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Abstract

We prove that every planar graph is the intersection graph of homothetic triangles in the plane[1].

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[1]This result was already annouced in [15].

1 Introduction

Here, an *intersection representation* is a collection of shapes in the plane. The *intersection graph* described by such a representation has one vertex per shape, and two vertices are adjacent if and only if the corresponding shapes intersect. In the following we only consider shapes that are homeomorphic to disks. In this context, if for an intersection representation the shapes are interior disjoint, we call such a representation a *contact representation.* In such a representation, a *contact point* is a point that is in the intersection of (at least) two shapes.

Research on contact representations of (planar) graphs with predefined shapes started with the work of Koebe in 1936, and was recently widely studied; see for example the literature for disks [3, 6, 17], triangles [5], homothetic triangles [4, 12, 15, 16, 24], rectangles [9, 20, 25], squares [18, 23], pentagons [11], hexagons [7], convex bodies [22], or (non-convex) axis aligned polygons [1, 2, 14]. In the present article, we focus on homothetic triangles. It has been shown that many planar graphs admit a contact representation with homothetic triangles.

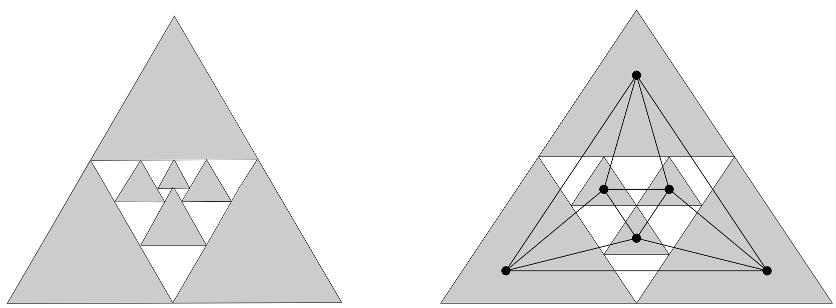


Figure 1: Contact representations with homothetic triangles.

Theorem 1 *Every 4-connected planar triangulation admits a contact representation with homothetic triangles.*

Note that one cannot drop the 4-connectedness requirement from Theorem 1. Indeed, in every contact representation of $K_{2,2,2}$ with homothetic triangles, there are three triangles intersecting in a point (see the right of Figure 1). This implies that the triangulation (not 4-connected) obtained from $K_{2,2,2}$ by adding a degree three vertex in every face does not admit a contact representation with homothetic triangles. Some questions related to this theorem remain open. For example, it is believed that if a triangulation T admits a contact representations with homothetic triangles, it is unique up to some choice for the triangles in the outer-boundary. However this statement is still not proved. Another line of research lies in giving another proof to Theorem 1 (a combinatorial one), or in providing a polynomial algorithm constructing such a representation [8, 24].

Theorem 1 has a nice consequence. It allowed Felsner and Francis [10] to prove that every planar graph has a contact representation with cubes in $\mathbb{R}^3$. In the present paper we remain in the plane. Theorem 1 is the building block for proving our main result. An intersection representation is said *simple* if every point belongs to at most two shapes.

Theorem 2 *A graph is planar if and only if it has a simple intersection representation with homothetic triangles.*

This answers a conjecture of Lehmann that planar graphs are max-tolerance graphs (as max-tolerance graphs have shown to be exactly the intersection graphs of homothetic triangles [16]). Müller et al. [19] proved that for some planar graphs, if the triangle corners have integer coordinates, then their intersection representation with homothetic triangles needs coordinates of order $2^{\Omega(n)}$, where n is the number of vertices. The following section is devoted to the proof of Theorem 2.

2 Intersection representations with homothetic triangles

It is well known that simple contact representations produce planar graphs. The following lemma is slightly stronger.

Lemma 3 *Consider a graph $G = (V, E)$ given with a simple intersection representation $\mathcal{C} = \{c(v) : v \in V\}$. If the shapes $c(v)$ are homeomorphic to disks, and if for any couple $(u, v) \in V^2$ the set $c(u) \setminus c(v)$ is non-empty and connected, then G is planar.*

Proof: Observe that since $\mathcal{C}$ is simple, the sets $c^\circ(u) = c(u) \setminus \left(\cup_{v \in V \setminus \{u\}} c(v)\right)$ are disjoint non-empty connected regions. Let us draw G by first choosing a point p_u inside $c^\circ(u)$, for representing each vertex u (see Figure 2). Then for each neighbor v of u, draw a curve inside $c^\circ(u)$ from p_u to the border of $c(u) \cap c(v)$ (in the border of $c^\circ(u)$) to represent the half-edge of uv incident to u. As the regions $c^\circ(u)$ are disjoint and connected, this can be done without crossings. Finally, for each edge uv it is easy to link its two half edges by drawing a curve inside $c(u) \cap c(v)$. As the obtained drawing has no crossings, the lemma follows. □

