy the same property w.r.t. homology as δ. For homology we have to consider walks as flows, but two walks going several times through a given vertex may have the same characteristic flow but different γ. This explains why δ is defined first. Now we adapt the results for γ.

The value of γ is related to δ by the next lemmas. Let C be a cycle of G with a direction of traversal. Let $W_L(C)$ be the closed walk of $\hat{G}^*$ just on the left of C and going in the same direction as C (i.e. $W_L(C)$ is composed of the dual edges of the edges of $\hat{G}$ incident to the left of $\hat{C}$, see Figure 15). Note that since the faces of $\hat{G}^*$ have exactly one incident vertex that is a primal-vertex, walk $W_L(C)$ is in fact a cycle of $\hat{G}^*$. Similarly, let $W_R(C)$ be the cycle of $\hat{G}^*$ just on the right of C.

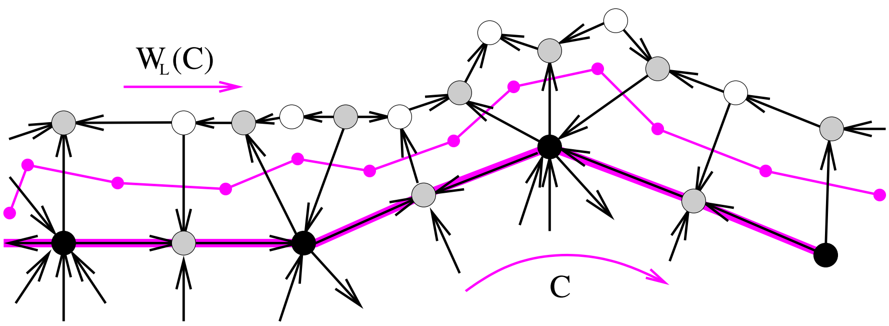


Figure 15: A cycle C of G, and the corresponding cycle $W_L(C)$ of $\hat{G}^*$.

Lemma 10 *Consider an orientation of $\hat{G}$ and a cycle C of G, then $\gamma(C) = \delta(W_L(C)) + \delta(W_R(C))$.*

Proof. We consider the different cases that can occur. An edge that is entering a primal-vertex of $\hat{C}$, is not counting in either $\gamma(C), \delta(W_L(C)), \delta(W_R(C))$. An edge that is leaving a primal-vertex of $\hat{C}$ from its right side (resp. left side) is counting $+1$ (resp. -1) for $\gamma(C)$ and $\delta(W_R(C))$ (resp. $\delta(W_L(C))$).

For edges incident to edge-vertices of $\hat{C}$ both sides have to be considered at the same time. Let v be an edge-vertex of $\hat{C}$. Vertex v is of degree four so it has exactly two edges incident to $\hat{C}$ and not on $\hat{C}$. One of these edges, e_L, is on the left side of $\hat{C}$ and dual to an edge of $W_L(C)$. The other edge, e_R, is on the right side of $\hat{C}$ and dual to an edge of $W_R(C)$. If e_L and e_R are both incoming edges for v, then e_R (resp. e_L) is counting -1 (resp. $+1$) for $\delta(W_R(C))$ (resp. $\delta(W_L(C))$) and not counting for $\gamma(C)$. If e_L and e_R are both outgoing edges for v, then e_R and e_L are not counting for both $\delta(W_R(C))$, $\delta(W_L(C))$ and sums to zero for $\gamma(C)$. If e_L is incoming and e_R is outgoing for v, then e_R (resp. e_L) is counting 0 (resp. $+1$) for $\delta(W_R(C))$ (resp. $\delta(W_L(C))$), and counting $+1$ (resp. 0) for $\gamma(C)$. The last case, e_L is outgoing and e_R is incoming, is symmetric and one can see that in the four cases we have that e_L and e_R count the same for $\gamma(C)$ and $\delta(W_L(C)) + \delta(W_R(C))$. We conclude $\gamma(C) = \delta(W_L(C)) + \delta(W_R(C))$. ☐

Lemma 11 *In a* mod$_3$*-orientation of G, a cycle C of G satisfies*

$$\delta(W_L(C)) \equiv 0 \pmod 3 \quad \textit{and} \quad \delta(W_R(C)) \equiv 0 \pmod 3 \iff \gamma(C) \equiv 0 \pmod 3$$

Proof. ($\Longrightarrow$) Clear by Lemma 10.

($\Longleftarrow$) Suppose that $\gamma(C) \equiv 0 \pmod 3$. Let x_L (resp. y_L) be the number of edges of $\hat{G}$ that are dual to edges of $W_L(C)$, that are outgoing for a primal-vertex of $\hat{C}$ (resp. incoming for an edge-vertex of $\hat{C}$). Similarly, let x_R (resp. y_R) be the number of edges of $\hat{G}$ that are dual to edges of $W_R(C)$, that are outgoing for a primal-vertex of $\hat{C}$ (resp. incoming for an edge-vertex of $\hat{C}$). So $\delta(W_L(C)) = y_L - x_L$ and $\delta(W_R(C)) = x_R - y_R$. So by Lemma 10,

$$\begin{aligned} \gamma(C) &\equiv 0 \pmod 3 \\ \delta(W_L(C)) + \delta(W_R(C)) &\equiv 0 \pmod 3 \\ (y_L + x_R) - (x_L + y_R) &\equiv 0 \pmod 3 \end{aligned}$$

Let k be the number of vertices of C. So $\hat{C}$ has k primal-vertices, k edge-vertices and $2k$ edges. Edge-vertices have out-degree $1 \pmod 3$ so their total number of outgoing edges on $\hat{C}$ is $-k + (y_L + y_R) \pmod 3$. Primal-vertices have out-degree $0 \pmod 3$ so their total number of outgoing edges on $\hat{C}$ is $-(x_L + x_R) \pmod 3$. So in total

$$\begin{aligned} -k + (y_L + y_R) - (x_L + x_R) &\equiv 2k \pmod 3 \\ (y_L + y_R) - (x_L + x_R) &\equiv 0 \pmod 3 \end{aligned}$$

By combining this with plus (resp. minus) $(y_L+x_R)-(x_L+y_R) \equiv 0 \pmod 3$, one obtains that $2\delta(W_L(C)) = 2(y_L - x_L) \equiv 0 \pmod 3$ (resp. $2\delta(W_R(C)) = 2(x_R - y_R) \equiv 0$ (mod 3)). Since $\delta(W_L(C))$ and $\delta(W_R(C))$ are integers we obtain $\delta(W_L(C)) \equiv 0 \pmod 3$ and $\delta(W_R(C)) \equiv 0 \pmod 3$. ☐

Finally we have the following characterization theorem concerning Schnyder orientations:

Theorem 4 *Consider a map G on an orientable surface of genus g. Let $\{B_1, \ldots, B_{2g}\}$ be a set of cycles of G that forms a basis for the homology. An orientation of $\hat{G}$ is a Schnyder orientation if and only if it is a mod_3-orientation such that $\gamma(B_i) \equiv 0 \pmod 3$, for all $1 \le i \le 2g$.*

Proof. ($\Longrightarrow$) Consider an EDGE angle labeling ℓ of G and the corresponding Schnyder orientation (see Figure 13). Type 0 edges correspond to edge-vertices of out-degree four, while type 1 and 2 edges correspond to edge-vertices of out-degree 1. Thus $d^+(v) \equiv 1 \pmod 3$ if v is an edge-vertex. By Lemma 1, the labeling is VERTEX and FACE. Thus $d^+(v) \equiv 0 \pmod 3$ if v is a primal- or dual-vertex. So the orientation is a mod_3-orientation. By Lemma 7, we have $\delta(W) \equiv 0 \pmod 3$ for each closed walk W of $\hat{G}^*$. So we have that $\delta(W_L(B_1)), \ldots, \delta(W_L(B_{2g}))$, $\delta(W_R(B_1)), \ldots, \delta(W_R(B_{2g}))$ are all congruent to $0 \pmod 3$. Thus, by Lemma 11, we have $\gamma(B_i) \equiv 0 \pmod 3$, for all $1 \le i \le 2g$.

