Approximating maximum uniquely restricted matchings in bipartite graphs

Julien Baste, Dieter Rautenbach, Ignasi Sau

Approximating Maximum Uniquely Restricted Matchings in Bipartite Graphs*

Julien Baste† Dieter Rautenbach‡ Ignasi Sau§¶

Abstract

A matching in a graph is *uniquely restricted* if no other matching covers exactly the same set of vertices. This notion was defined by Golumbic, Hirst, and Lewenstein [Algorithmica, 2001] and studied in a number of articles. We provide approximation algorithms for computing a uniquely restricted matching of maximum size in some bipartite graphs, namely those excluding a C_4 or with maximum degree at most three. In particular, we achieve a ratio of 5/9 for subcubic bipartite graphs, improving over a 1/2-approximation algorithm proposed by Mishra [Electron. Notes Discrete Math, 2011].

Keywords: uniquely restricted matching; bipartite graph; approximation algorithm; subcubic graph.

1 Introduction

Matchings in graphs are among the most fundamental and well-studied objects in combinatorial optimization [16,21]. While classical matchings lead to many efficiently solvable problems, more restricted types of matchings [20] are often intractable; induced matchings [1–3,5,6,9–12,17] being a prominent example. Here we study the so-called uniquely restricted matchings, which were introduced by Golumbic, Hirst, and Lewenstein [8] and studied in a number of papers [7,13–15,18,19]. We also consider the corresponding edge coloring notion.

Before we explain our contribution and discuss related research, we collect some terminology and notation (cf. e.g. [4] for undefined terms). We consider finite, simple, and undirected graphs. A *matching* in a graph G [16] is a set of pairwise non-adjacent

*Emails: julien.baste@uni-ulm.de, dieter.rautenbach@uni-ulm.de, ignasi.sau@lirmm.fr.

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†Université de Montpellier, LIRMM, Montpellier, France.

‡Institute of Optimization and Operations Research, Ulm University, Ulm, Germany.

§CNRS, LIRMM, Université de Montpellier, Montpellier, France.

¶Departamento de Matemática, Universidade Federal do Ceará, Fortaleza, Brazil.

edges of G. For a matching M, let $V(M)$ be the set of vertices incident with an edge in M. A matching M in a graph G is *induced* [5] if the subgraph $G[V(M)]$ of G induced by $V(M)$ is 1-regular. Golumbic, Hirst, and Lewenstein [8] define a matching M in a graph G to be *uniquely restricted* if there is no matching M' in G with $M' \neq M$ and $V(M') = V(M)$, that is, no other matching covers exactly the same set of vertices. It is easy to see that a matching M in G is uniquely restricted if and only if there is no *M-alternating cycle* in G, which is a cycle in G that alternates between edges in M and edges not in M. Let the *matching number* $\nu(G)$, the *strong matching number* $\nu_s(G)$, and the *uniquely restricted matching number* $\nu_{ur}(G)$ of G be the maximum size of a matching, an induced matching, and a uniquely restricted matching in G, respectively. Since every induced matching is uniquely restricted, we obtain

$$\nu_s(G) \leq \nu_{ur}(G) \leq \nu(G)$$

for every graph G.

It is worth mentioning that, as discussed by Golumbic, Hirst, and Lewenstein [8], maximum uniquely restricted matchings in bipartite graphs correspond to largest possible upper triangular submatrices that can be obtained by permuting rows and columns of a given matrix. Upper triangular submatrices are interesting objects, since they allow the associated systems of linear equations to be solved quickly; see [8] for more details.

Stockmeyer and Vazirani [20] showed that computing the strong matching number is NP-hard. Their result was strengthened in many ways, and also restricted graph classes where the strong matching number can be determined efficiently were studied [1–3, 17]. Golumbic, Hirst, and Lewenstein [8] showed that it is NP-hard to determine $\nu_{ur}(G)$ for a given bipartite or split graph G. Mishra [18] strengthened this by showing that it is not possible to approximate $\nu_{ur}(G)$ within a factor of $O(n^{\frac{1}{3}-\epsilon})$ for any $\epsilon > 0$, unless NP=ZPP, even when restricted to bipartite, split, chordal or comparability graphs of order n. Furthermore, he showed that $\nu_{ur}(G)$ is APX-complete for subcubic bipartite graphs.