Note that for any two homothetic triangles Δ and Δ', the set $\Delta \setminus \Delta'$ is connected. Lemma 3 thus implies the sufficiency of Theorem 2. For proving Theorem 2 it thus suffices to construct an intersection representation with homothetic triangles for any planar graph G. In fact we restrict ourselves to (planar) triangulations because any such G is an induced subgraph of a triangulation T (an intersection representation of T thus contains a representation of G). The following Proposition 4 thus implies Theorem 2.

From now on we consider a particular triangle. Given a Cartesian coordinate system, let Δ be the triangle with corners at coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$ (see Figure 3.(a)). Thus the homothets of Δ have corners of the form (x, y),

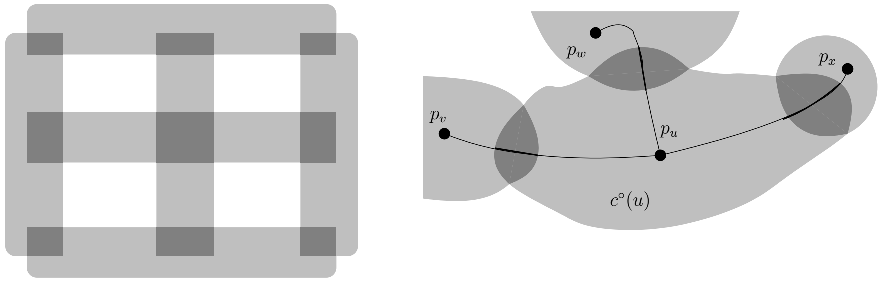


Figure 2: (left) A non-simple intersection representation of $K_{3,3}$ (right) How to draw a graph planarly from its simple intersection representation.

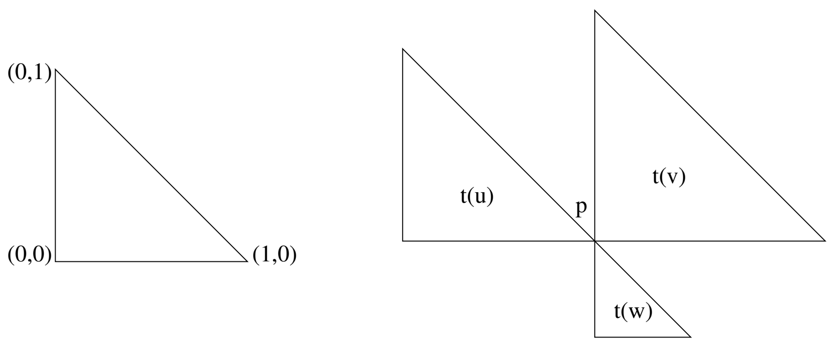


Figure 3: (a) The triangle Δ (b) The triangles $t(u)$, $t(v)$ and $t(w)$.

$(x, y+h)$ and $(x+h, y)$ with $h > 0$, and we call (x, y) their *right corner* and h their *height*.

Proposition 4 *For any triangulation T with outer vertices a, b and c, for any three triangles $t(a)$, $t(b)$, and $t(c)$ homothetic to Δ, that pairewise intersect but do not intersect (i.e. $t(a) \cap t(b) \cap t(c) = \emptyset$), and for any $\epsilon > 0$, there exists an intersection representation $\mathcal{T} = \{t(v) : v \in V(T)\}$ of T with homothets of Δ such that:*

(a) No three triangles intersect.

(b) The representation is bounded by $t(a)$, $t(b)$, and $t(c)$ and the inner triangles intersecting those outer triangles intersect them on a point or on a triangle of height less than ϵ.

Proof: Let us first prove the proposition for 4-connected triangulations. Theorem 1 tells us that 4-connected triangulations have such a representation if we relax condition (a) by allowing 3 triangles $t(u)$, $t(v)$ and $t(w)$ to intersect if they pairewise intersect in the same single point p (i.e. $t(u) \cap t(v) = t(u) \cap t(w) = t(v) \cap t(w) = p$). We call (a') this relaxation of condition (a), and we call "bad points", the points at the intersection of 3 triangles. Let us now reduce their number (to zero) as follows (and thus fulfill condition (a)).

Note that the corners of the outer triangles do not intersect inner triangles. This property will be preserved along the construction below.