($\Longleftarrow$) Consider a mod_3-orientation of G such that $\gamma(B_i) \equiv 0 \pmod 3$, for all $1 \le i \le 2g$. By Lemma 11, we have $\delta(W_L(B_i)) \equiv 0 \pmod 3$ for all $1 \le i \le 2g$. Moreover $\{W_L(B_1), \ldots, W_L(B_{2g})\}$ forms a basis for the homology. So by Lemma 9, $\delta(W) \equiv 0 \pmod 3$ for each closed walk W of $\hat{G}^*$. So the orientation is a Schnyder orientation by Lemma 7. □

The condition of Theorem 4 is easy to check: choose $2g$ cycles that form a basis for the homology and check whether γ is congruent to 0 mod 3 for each of them.

When restricted to triangulations and to edges of type 1 only, the definition of γ can be simplified. Consider a triangulation G on an orientable surface of genus g and an orientation of the edges of G. Figure 16 shows how to transform the orientation of G into an orientation of $\hat{G}$. Note that all the edge-vertices have out-degree exactly 1. Furthermore, all the dual-vertices only have outgoing edges and since we are considering triangulations they have out-degree exactly three.

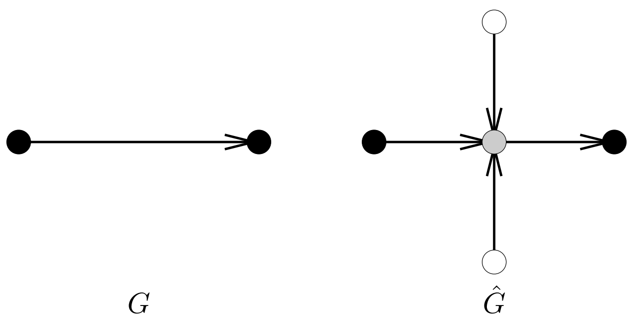


Figure 16: How to transform an orientation of a triangulation G into an orientation of $\hat{G}$.

Then the definition of γ can be simplified by the following:

$$\gamma(C) = \#\text{ edges of } G \text{ leaving } C \text{ on its right} - \#\text{ edges of } G \text{ leaving } C \text{ on its left}$$

Note that comparing to the general definition of γ, only the symbols $\hat{}$ have been removed.

The orientation of the toroidal triangulation on the left of Figure 5 is an example of a 3-orientation of a toroidal triangulation where some non-contractible cycles have value γ not congruent to 0 mod 3. The value of γ for the three loops is $2, 0$ and -2. This explains why this orientation does not correspond to a Schnyder wood. On the contrary, on the right of the figure, the three loops have γ equal to 0 and we have a Schnyder wood.

4 Structure of Schnyder orientations

We now investigate how two Schnyder orientations can differ, that is for which set of edges their orientations differ. We will see that these sets have homological properties that are defined at the beginning of the section. We will then see that given a Schnyder orientation, the set of orientations that differ on a 0-homologous subgraph are also Schnyder orientations, and we will see in the second subsection how those form a distributive lattice, as in the planar case.

4.1 Transformations between Schnyder orientations

We investigate the structure of the set of Schnyder orientations of a given graph. For that purpose we need some definitions that are given on a general map G and then applied to $\hat{G}$.

Consider a map G on an orientable surface of genus g. Given two orientations D and D' of G, let $D \setminus D'$ denote the subgraph of D induced by the edges that are not oriented as in D'.

An oriented subgraph T of G is *partitionable* if its edge set can be partitioned into three sets T_0, T_1, T_2 such that all the T_i are pairwise homologous, i.e. $T_i - T_j \in \mathbb{F}$ for $i, j \in \{0, 1, 2\}$. An oriented subgraph T of G is called a *topological Tutte-orientation* if $\beta(T, W) \equiv 0 \pmod 3$ for every closed walk W in G^* (more intuitively, the number of edges crossing W from left to right minus the number of those crossing W from right to left is divisible by three).

The name "topological Tutte-orientation" comes from the fact that an oriented graph T is called a *Tutte-orientation* if the difference of out-degree and in-degree is divisible by three, i.e. $d^+(v) - d^-(v) \equiv 0 \pmod 3$, for every vertex v. So a topological Tutte-orientation is a Tutte orientation, since the latter requires the condition of the topological Tutte orientation only for the walks W of G^* going around a vertex v of G.

The notions of partitionable and topological Tutte-orientation are equivalent:

Lemma 12 *An oriented subgraph of G is partitionable if and only if it is a topological Tutte-orientation.*

Proof. ($\Longrightarrow$) If T is partitionable, then by definition it is the disjoint union of three homologous edge sets T_0, T_1, and T_2. Hence by Lemma 5, $\beta(T_0, W) = \beta(T_1, W) = \beta(T_2, W)$ for each closed walk W of G^*. By linearity of β this implies that $\beta(T, W) \equiv 0 \pmod 3$ for any closed walk W of G^*. So T is a topological Tutte-orientation.

($\Longleftarrow$) Let T be a topological Tutte-orientation of G, i.e. $\beta(T, W) \equiv 0 \pmod 3$ for each closed walk W of G^*. In the following, *T-faces* are the faces of T considered as an embedded graph. Note that T-faces are not necessarily disks. Let us introduce a $\{0, 1, 2\}$-labeling of the T-faces. Label an arbitrary T-face F_0 by 0. For each T-face F, find a path P of G^* from F_0 to F. Label F with $\beta(T, P) \pmod 3$. Note that the label of F is independent from the choice of P by our assumption on closed walks. For $0 \leq i \leq 2$, let T_i be the set of edges of T with two incident T-faces labeled $i-1$ and $i+1$. Note that an edge of T_i has label $i-1$ on its left and label $i+1$ on its right. The sets T_i form a partition of the edges of T. Let $\mathcal{F}_i$ be the counterclockwise facial walks of G that are in a T-face labeled i. We have $\phi(T_{i+1}) - \phi(T_{i-1}) = \sum_{F \in \mathcal{F}_i} \phi(F)$, so the T_i are homologous. □

Let us refine the notion of partitionable. Denote by $\mathcal{E}$ the set of *oriented Eulerian subgraphs* of G (i.e. the oriented subgraphs of G where each vertex has the same in- and out-degree). Consider a partitionable oriented subgraph T of G, with homologous edge sets T_0, T_1, T_2. We say that T is *Eulerian-partitionable* if $T_i \in \mathcal{E}$ for all $0 \leq i \leq 2$. Note that if T is Eulerian-partitionable then it is Eulerian. Note that an oriented subgraph T of G that is 0-homologous is also Eulerian and thus Eulerian-partitionable (with the partition $T, \emptyset, \emptyset$).