On the positive side, Golumbic, Hirst, and Lewenstein [8] described efficient algorithms that determine $\nu_{ur}(G)$ for cacti, threshold graphs, and proper interval graphs. Solving a problem from [8], Francis, Jacob, and Jana [7] described an efficient algorithm for $\nu_{ur}(G)$ in interval graphs. Solving yet another problem from [8], Penso, Rautenbach, and Souza [19] showed that the graphs G with $\nu(G) = \nu_{ur}(G)$ can be recognized in polynomial time. Complementing his hardness results, Mishra [18] proposed a 2-approximation algorithm for cubic bipartite graphs.

In this article, we present approximation algorithms for $\nu_{ur}(G)$ in some bipartite graphs. Namely, improving on Mishra's approximation algorithm [18], we describe in Section 3 a 5/9-approximation algorithm for computing $\nu_{ur}(G)$ of a given bipartite subcubic graph G. This algorithm requires some complicated preprocessing based on detailed local analysis. In order to illustrate our general approach in a cleaner setting, we first describe in Section 2 an approximation algorithm for C_4-free bipartite graphs of arbitrary maximum degree. We conclude with some open problems in Section 4.

2 Approximation algorithms for C_4-free bipartite graphs

Before we proceed to the 5/9-approximation algorithm for subcubic bipartite graphs in Section 3, we first describe in this section an approximation algorithm for the C_4-free bipartite graphs with an arbitrary bound on the maximum degree. The proof of the next lemma contains the main algorithmic ingredients. Note that the size of the smaller partite set in a bipartite graph is always an upper bound on the uniquely restricted matching number.

For an integer k, let $[k]$ denote the set of positive integers between 1 and k. For a graph G, let $n(G)$ denote its number of vertices.

Lemma 1. *Let $\Delta \geq 3$ be an integer. If G is a connected C_4-free bipartite graph of maximum degree at most Δ with partite sets A and B such that every vertex in A has degree at least 2, and some vertex in B has degree less than Δ, then G has a uniquely restricted matching M of size at least $\frac{(\Delta-1)^2+(\Delta-2)}{(\Delta-1)^3+(\Delta-2)}|A|$. Furthermore, such a matching can be found in polynomial time.*

Proof: We give an algorithmic proof of the lower bound such that the running time of the corresponding algorithm is polynomial in $n(G)$, which immediately implies the second part of the statement. Therefore, let G be as in the statement. Throughout the execution of our algorithm, as illustrated in Figure 1, we maintain a pair (U, M) such that

(a) U is a subset of $V(G)$,

(b) M is a uniquely restricted matching with $V(M) \subseteq U$,

(c) every vertex in $B \cap U$ has all its neighbors in $A \cap U$,

(d) every vertex in $B \setminus U$ has a neighbor in $A \setminus U$,

(e) if

s vertices in $A \cap U$ are incident with an edge in M,

d vertices in $A \cap U$ are not incident with an edge in M but have a neighbor in $B \setminus U$, and

f vertices in $A \cap U$ are neither incident with an edge in M nor have a neighbor in $B \setminus U$, then

$$(\Delta - 1)^2\Big((\Delta - 2)s - (d + f)\Big) \geq (\Delta - 2)f. \tag{1}$$

Initially, let U and M be empty sets. Note that properties (a) to (e) hold.

As long as U is a proper subset of $V(G)$, we iteratively replace the pair (U, M) with a pair (U', M') such that U is a proper subset of U', M is a proper subset of M', and properties (a) to (e) hold for (U', M'). Let s', d', and f' denote the updated values

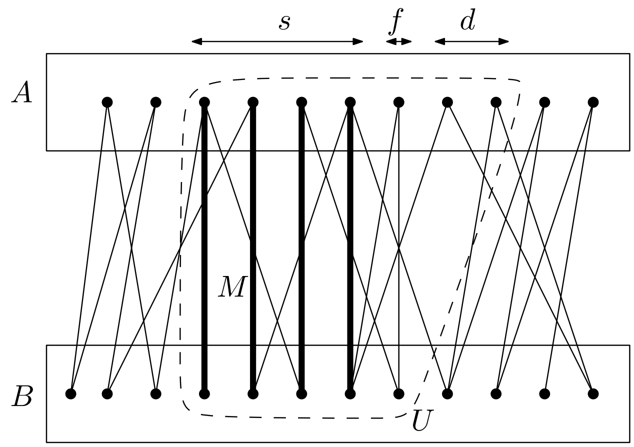


Figure 1: Example for $\Delta = 3$ of the parameters defined in the proof of Lemma 1. The set U is dashed, and the uniquely restricted matching M corresponds to the thicker edges.

considered in (e). Once $U = V(G)$, we have $s = |M|$, $d = 0$, and $f = |A| - |M|$, and (1) implies the stated lower bound on $|M|$.