Let $p = (x_p, y_p)$ be the highest (i.e. maximizing y_p) bad point. If there are several bad points at the same height, take among those the leftmost one (i.e. minimizing x_p). Then let $t(u)$, $t(v)$ and $t(w)$ be the three triangles pairewise intersecting at p. Let us denote the coordinates of their right corners by (x_u, y_u), (x_v, y_v) and (x_w, y_w), and their height by h_u, h_v and h_w. Without loss of generality we let $p = (x_u + h_u, y_u) = (x_v, y_v) = (x_w, y_w + h_w)$ (see Figure 3.(b)). By definition of p it is clear that p is the only bad point around $t(u)$. Note also that none of $t(u)$, $t(v)$ and $t(w)$ is an outer triangle.

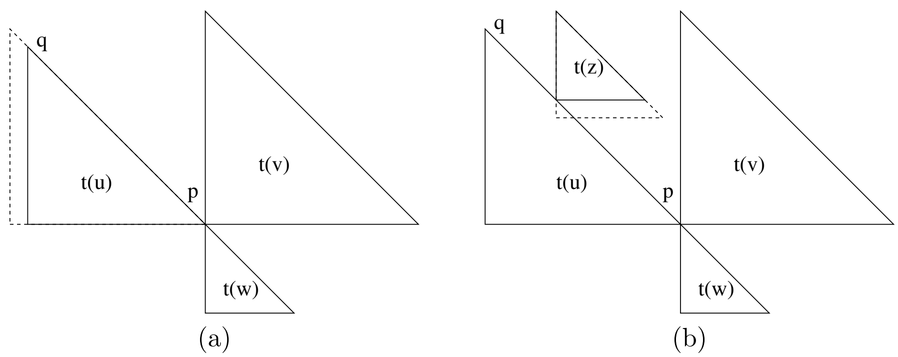


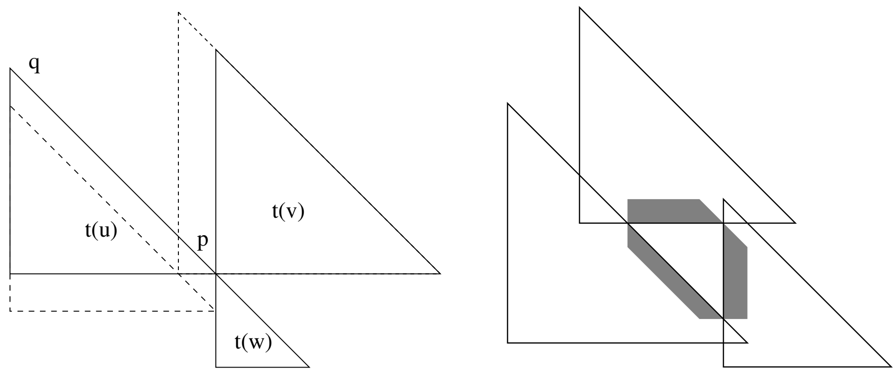
Figure 4: (a) Step 1 (b) Step 2

Step 1: By definition of p and $t(u)$, the corner $q = (x_u, y_u + h_u)$ of $t(u)$ is not a bad point. Now inflate $t(u)$ in order to have its right angle in $(x_u - \epsilon_1, y_u)$ and height $h_u + \epsilon_1$, for a sufficiently small $\epsilon_1 > 0$ (see Figure 4.(a)). Here ϵ_1 is sufficiently small to avoid new pairs of intersecting triangles, new triples of intersecting triangles, or an intersection between $t(u)$ and an outer triangle on a too big triangle (with height $\geq \epsilon$). Since the new $t(u)$ contains the old one, the triangles originally intersected by $t(u)$ are still intersected. Hence, $t(u)$ intersects the same set of triangles, and the new representation is still a representation of T. Since there was no bad point distinct from p around $t(u)$, it is clear by the choice of $\epsilon_1 > 0$ that the new representation still fulfills (a') and (b). After this step we have the following.

Claim 5 *The top corner of* $t(u)$ *is not a contact point.*

Step 2: For every triangle $t(z)$ that intersects $t(u)$ on a single point of the open segment $]p, q[$ do the following. Denote (x_z, y_z) the right corner of $t(z)$, and h_z its height. Note that $t(z)$ is an inner triangle of the representation and that by definition of p there is no bad point involving $t(z)$. Now inflate $t(z)$ in order to have its right corner at $(x_z, y_z - \epsilon_2)$, and height $h_z + \epsilon_2$, for a sufficiently small $\epsilon_2 > 0$ (see Figure 4.(b)). Here ϵ_2 is again sufficiently small to avoid new pairs or new triples of intersecting triangles, and to preserve (b). Since $t(z)$ was not involved in a bad point, the new representation still fulfills (a'). Since the new $t(z)$ contains the old one, the triangles originally intersected by $t(z)$ are still intersected. Hence, $t(z)$ intersects the same set of triangles, and the new representation is still a representation of T. After doing this to every $t(z)$ we have the following.