We now investigate the structure of Schnyder orientations. For that purpose, consider a map G on an orientable surface of genus g and apply the above definitions and results to orientations of $\hat{G}$.

Let D, D' be two orientations of $\hat{G}$ such that D is a Schnyder orientation and $T = D \setminus D'$. Note that summing $\phi(D)$, and $-\phi(D')$ the edges of T are counted twice, while the other edges cancel out. Thus $2\phi(T) = \phi(D) - \phi(D')$, and T is 0-homologous if and only if D, D' are homologous. Let $Out = \{(u, v) \in E(D) \mid v \text{ is an edge-vertex}\}$. Similarly, let $Out' = \{(u, v) \in E(D') \mid v \text{ is an edge-vertex}\}$. Note that an edge of T is either in Out or in Out' (but oriented in the opposite direction), so $\phi(T) = \phi(Out) - \phi(Out')$. By Lemma 7, for any closed walk W of $\hat{G}^*$, $\beta(Out, W) \equiv 0 \pmod 3$. The three following lemmas give necessary and sufficient conditions on T for D' being a Schnyder orientation.

Lemma 13 *D' is a Schnyder orientation if and only if T is partitionable.*

Proof. Let D' is a Schnyder orientation. By Lemma 7, this is equivalent to the fact that for each closed walk W of $\hat{G}^*$, we have $\beta(Out', W) \equiv 0 \pmod 3$. Since $\beta(Out, W) \equiv 0 \pmod 3$, this is equivalent to the fact that for any closed walk W of $\hat{G}^*$, we have $\beta(T, W) \equiv 0 \pmod 3$. Finally, by Lemma 12 this is equivalent to T being partitionable. □

Lemma 14 *D' is a Schnyder orientation having the same out-degrees as D if and only if T is Eulerian-partitionable.*

Proof. ($\Longrightarrow$) Suppose D' is a Schnyder orientation having the same out-degrees as D. Lemma 13 implies that T is partitionable into homologous T_0, T_1, T_2. By Lemma 5, for each closed walk W of $\hat{G}^*$, we have $\beta(T_0, W) = \beta(T_1, W) = \beta(T_2, W)$. Since D, D' have the same out-degrees, we have that T is Eulerian. Consider a vertex v of $\hat{G}$ and a walk W_v of $\hat{G}^*$ going counterclockwise around v. For each oriented subgraph H of $\hat{G}^*$, we have $d^+_H(v) - d^-_H(v) = \beta(H, W_v)$, where $d^+_H(v)$ and $d^-_H(v)$ denote the out-degree and in-degree of v restricted to H, respectively. Since T is Eulerian, we have $\beta(T, W_v) = 0$. Since $\beta(T_0, W_v) = \beta(T_1, W_v) = \beta(T_2, W_v)$ and $\sum \beta(T_i, W_v) = \beta(T, W_v) = 0$, we obtain that $\beta(T_0, W_v) = \beta(T_1, W_v) = \beta(T_2, W_v) = 0$. So each T_i is Eulerian.

($\Longleftarrow$) Suppose T is Eulerian-partitionable. Then Lemma 13 implies that D' is a Schnyder orientation. Since T is Eulerian, the two orientations D, D' have the same out-degrees. $\square$

Consider $\{B_1, \ldots, B_{2g}\}$ a set of cycles of G that forms a basis for the homology. For $\Gamma \in \mathbb{Z}^{2g}$, an orientation of $\hat{G}$ is of *type* Γ if $\gamma(B_i) = \Gamma_i$ for all $1 \leq i \leq 2g$.

Lemma 15 *D' is a Schnyder orientation having the same out-degrees and the same type as D (for the considered basis) if and only if T is 0-homologous (i.e. D, D' are homologous).*

Proof. ($\Longrightarrow$) Suppose D' is a Schnyder orientation having the same out-degrees and the same type as D. Then, Lemma 14 implies that T is Eulerian-partitionable and thus Eulerian. As already observed in the proof of Lemma 14, this implies that for each $F \in \hat{\mathcal{F}}^*$, we have $\beta(T, F) = 0$. Moreover, for $1 \leq i \leq 2g$, consider the region R_i between $W_L(B_i)$ and $W_R(B_i)$ containing B_i. Since T is Eulerian, it is going in and out of R_i the same number of times. So $\beta(T, W_L(B_i) - W_R(B_i)) = 0$. Since D, D' have the same type, we have $\gamma_D(B_i) = \gamma_{D'}(B_i)$. So by Lemma 10, $\delta_D(W_L(B_i)) + \delta_D(W_R(B_i)) = \delta_{D'}(W_L(B_i)) + \delta_{D'}(W_R(B_i))$. Thus $\beta(T, W_L(B_i) + W_R(B_i)) = \beta(Out - Out', W_L(B_i) + W_R(B_i)) = \delta_D(W_L(B_i)) + \delta_D(W_R(B_i)) - \delta_{D'}(W_L(B_i)) - \delta_{D'}(W_R(B_i)) = 0$. By combining this with the previous equality, we obtain $\beta(T, W_L(B_i)) = \beta(T, W_R(B_i)) = 0$ for all $1 \leq i \leq 2g$. As $\{W_L(B_1), \ldots, W_L(B_{2g})\}$ forms a basis for the homology, we thus have by Lemma 6 that T is 0-homologous.

($\Longleftarrow$) Suppose that T is 0-homologous. Then T is in particular Eulerian-parti-tionable (with the partition $T, \emptyset, \emptyset$). So Lemma 14 implies that D' is a Schnyder orientation with the same out-degrees as D. Since T is 0-homologous, by Lemma 6, for all $1 \leq i \leq 2g$, we have $\beta(T, W_L(B_i)) = \beta(T, W_R(B_i)) = 0$. Thus $\delta_D(W_L(B_i)) = \beta(Out, W_L(B_i)) = \beta(Out', W_L(B_i)) = \delta_{D'}(W_L(B_i))$ and $\delta_D(W_R(B_i)) = \beta(Out, W_R(B_i)) = \beta(Out', W_R(B_i)) = \delta_{D'}(W_R(B_i))$. So by Lemma 10, $\gamma_D(B_i) = \delta_D(W_L(B_i)) + \delta_D(W_R(B_i)) = \delta_{D'}(W_L(B_i)) + \delta_{D'}(W_R(B_i)) = \gamma_{D'}(B_i)$. So D, D' have the same type. $\square$

Lemma 15 implies that when you consider Schnyder orientations having the same out-degrees the property that they have the same type does not depend on the choice of the basis since being homologous does not depend on the basis. So we have the following:

Lemma 16 *If two Schnyder orientations have the same out-degrees and the same type (for the considered basis), then they have the same type for any basis.*

Lemma 13, 14 and 15 are summarized in the following theorem (where by Lemma 16 we do not have to assume a particular choice of a basis for the third item):

Theorem 5 *Let G be a map on an orientable surface and D, D' orientations of $\hat{G}$ such that D is a Schnyder orientation and $T = D \setminus D'$. We have the following:*

- *D' is a Schnyder orientation if and only if T is partitionable.*
- *D' is a Schnyder orientation having the same out-degrees as D if and only if T is Eulerian-partitionable.*
- *D' is a Schnyder orientation having the same out-degrees and the same type as D if and only if T is 0-homologous (i.e. D, D' are homologous).*

We show in the next section that the set of Schnyder orientations that are homologous (see third item of Theorem 5) carries a structure of distributive lattice.