We proceed to the description of the extension operations. Therefore, suppose that U is a proper subset of $V(G)$. Since G is connected, and some vertex in B has degree less than Δ, there exists at least one vertex u in $B \setminus U$ having less than Δ neighbors in $A \setminus U$, that is, if $d_{\bar{U}}(u) = |N_G(u) \setminus U|$, then $1 \leq d_{\bar{U}}(u) \leq \Delta - 1$, where the existence of u and the first inequality follow from property (d). We choose $u \in B \setminus U$ such that $d_{\bar{U}}(u)$ is as small as possible. Note that by assumption, u has a neighbor in $A \setminus U$ and $d_{\bar{U}}(u) \geq 1$.

Case 1: $d_{\bar{U}}(u) = 1$.

Let v be the unique neighbor of u in $A \setminus U$. Let $\{u_1, \ldots, u_k\}$ be the set of all vertices u in $B \setminus U$ with $N_G(u) \setminus U = \{v\}$, and note that $1 \leq k \leq \Delta$. Let $U' = U \cup \{u_1, \ldots, u_k, v\}$. For some integer $0 \leq \ell \leq k$, we may assume that $\{u_1, \ldots, u_\ell\}$ is the set of those u_i with $i \in [k]$ such that u_i has a neighbor in $A \cap U$, and no neighbor of u_i in $A \cap U$ is incident with M. Note that every vertex u_i with $i \in [k] \setminus [\ell]$ either has no neighbor in $A \cap U$ or has some neighbor in $A \cap U$ that is incident with M.

First, suppose that $\ell \geq 2$. For each $i \in [\ell]$, we fix $w_i \in A \cap U$ to be a neighbor of u_i, and let $M' = M \cup \{u_i w_i : i \in [\ell]\}$. Note that all these neighbors w_i in $A \cap U$ are distinct. Indeed, if two vertices u_i and u_j have a common neighbor w in $A \cap U$, then the set of vertices $\{v, u_i, u_j, w\}$ would induce a C_4 in G. Note also that M' is indeed a uniquely restricted matching, as if there exists an edge $u_i w_j$ with $i, j \in [\ell]$ and $i \neq j$ that could potentially create an M'-alternating cycle, then the set of vertices $\{v, u_i, u_j, w_j\}$ would again induce a C_4 in G. Clearly, replacing (U, M) with (U', M'), we maintain properties (a) to (d), and $s' = s + \ell$. Let n_d be the number of vertices in $A \cap U$ that are not incident with an edge in M', have a neighbor in $B \setminus U$, but do not have a neighbor in $B \setminus U'$; note that each such vertex has a neighbor in the set $\{u_1, \dots, u_k\}$. As every vertex in $\{u_1, \dots, u_k\}$ is neighbor of v and of a vertex incident with an edge in M', it holds that $n_d \leq k(\Delta - 2) \leq \Delta(\Delta - 2)$. If v has a neighbor in $B \setminus U'$, then $d' = d - n_d + 1$ and $f' = f + n_d$, and, if v has no neighbor in $B \setminus U'$, then $d' = d - n_d$ and $f' = f + n_d + 1$. In both cases $d' + f' = d + f + 1$ and $f' \leq f + n_d + 1$. Since $\frac{(\Delta-1)^2}{\Delta-2}\Big((\Delta - 2)\ell - 1\Big) \geq \Delta(\Delta - 2) + 1 \geq n_d + 1$, property (e) is maintained.

Next, suppose that $0 \leq \ell \leq 1$. Let M' arise from M by adding the edge u_1v. Clearly, replacing (U, M) with (U', M'), we maintain properties (a) to (d), and $s' = s + 1$. Defining n_d exactly as above, we obtain $n_d \leq k(\Delta - 2) + \ell \leq \Delta(\Delta - 2) + 1$, $d' = d - n_d$, and $f' = f + n_d$. Since $\frac{(\Delta-1)^2}{\Delta-2}(\Delta - 2) \geq \Delta(\Delta - 2) + 1 \geq n_d$, property (e) is maintained.