Claim 6 *There is no contact point on* $]p, q]$.



(a) (b)

Figure 5: (a) Step 3 (b) Condition (c)

Step 3: Now translate $t(u)$ downwards in order to have its right corner in $(x_u, y_u - \epsilon_3)$, and inflate $t(v)$ in order to have its right angle in $(x_v - \epsilon_3, y_v)$, and height $h_v + \epsilon_3$, for a sufficiently small $\epsilon_3 > 0$ (see Figure 5.(a)). Here ϵ_3 is again sufficiently small to avoid new pairs or triples of intersecting triangles, and to preserve (b) but it is also sufficiently small to preserve the existing pairs of intersecting triangles. This last requirement can be fulfilled because the only intersections that $t(u)$ could loose would be contact points on $]p, q]$, which do not exist.

After these three steps, it is clear that the new representation has one bad point less and induces the same graph. This proves the proposition for 4-connected triangulations. The conditions (a) and (b) imply the following property.

(c) For every inner face xyz of T, there exists a triangle $t(xyz)$, negatively homothetic to Δ, which interior is disjoint to any triangle $t(v)$ but which

3 sides are respectively contained in the sides of $t(x)$, $t(y)$ and $t(z)$. Furthermore, there exists an $\epsilon' > 0$ such that any triangle t homothetic to Δ of height ϵ' with a side in $t(x) \cap t(xyz)$ does not intersect any triangle $t(v)$ with $v \neq x$, and similarly for y and z (see Figure 5.(b) where the grey regions represent the union of all these triangles).

We are now ready to prove the proposition for any triangulation T. We prove this by induction on the number of separating triangles. We just proved the initial case of that induction, when T has no separating triangle (i.e. when T is 4-connected). For the inductive step we consider a separating triangle (u, v, w) and we call T_{in} (resp. T_{out}) the triangulation induced by the edges on or inside (resp. on or outside) the cycle (u, v, w). By induction hypothesis T_{out} has a representation fulfilling (a), (b), and (c). Here we choose arbitrarily the outer triangles and ϵ. Since uvw is an inner face of T_{out} there exists a triangle $t(uvw)$ and an $\epsilon' > 0$ (with respect to the inner face uvw) as described in (c). Then it suffices to apply the induction hypothesis for T_{in} (which outer vertices are u, v and w), with the already existing triangles $t(u)$, $t(v)$, and $t(w)$, and for $\epsilon'' = \min(\epsilon, \epsilon')$. Then one can easily check that the obtained representation fulfills (a), (b), and (c). This completes the proof of the proposition. □

3 Conclusion

Given a graph G its *incidence poset* is defined on $V(G) \cup E(G)$ and it is such that x is greater than y if and only if x is an edge with an end at y. A *triangle poset* is a poset which elements correspond to homothetic triangles, and such that x is greater than y if and only if x is contained inside y. It has been shown that a graph is planar if and only if its incidence poset is a triangle poset [21][2]. Theorem 2 improves on this result. Indeed, in the obtained representation the triangles $t(u)$ corresponding to vertices intersect only if those vertices are adjacent, and the triangles corresponding to edges uv, $t(u) \cap t(v)$, are disjoint.

In $\mathbb{R}^3$, one can define *tetrahedral posets* as those which elements correspond to homothetic tetrahedrons in $\mathbb{R}^3$, and such that x is greater than y if and only if x is contained inside y. Unfortunately, graphs whose incidence poset is tetrahedral do not always admit an intersection representation in $\mathbb{R}^3$ with homothetic tetrahedrons. This is the case for the complete bipartite graph $K_{n,n}$, for a sufficiently large n. It is easy to show that its incidence poset is tetrahedral. In an intersection representation with homothetic tetrahedrons, let us prove that the smallest tetrahedron t has a limited number of neighbors that induce a stable set. Let t' be the tetrahedron centered at t and with three times its size. Note that every other tetrahedron intersecting t, intersects t' on a tetrahedron at least as large as t. The limited space in t' implies that one cannot avoid intersections among the neighbors of t, if they are too many. The interested reader will see in [13] that these graphs defined by tetrahedral incidence posets also escape a characterization as TD-Delaunay graphs.

[2]Triangle posets are exactly dimension three posets.

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