4.2 The distributive lattice of homologous orientations

Consider a partial order $\leq$ on a set S. Given two elements x, y of S, let $m(x, y)$ (resp. $M(x, y)$) be the set of elements z of S such that $z \leq x$ and $z \leq y$ (resp. $z \geq x$ and $z \geq y$). If $m(x, y)$ (resp. $M(x, y)$) is not empty and admits a unique maximal (resp. minimal) element, we say that x and y admit a *meet* (resp. a *join*), noted $x \vee y$ (resp. $x \wedge y$). Now, $(S, \leq)$ is a *lattice* if each pair of elements of S admits a meet and a join. Thus, in particular a (finite) lattice has a unique minimal (resp. maximal) element. A lattice is *distributive* if the two operators $\vee$ and $\wedge$ are distributive over each other.

For the sake of generality, in this subsection we consider that maps may have loops or multiple edges. Consider a map G on an orientable surface and a given orientation D_0 of G. Let $O(G, D_0)$ be the set of all the orientations of G that are homologous to D_0. In this section we prove that $O(G, D_0)$ forms a distributive lattice. We show some additional interesting properties that are used in a recent paper by Despré, the first author, and the third author [8]. This generalizes results for the plane obtained by Ossona de Mendez [24] and Felsner [11]. The distributive lattice structure can also be derived from a result of Propp [26] interpreted on the dual map, see the discussion below Theorem 6.

In order to define an order on $O(G, D_0)$, fix an arbitrary face f_0 of G and let F_0 be its counterclockwise facial walk. Let $\mathcal{F}' = \mathcal{F} \setminus \{F_0\}$ (where $\mathcal{F}$ is the set of counterclockwise facial walks of G as defined earlier). Note that $\phi(F_0) = -\sum_{F\in\mathcal{F}'}\phi(F)$. Since the characteristic flows of $\mathcal{F}'$ are linearly independent, any oriented subgraph of G has at most one representation as a combination of characteristic flows of $\mathcal{F}'$. Moreover the 0-homologous oriented subgraphs of G are precisely the oriented subgraphs that have such a representation. We say that a 0-homologous oriented subgraph T of G is *counterclockwise* (resp. *clockwise*) if its characteristic flow can be written as a combination with positive (resp. negative) coefficients of characteristic flows of $\mathcal{F}'$, i.e. $\phi(T) = \sum_{F\in\mathcal{F}'}\lambda_F\phi(F)$, with $\lambda \in \mathbb{N}^{|\mathcal{F}'|}$ (resp. $-\lambda \in \mathbb{N}^{|\mathcal{F}'|}$). Given two orientations D, D', of G we set $D \leq_{f_0} D'$ if and only if $D \setminus D'$ is counterclockwise. Then we have the following theorem.

Theorem 6 ([26]) *Let G be a map on an orientable surface given with a particular orientation D_0 and a particular face f_0. Let $O(G, D_0)$ the set of all the orientations of G that are homologous to D_0. We have $(O(G, D_0), \leq_{f_0})$ is a distributive lattice.*

We attribute Theorem 6 to Propp even if it is not presented in this form in [26]. Here we do not introduce Propp's formalism, but provide a new proof of Theorem 6 (as a consequence of the forthcoming Proposition 3). This allows us to introduce notions used later in the study of this lattice. It is notable that the study of this lattice found applications in [8], where the authors found a bijection between toroidal triangulations and unicellular toroidal maps.

To prove Theorem 6, we need to define the elementary flips that generates the lattice. We start by reducing the graph G. We call an edge of G *rigid with respect to* $O(G, D_0)$ if it has the same orientation in all elements of $O(G, D_0)$. Rigid edges do not play a role for the structure of $O(G, D_0)$. We delete them from G and call the obtained embedded graph $\widetilde{G}$. This graph is embedded but it is not necessarily a map, as some faces may not be homeomorphic to open disks. Note that if all the edges are rigid, i.e. $|O(G, D_0)| = 1$, then $\widetilde{G}$ has no edges.

Lemma 17 *Given an edge e of G, the following are equivalent:*

1. *e is non-rigid*
2. *e is contained in a 0-homologous oriented subgraph of D_0*
3. *e is contained in a 0-homologous oriented subgraph of D, for each $D \in O(G, D_0)$*

Proof. ($1 \Longrightarrow 3$) Let $D \in O(G, D_0)$. If e is non-rigid, then it has a different orientation in two elements D', D'' of $O(G, D_0)$. Then we can assume by symmetry that e has a different orientation in D and D' (otherwise in D and D'' by symmetry). Since D, D' are homologous to D_0, they are also homologous to each other. So $T = D \setminus D'$ is a 0-homologous oriented subgraph of D that contains e.

($3 \Longrightarrow 2$) Trivial since $D_0 \in O(G, D_0)$

($2 \Longrightarrow 1$) If an edge e is contained in a 0-homologous oriented subgraph T of D_0, then let D be the element of $O(G, D_0)$ such that $T = D_0 \setminus D$. Clearly e is oriented differently in D and D_0, thus it is non-rigid. □

Note that by Lemma 17, one can build $\widetilde{G}$ by keeping only the edges that are contained in a 0-homologous oriented subgraph of D_0. Denote by $\widetilde{\mathcal{F}}$ the set of oriented subgraphs of $\widetilde{G}$ corresponding to the boundaries of faces of $\widetilde{G}$ considered counterclockwise. Note that for each $\widetilde{F} \in \widetilde{\mathcal{F}}$, its characteristic flow can be written $\phi(\widetilde{F}) = \sum_{F \in X_{\widetilde{F}}} \phi(F)$, where $X_{\widetilde{F}}$ corresponds to the faces of $\mathcal{F}$ that lie inside $\widetilde{F}$. Let $\widetilde{f_0}$ be the face of $\widetilde{G}$ containing f_0 and $\widetilde{F_0}$ be the element of $\widetilde{\mathcal{F}}$ corresponding to the boundary of $\widetilde{f_0}$. Let $\widetilde{\mathcal{F}}' = \widetilde{\mathcal{F}} \setminus \{\widetilde{F_0}\}$. Note that for $\widetilde{F} \in \widetilde{\mathcal{F}}'$ we have $\phi(\widetilde{F}) = \sum_{F \in X_{\widetilde{F}}} \phi(F)$, where $X_{\widetilde{F}} \subseteq \mathcal{F}'$ (i.e. $f_0 \notin X_{\widetilde{F}}$). We thus have that every element of $\widetilde{\mathcal{F}}'$ is counterclockwise.