Case 2: $2 \leq d_{\bar{U}}(u) \leq \Delta - 1$.

Let $\{v_1, \dots, v_k\} = N_G(u) \setminus U$ and let $U' = U \cup \{u, v_1, \dots, v_k\}$. Note that $2 \leq k \leq \Delta - 1$.

First, suppose that u has a neighbor v in $A \cap U$, and that no neighbor of u in $A \cap U$ is incident with M. Let M' arise from M by adding the edge uv. Clearly, replacing (U, M) with (U', M'), we maintain properties (a) to (c), and $s' = s + 1$. Let us prove that property (d) is also maintained. Since G has no C_4 and $k \geq 2$, no vertex in $B \setminus U$ that is distinct from u can have more than one neighbor among $v_1, \dots, v_k$. Since we are in Case 2, every vertex in $B \setminus U$ has more than one neighbor in $A \setminus U$, hence property (d) remains true. Similarly as above, let n_d be the number of vertices in $A \cap U$ that are not incident with an edge in M', have a neighbor in $B \setminus U$, but do not have a neighbor in $B \setminus U'$. Note that $n_d \leq \Delta - k - 1$, $d' = d + k - n_d - 1$, and $f' = f + n_d$. Since $\frac{(\Delta-1)^2}{\Delta-2}\Big((\Delta - 2) - (k - 1)\Big) \geq \Delta - k - 1 \geq n_d$, property (e) is maintained.

Next, suppose that u has no neighbor in $A \cap U$ or some neighbor of u in $A \cap U$ is incident with M. Let M' arise from M by adding the edge uv_1. Clearly, replacing (U, M) with (U', M'), we again maintain properties (a) to (d), and $s' = s + 1$. Note that, in the case where u has a neighbor in $A \cap U$, v_1 does not have neighbors in $V(M)$ because of property (c), which guarantees that M' is indeed a uniquely restricted matching. Defining n_d exactly as above, we obtain $n_d \leq \Delta - k - 1$. Indeed, if u has no neighbor in $A \cap U$, then $n_d = 0$. On the other hand, if u has a neighbor in $A \cap U$ that is incident with M, then $n_d \leq \Delta - k - 1$. As $k \leq \Delta - 1$, in both cases it holds that $n_d \leq \Delta - k - 1$. Also, we get that $d' = d + k - n_d + 1$ and $f' = f + n_d$, and the same calculation as above implies that property (e) is maintained.

Since the considered cases exhaust all possibilities, and in each case we described an extension that maintains the relevant properties, the proof is complete up to the running time of the algorithm, which we proceed to analyze. One can easily check that each extension operation takes time $O(\Delta n)$, where $n = n(G)$. As in each extension operation, the size of U is incremented by at least one, it follows that the overall running time of the algorithm is $O(\Delta n^2)$. $\square$

With Lemma 1 at hand, we proceed to our first approximation algorithm.

Theorem 1. *Let $\Delta \geq 3$ be an integer. For a given connected C_4-free bipartite graph G of maximum degree at most Δ, one can find in polynomial time a uniquely restricted matching M of G of size at least $\frac{(\Delta-1)^2+(\Delta-2)}{(\Delta-1)^3+(\Delta-2)}\nu_{ur}(G)$.*

Proof: Let $\alpha = \frac{(\Delta-1)^2+(\Delta-2)}{(\Delta-1)^3+(\Delta-2)}$ and let $\mathcal{G}$ be the set of all C_4-free bipartite graphs G of maximum degree at most Δ such that every component of G has a vertex of degree less than Δ. First, we prove that, for every given graph G in $\mathcal{G}$, one can find in polynomial time a uniquely restricted matching M of size at least $\alpha\nu_{ur}(G)$. Therefore, let G be in $\mathcal{G}$.

If G has a vertex u of degree 1, and v is the unique neighbor of u, then let $G' = G - \{u, v\}$. Clearly, $\nu_{ur}(G') = \nu_{ur}(G) - 1$, and if M' is a uniquely restricted matching of G', then $M' \cup \{uv\}$ is a uniquely restricted matching of G. Note that G' belongs to $\mathcal{G}$. Let G'' be the graph obtained from G' by removing every isolated vertex. Clearly, $\nu_{ur}(G'') = \nu_{ur}(G')$, if M'' is a uniquely restricted matching of G'', then M'' is a uniquely restricted matching of G', and G'' belongs to $\mathcal{G}$.