We prove in Proposition 3 that the elements of $\widetilde{\mathcal{F}}'$ are precisely the elementary flips which suffice to generate the entire distributive lattice $(O(G, D_0), \leq_{f_0})$. Towards this aim we first prove two technical lemmas concerning $\widetilde{\mathcal{F}}'$:

Lemma 18 *Let $D \in O(G, D_0)$ and T be a non-empty 0-homologous oriented subgraph of D. Then there exist a partition of T into 0-homologous oriented subgraphs $T_1, \ldots, T_k$ such that $\phi(T) = \sum_{1 \leq i \leq k} \phi(T_i)$, and, for $1 \leq i \leq k$, there exists $\widetilde{X_i} \subseteq \widetilde{\mathcal{F}}'$ and $\epsilon_i \in \{-1, 1\}$ such that $\phi(T_i) = \epsilon_i \sum_{\widetilde{F} \in \widetilde{X_i}} \phi(\widetilde{F})$.*

Proof. Since T is 0-homologous, we have $\phi(T) = \sum_{F \in \mathcal{F}'} \lambda_F \phi(F)$, for $\lambda \in \mathbb{Z}^{|\mathcal{F}'|}$. Let $\lambda_{f_0} = 0$. Thus we have $\phi(T) = \sum_{F \in \mathcal{F}} \lambda_F \phi(F)$. Let $\lambda_{\min} = \min_{F \in \mathcal{F}} \lambda_F$ and $\lambda_{\max} = \max_{F \in \mathcal{F}} \lambda_F$. We may have $\lambda_{\min} = 0$ or $\lambda_{\max} = 0$ but not both since T is non-empty. For $1 \leq i \leq \lambda_{\max}$, let $X_i = \{F \in \mathcal{F}' \,|\, \lambda_F \geq i\}$ and $\epsilon_i = 1$. Let $X_0 = \emptyset$ and $\epsilon_0 = 1$. For $\lambda_{\min} \leq i \leq -1$, let $X_i = \{F \in \mathcal{F}' \,|\, \lambda_F \leq i\}$ and $\epsilon_i = -1$. For $\lambda_{\min} \leq i \leq \lambda_{\max}$, let T_i be the 0-homologous oriented subgraph such that $\phi(T_i) = \epsilon_i \sum_{F \in X_i} \phi(F)$. Note that we have $\phi(T) = \sum_{\lambda_{\min} \leq i \leq \lambda_{\max}} \phi(T_i)$.

Since T is an oriented subgraph, we have $\phi(T) \in \{-1, 0, 1\}^{|E(G)|}$. Thus for each edge of G, incident to faces F_1 and F_2, we have $(\lambda_{F_1} - \lambda_{F_2}) \in \{-1, 0, 1\}$. So, for $1 \leq i \leq \lambda_{\max}$, the oriented graph T_i is the border between the faces with λ value equal to i and $i - 1$. Symmetrically, for $\lambda_{\min} \leq i \leq -1$, the oriented graph T_i is the border between the faces with λ value equal to i and $i + 1$. So every edge of T is contained in exactly one T_i, and all the T_i are edge disjoint and are oriented subgraphs of T.

Let $\widetilde{X_i} = \{\widetilde{F} \in \widetilde{\mathcal{F}}' \,|\, \phi(\widetilde{F}) = \sum_{F \in X'} \phi(F) \text{ for some } X' \subseteq X_i\}$. Since T_i is 0-homologous, the edges of T_i can be reversed in D to obtain another element of $O(G, D_0)$. Thus there is no rigid edge in T_i. Thus $\phi(T_i) = \epsilon_i \sum_{F \in X_i} \phi(F) = \epsilon_i \sum_{\widetilde{F} \in \widetilde{X_i}} \phi(\widetilde{F})$. □

Lemma 19 *Let $D \in O(G, D_0)$ and T be a non-empty 0-homologous oriented subgraph of D such that there exists $\widetilde{X} \subseteq \widetilde{\mathcal{F}}'$ and $\epsilon \in \{-1, 1\}$ satisfying $\phi(T) = \epsilon \sum_{\widetilde{F} \in \widetilde{X}} \phi(\widetilde{F})$. Then there exists $\widetilde{F} \in \widetilde{X}$ such that $\epsilon\, \phi(\widetilde{F})$ corresponds to an oriented subgraph of T.*

Proof. The proof is done by induction on $|\widetilde{X}|$. Assume that $\epsilon = 1$ (the case $\epsilon = -1$ is proved similarly).

If $|\widetilde{X}| = 1$, then the conclusion is clear since $\phi(T) = \sum_{\widetilde{F}\in\widetilde{X}} \phi(\widetilde{F})$. We now assume that $|\widetilde{X}| > 1$. Towards a contradiction, suppose that for each $\widetilde{F} \in \widetilde{X}$ we do not have the conclusion, i.e $\phi(\widetilde{F})_e \neq \phi(T)_e$ for some $e \in \widetilde{F}$. Let $\widetilde{F_1} \in \widetilde{X}$ and $e \in \widetilde{F_1}$ such that $\phi(\widetilde{F_1})_e \neq \phi(T)_e$. Since $\widetilde{F_1}$ is counterclockwise, we have $\widetilde{F_1}$ on the left of e. Let $\widetilde{F_2} \in \widetilde{\mathcal{F}}$ that is on the right of e. Note that $\phi(\widetilde{F_1})_e = -\phi(\widetilde{F_2})_e$ and for any other face $\widetilde{F} \in \widetilde{\mathcal{F}}$, we have $\phi(\widetilde{F})_e = 0$. Since $\phi(T) = \sum_{\widetilde{F}\in\widetilde{X}} \phi(\widetilde{F})$, we have $\widetilde{F_2} \in \widetilde{X}$ and $\phi(T)_e = 0$. By possibly swapping the role of $\widetilde{F_1}$ and $\widetilde{F_2}$, we can assume that $\phi(D)_e = \phi(\widetilde{F_1})_e$, i.e., e is oriented the same way in $\widetilde{F_1}$ and D. Since e is not rigid, there exists an orientation D' in $O(G, D_0)$ such that $\phi(D)_e = -\phi(D')_e$.

Let T' be the non-empty 0-homologous oriented subgraph of D such that $T' = D \setminus D'$. Lemma 18 implies that T' admits a partition into 0-homologous oriented subgraphs $T_1, \ldots, T_k$ such that $\phi(T') = \sum_{1\le i\le k} \phi(T_i)$, and, for $1 \le i \le k$, there exists $\widetilde{X_i} \subseteq \widetilde{\mathcal{F}}'$ and $\epsilon_i \in \{-1, 1\}$ such that $\phi(T_i) = \epsilon_i \sum_{\widetilde{F}\in\widetilde{X_i}} \phi(\widetilde{F})$. Since T' is the disjoint union of $T_1, \ldots, T_k$, there exists $1 \le i \le k$, such that e is an edge of T_i. Assume by symmetry that e is an edge of T_1. Since $\phi(T_1)_e = \phi(D)_e = \phi(\widetilde{F_1})_e$, we have $\epsilon_1 = 1$, $\widetilde{F_1} \in \widetilde{X_1}$ and $\widetilde{F_2} \notin \widetilde{X_1}$.

Let $\widetilde{Y} = \widetilde{X} \cap \widetilde{X_1}$. Thus $\widetilde{F_1} \in \widetilde{Y}$ and $\widetilde{F_2} \notin \widetilde{Y}$. So $|\widetilde{Y}| < |\widetilde{X}|$. Let $T_{\widetilde{Y}}$ be the oriented subgraph of G such that $T_{\widetilde{Y}} = \sum_{\widetilde{F}\in\widetilde{Y}} \phi(\widetilde{F})$. Note that the edges of T (resp. T_1) are those incident to exactly one face of $\widetilde{X}$ (resp. $\widetilde{X_1}$). Similarly every edge of $T_{\widetilde{Y}}$ is incident to exactly one face of $\widetilde{Y} = \widetilde{X} \cap \widetilde{X_1}$, i.e. it has one incident face in $\widetilde{Y} = \widetilde{X} \cap \widetilde{X_1}$ and the other incident face not in $\widetilde{X}$ or not in $\widetilde{X_1}$. In the first case this edge is in T, otherwise it is in T_1. So every edge of $T_{\widetilde{Y}}$ is an edge of $T \cup T_1$. Hence $T_{\widetilde{Y}}$ is an oriented subgraph of D. So we can apply the induction hypothesis on $T_{\widetilde{Y}}$. This implies that there exists $\widetilde{F} \in \widetilde{Y}$ such that $\widetilde{F}$ is an oriented subgraph of D. Since $\widetilde{Y} \subseteq \widetilde{X}$, this is a contradiction to our assumption. □

We need the following characterization of distributive lattice from [12]:

Theorem 7 ([12]) *An oriented graph $\mathcal{H} = (V, E)$ is the Hasse diagram of a distributive lattice if and only if it is connected, acyclic, and admits an edge-labeling c of the edges such that:*

- *if $(u, v), (u, w) \in E$, with $v \neq w$, then*
  - *(U1) $c(u, v) \neq c(u, w)$ and*
  - *(U2) there is $z \in V$ such that $(v, z), (w, z) \in E$, $c(u, v) = c(w, z)$, and $c(u, w) = c(v, z)$.*
- *if $(v, z), (w, z) \in E$, with $v \neq w$, then*
  - *(L1) $c(v, z) \neq c(w, z)$ and*
  - *(L2) there is $u \in V$ such that $(u, v), (u, w) \in E$, $c(u, v) = c(w, z)$, and $c(u, w) = c(v, z)$.*

We define the directed graph $\mathcal{H}$ with vertex set $O(G, D_0)$. There is an oriented edge from D_1 to D_2 in $\mathcal{H}$ (with $D_1 \leq_{f_0} D_2$) if and only if $D_1 \setminus D_2 \in \widetilde{\mathcal{F}}'$. We define the label of that edge as $c(D_1, D_2) = D_1 \setminus D_2$. We show that $\mathcal{H}$ fulfills all the conditions of Theorem 7, and thus obtain the following:

Proposition 3 *$\mathcal{H}$ is the Hasse diagram of a distributive lattice.*

Proof. The characteristic flows of elements of $\widetilde{\mathcal{F}}'$ form an independent set, hence the digraph $\mathcal{H}$ is acyclic. By definition all outgoing and all incoming edges of a vertex of $\mathcal{H}$ have different labels, i.e. the labeling c satisfies (U1) and (L1). If (D_u, D_v) and (D_u, D_w) belong to $\mathcal{H}$, then $T_v = D_u \setminus D_v$ and $T_w = D_u \setminus D_w$ are both elements of $\widetilde{\mathcal{F}}'$, so they must be edge disjoint. Thus, the orientation D_z obtained from reversing the edges of T_w in D_v or equivalently T_v in D_w is in $O(G, D_0)$. This gives (U2). The same reasoning gives (L2). It remains to show that $\mathcal{H}$ is connected.

Given a 0-homologous oriented subgraph T of G, such that $T = \sum_{F \in \mathcal{F}'} \lambda_F \phi(F)$, we define $s(T) = \sum_{F \in \mathcal{F}'} |\lambda_F|$.

Let D, D' be two orientations in $O(G, D_0)$, and $T = D \setminus D'$. We prove by induction on $s(T)$ that D, D' are connected in $\mathcal{H}$. This is clear if $s(T) = 0$ as then $D = D'$. So we now assume that $s(T) \neq 0$ and so that D, D' are distinct. Lemma 18 implies that T admits a partition into 0-homologous oriented subgraphs $T_1, \dots, T_k$ such that $\phi(T) = \sum_{1 \leq i \leq k} \phi(T_i)$, and, for $1 \leq i \leq k$, there exists $\widetilde{X_i} \subseteq \widetilde{\mathcal{F}}'$ and $\epsilon_i \in \{-1, 1\}$ such that $\phi(T_i) = \epsilon_i \sum_{\widetilde{F} \in \widetilde{X_i}} \phi(\widetilde{F})$. Lemma 19 applied to T_1 implies that there exists $\widetilde{F_1} \in \widetilde{X_1}$ such that $\epsilon_1 \phi(\widetilde{F_1})$ corresponds to an oriented subgraph of T_1. Let T' be the oriented subgraph obtained from T by removing the border of $\widetilde{F_1}$. As $\phi(T') = \phi(T) - \epsilon_1 \phi(\widetilde{F_1})$, we have that T' also is 0-homologous. Let D'' be such that $\epsilon_1 \widetilde{F_1} = D \setminus D''$. So we have $D'' \in O(G, D_0)$ and there is an edge between D and D'' in $\mathcal{H}$. Moreover $T' = D'' \setminus D'$ and $s(T') = s(T) - |X_{\widetilde{F_1}}| < s(T)$, where $X_{\widetilde{F_1}}$ corresponds to the faces of $\mathcal{F}'$ that lie inside $\widetilde{F_1}$. So the induction hypothesis on D'', D' implies that they are connected in $\mathcal{H}$. So D, D' are also connected in $\mathcal{H}$. □

Note that Proposition 3 gives a proof of Theorem 6 independent from Propp [26].

We continue to further investigate the set $O(G, D_0)$.

Proposition 4 *For every element $\widetilde{F} \in \widetilde{\mathcal{F}}$, there exists D in $O(G, D_0)$ such that $\widetilde{F}$ is an oriented subgraph of D.*

Proof. Let $\widetilde{F} \in \widetilde{\mathcal{F}}$. Let D be an element of $O(G, D_0)$ that maximizes the number of edges of $\widetilde{F}$ that have the same orientation in $\widetilde{F}$ and D, i.e. D maximizes the number of edges oriented counterclockwise on the boundary of the face of $\widetilde{G}$ corresponding to $\widetilde{F}$. Towards a contradiction, suppose that there is an edge e of $\widetilde{F}$ that does not have the same orientation in $\widetilde{F}$ and D. The edge e is in $\widetilde{G}$ so it is non-rigid. Let $D' \in O(G, D_0)$ such that e is oriented differently in D and D'. Let $T = D \setminus D'$. By Lemma 18, there exist a partition of T into 0-homologous oriented subgraphs $T_1, \dots, T_k$ such that $\phi(T) = \sum_{1\le i\le k} \phi(T_i)$, and, for $1 \le i \le k$, there exists $\widetilde{X}_i \subseteq \widetilde{\mathcal{F}}'$ and $\epsilon_i \in \{-1, 1\}$ such that $\phi(T_i) = \epsilon_i \sum_{\widetilde{F}'\in\widetilde{X}_i} \phi(\widetilde{F}')$. W.l.o.g., we can assume that e is an edge of T_1. Let D'' be the element of $O(G, D_0)$ such that $T_1 = D \setminus D''$. The oriented subgraph T_1 intersects $\widetilde{F}$ only on edges of D oriented clockwise on the border of $\widetilde{F}$. So D'' contains strictly more edges oriented counterclockwise on the border of the face $\widetilde{F}$ than D, a contradiction. So all the edges of $\widetilde{F}$ have the same orientation in D. So $\widetilde{F}$ is a 0-homologous oriented subgraph of D. □

By Proposition 4, for every element $\widetilde{F} \in \widetilde{\mathcal{F}}'$ there exists D in $O(G, D_0)$ such that $\widetilde{F}$ is an oriented subgraph of D. Thus there exists D' such that $\widetilde{F} = D \setminus D'$ and D, D' are linked in $\mathcal{H}$. Thus, $\widetilde{\mathcal{F}}'$ is a minimal set that generates the lattice.

A distributive lattice has a unique maximal (resp. minimal) element. Let $D_{\max}$ (resp. $D_{\min}$) be the maximal (resp. minimal) element of $(O(G, D_0), \le_{f_0})$.

Proposition 5 *$\widetilde{F}_0$ (resp. $-\widetilde{F}_0$) is an oriented subgraph of $D_{\max}$ (resp. $D_{\min}$).*

Proof. By Proposition 4, there exists D in $O(G, D_0)$ such that $\widetilde{F_0}$ is an oriented subgraph of D. Let $T = D \setminus D_{\max}$. Since $D \le_{f_0} D_{\max}$, the characteristic flow of T can be written as a combination with positive coefficients of characteristic flows of $\widetilde{\mathcal{F}}'$, i.e. $\phi(T) = \sum_{\widetilde{F}\in\widetilde{\mathcal{F}}'} \lambda_F \phi(\widetilde{F})$ with $\lambda \in \mathbb{N}^{|\mathcal{F}'|}$. So T is disjoint from $\widetilde{F}_0$. Thus $\widetilde{F}_0$ is an oriented subgraph of $D_{\max}$. The proof is analogous for $D_{\min}$. □

Proposition 6 *$D_{\max}$ (resp. $D_{\min}$) contains no counterclockwise (resp. clockwise) non-empty 0-homologous oriented subgraph.*

Proof. Towards a contradiction, suppose that $D_{\max}$ contains a counterclockwise non-empty 0-homologous oriented subgraph T. Then there exists $D \in O(G, D_0)$ distinct from $D_{\max}$ such that $T = D_{\max} \setminus D$. We have $D_{\max} \le_{f_0} D$ by definition of $\le_{f_0}$, a contradiction to the maximality of $D_{\max}$. □

In the definition of counterclockwise (resp. clockwise) non-empty 0-homologous oriented subgraph, used in Proposition 6, the sum is taken over elements of $\mathcal{F}'$ and thus does not use F_0. In particular, $D_{\max}$ (resp. $D_{\min}$) may contain regions whose boundary is oriented counterclockwise (resp. clockwise) according to the region but then such a region contains F_0.

We conclude this section by applying Theorem 6 to Schnyder orientations:

Theorem 8 *Let G be a map on an orientable surface given with a particular Schnyder orientation D_0 of $\hat{G}$ and a particular face f_0 of $\hat{G}$. Let $S(\hat{G}, D_0)$ be the set of all the Schnyder orientations of $\hat{G}$ that have the same out-degrees and same type as D_0. We have that $(S(\hat{G}, D_0), \le_{f_0})$ is a distributive lattice.*

Proof. By the third item of Theorem 5, we have $S(\hat{G}, D_0) = O(\hat{G}, D_0)$. Then the conclusion holds by Theorem 6. □

Note that the minimal element of the lattice and its properties (Proposition 3 to 6) are used in [8] to obtain a new bijection concerning toroidal triangulations.

5 Toroidal triangulations

In this section we look specifically at the case of toroidal triangulations, for which we identify a particular lattice characterized in terms of its type (for any choice of a basis). We prove that this lattice contains all the *crossing* Schnyder woods [18] - to be defined below.

According to Euler's formula, toroidal maps are such that $n - m + f = 0$. Thus when generalizing Schnyder woods, one can avoid vertices satisfying several times the Schnyder property (see Figure 3), avoid edges of type 0 (two incoming edges with the same color, see Figure 6), and the general definition of Schnyder woods (see Definition 6) can be simplified for toroidal triangulation.

Indeed an EDGE, $\mathbb{N}^*$-VERTEX, $\mathbb{N}^*$-FACE angle labeling of a toroidal triangulation is in fact a 1-EDGE, 1-VERTEX, 1-FACE angle labeling. One gets this by counting label changes on the angles around vertices, faces and edges. Around vertices and faces there are at least 3 changes. Around an edge there are at most 3 changes (see Figure 6). So $3n+3f \leq 3m$. By Euler's formula we have equality $3n+3f = 3m$. So around all vertices, faces and edges there are exactly 3 changes. This implies that vertices and faces are of type 1, and that edges are of type 1 or 2. Since for triangulations $3f = 2m$, we have that $m = 3n$. Since every vertex is of type 1, there are exactly $3n$ arcs and thus all the edges carry exactly one arc each, that is all the edges are of type 1.

Let G be a toroidal triangulation given with a Schnyder wood. The graph G_i denotes the directed graph induced by the edges of color i. Each graph G_i has exactly n edges. For each connected component of G_i, every vertex has exactly one outgoing edge. Thus each connected component of G_i has exactly one directed cycle that is called a *monochromatic cycle* of color i. By Lemma 3 and the Schnyder property, it is not difficult to see that for a given color $i \in \{0, 1, 2\}$, all i-cycles are non-contractible, non intersecting and weakly homologous. Moreover, two monochromatic cycles of distinct colors are intersecting if and only if they are not weakly homologous.

Monochromatic cycles capture some "global" property that a Schnyder wood may have. A Schnyder wood of a toroidal triangulation is said to be *crossing*, if for each pair i, j of different colors, there exists a monochromatic cycle of color i intersecting a monochromatic cycle of color j. In [18], the following theorem is proved:

Theorem 9 ([18]) *Every toroidal triangulation admits a crossing Schnyder wood.*

Note that the right side of Figure 8 is an example of a crossing Schnyder wood of a toroidal triangulation. On the contrary Figure 17 gives an example of a non-crossing Schnyder wood of a toroidal triangulation (all the monochromatic cycles are vertical).

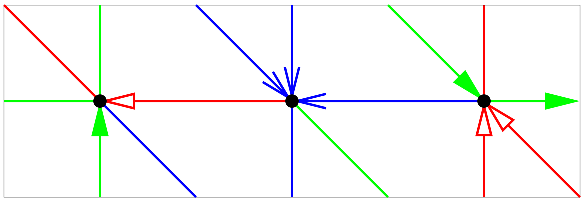


Figure 17: A Schnyder wood of a toroidal triangulation that is non-crossing.

Crossing Schnyder woods play a particular role w.r.t. to the type of a Schnyder orientation as defined in Section 4.1. Consider a toroidal triangulation G and a pair $\{B_1, B_2\}$ of cycles that form a basis for the homology. Figure 16 shows how to transform an orientation of G into an orientation of $\hat{G}$. With this transformation a Schnyder wood of G naturally corresponds to a Schnyder orientation of $\hat{G}$. This allows us to not distinguish between a Schnyder wood or the corresponding Schnyder orientation of $\hat{G}$. Recall, that the type of a Schnyder orientation of $\hat{G}$ in the basis $\{B_1, B_2\}$ is the pair $(\gamma(B_1), \gamma(B_2))$. Then we have the following lemma:

Lemma 20 *A crossing Schnyder wood is of type* $(0, 0)$ *(for the considered basis).*

Proof. Let us prove that for any non-contractible cycle B we have $\gamma(B) = 0$. As observed at the end of Section 3, for triangulations $\gamma(B) = 0$ if and only if the vertices of B have as many edges leaving on one side as edges leaving on the other side. Consider an infinite walk B^∞ of G^∞ corresponding to the cycle B. If B is homologous to a monochromatic cycle (colored i) C_i of G, then it is not homologous to the monochromatic cycles of color $i+1$ or $i-1$ (See Theorem 7 in [18]). We thus assume without loss of generality that B is not homologous to a monochromatic cycle (colored 1) C_1 of G. Let $v \in B \cap C_1$ be a vertex such that for any copy of v, $v_i \in B^\infty$, the path $P_1(v_i)$ and B^∞ only intersect at v_i. Let now v_0, v_1 be two consecutive copies of v in B^∞. As B and C_1 are not homologous, the paths $P_1(v_0)$ and $P_1(v_1)$ do not intersect. Consider now two vertices $u_0 \in P_1(v_0)$ and $u_1 \in P_1(v_1)$ such that either $u_0 \in P_2(u_1)$ or $u_1 \in P_2(u_0)$. By inverting v_0 and v_1 if necessary, we assume that $u_1 \in P_2(u_0)$ (see Figure 18). We also assume that the subpath of $P_2(u_0)$ between u_0 and u_1 does not intersect B^∞. This can be achieved by considering a vertex u_0 further away from v_0 on $P_1(v_0)$.

Consider now the finite plane graph H, bounded by B^∞, $P_1(v_0)$, $P_1(v_1)$, and $P_2(u_0)$. Note that by construction, and by Lemma 4, the outer-boundary of H is a cycle. It is well known by Euler's formula that if H has n inner vertices and k outer-vertices, it has exactly $3n + 2k - 3$ edges. This implies that in the considered orientation of G, the vertices on the boundary have exactly $k - 3$ edges oriented towards the interior.

By the Schnyder property u_0 has no edge leaving towards the interior of H. For the same reason each of the other vertices on the boundary of H and that do not belong

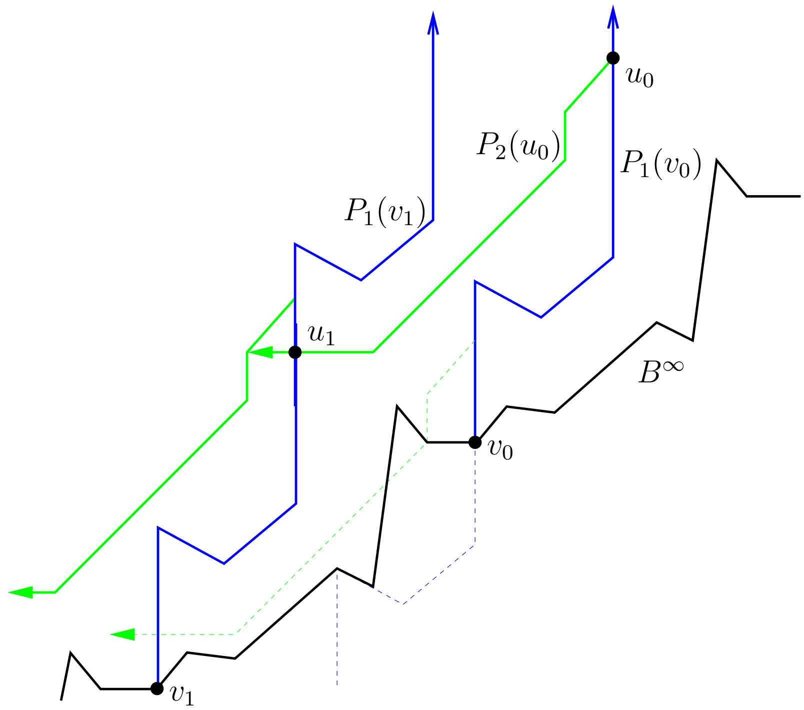


Figure 18: A path B^∞ in the universal cover of a toroidal triangulation G with a crossing Schnyder wood.

to B^∞ has exactly one such edge. This implies that if the subpath of B^∞ between v_0, v_1 has length ℓ (and has $\ell+1$ vertices), its vertices have $\ell-1$ such edges. Adding the edge colored 1 leaving from v_0 you obtain that the ℓ vertices of B^∞ between v_0 (including) and v_1 (excluding) have exactly ℓ edges leaving on some side of B^∞ (the side corresponding to the interior of H). Since they have ℓ edges on B^∞, we deduce that they have ℓ edges leaving on the other side of B^∞. Thus $\gamma(B) = 0$. □

A consequence of Lemma 20 is the following:

Theorem 10 (Existence of a canonical lattice) *Let G be a toroidal triangulation, given with a particular crossing Schnyder wood D_0, then the set $T(G, D_0)$ of all Schnyder woods of G that have the same type as D_0 contains all the crossing Schnyder woods of G.*

Recall from Section 4.2, that the set $T(G, D_0)$ carries the structure of a distributive lattice. This lattice contains all the crossing Schnyder woods. Thus Theorem 10 shows the existence of a canonical lattice that is used to build a bijection between toroidal triangulations and particular maps [8].

Note that $T(G, D_0)$ may contain Schnyder woods that are not crossing. The Schnyder wood of Figure 17 is an example where $\gamma(C) = 0$ for each non-contractible cycle C. So it is of the same type as any crossing Schnyder wood but it is not crossing.

Note also that there exist Schnyder woods not in $T(G, D_0)$. The Schnyder wood of Figure 19 is an example where the horizontal cycle has value γ equal to ± 6, and so this Schnyder wood is not in $T(G, D_0)$.

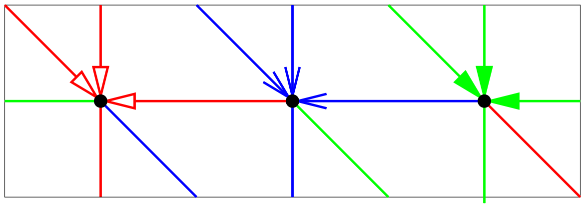


Figure 19: A Schnyder wood of a toroidal triangulation where $\gamma(C) \neq 0$ for a non-contractible cycle C.

See [19] for a full representation of the lattice $T(G, D_0)$ for the toroidal triangulation of Figures 17-19.

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