Iteratively repeating these reductions, we efficiently obtain a set M_1 of edges of G as well as a subgraph G_2 of G such that $G_2 \in \mathcal{G}$, $\nu_{ur}(G_2) = \nu_{ur}(G) - |M_1|$, $M_1 \cup M_2$ is a uniquely restricted matching of G for every uniquely restricted matching M_2 of G_2, and either $n(G_2) = 0$ or the minimum degree of G_2, denoted $\delta(G_2)$, is at least 2. Note that if G has minimum degree at least 2, then we may choose M_1 empty and G_2 equal to G. Now, by suitably choosing the bipartition of each component K of G_2, and applying Lemma 1 to K, one can determine in polynomial time a uniquely restricted matching M_2 of G_2 with $|M_2| \geq \alpha\nu_{ur}(G_2)$. Since the set $M_1 \cup M_2$ is a uniquely restricted matching of G of size at least $|M_1| + \alpha\nu_{ur}(G_2) \geq |M_1| + \alpha(\nu_{ur}(G) - |M_1|) \geq \alpha\nu_{ur}(G)$, the proof of our claim about $\mathcal{G}$ is complete.

Now, let G be a given connected C_4-free bipartite graph of maximum degree at most Δ. If G is not Δ-regular, then $G \in \mathcal{G}$, and the desired statement already follows. Hence, we may assume that G is Δ-regular, which implies that its two partite sets A and B are of the same order. By [19], we can efficiently decide whether $\nu_{ur}(G) = \nu(G)$. Furthermore, if $\nu_{ur}(G) = \nu(G)$, then, again by [19], we can efficiently determine a maximum matching that is uniquely restricted. Hence, we may assume that $\nu_{ur}(G) < \nu(G)$. This implies that $\nu_{ur}(G) < |A|$, and, hence, there is some vertex $u \in V(G)$ with $\nu_{ur}(G-u) = \nu_{ur}(G)$. Since $G - u \in \mathcal{G}$ for every vertex u of G, considering the $n(G)$ induced subgraphs $G - u$ for $u \in V(G)$, one can determine in polynomial time a uniquely restricted matching M of G with $|M| \geq \max\{\alpha\nu_{ur}(G - u) : u \in V(G)\} = \alpha\nu_{ur}(G)$. The desired statement follows. $\square$

3 A 5/9-approximation for subcubic bipartite graphs

In view of Theorem 1, it is natural to ask whether C_4-freeness is an essential assumption in order to obtain an approximation factor larger than 1/2. In this section we show that, at least for $\Delta = 3$, this assumption can de dropped. Namely, this section is devoted to proving the following theorem.

Theorem 2. *For a given connected subcubic bipartite graph G, one can find in polynomial time a uniquely restricted matching of G of size at least $\frac{5}{9}\nu_{ur}(G)$.*

In order to ease the presentation, in this section we will use figures to describe some of the "patterns" considered by the algorithms. More formally, given a graph G, a *pattern* P is a subgraph of G in which the set of vertices that have neighbors in $V(G) \setminus V(P)$ is fixed.

In all these figures, the partition of the corresponding bipartite subcubic graph into two sets A and B is represented by using squares and circles, respectively. The half-edges specify which vertices in a pattern have neighbors outside of it.

The following lemma is crucial in order to prove Theorem 2; it plays a role similar to the one played by Lemma 1 for proving Theorem 1. More precisely, we prove in Lemma 2 that we can achieve the desired approximation ratio provided that the input graph satisfies some simple conditions and, more importantly, contains none of the six patterns depicted in Figure 2. The proof of Lemma 2 is quite technical, as several cases need to be distinguished. The analysis of these cases naturally leads to considering a number of other patterns, illustrated in Figures 3-10. With Lemma 2 at hand, the proof of Theorem 1 is not difficult. Namely, we apply exhaustively to the input graph a series of basic reduction rules to guarantee that the conditions of Lemma 2 are fulfilled, while keeping control on the effect of these reduction rules on the parameter ν_{ur}.

Lemma 2. *If G is a connected subcubic bipartite graph with partite sets A and B such that*

(1) G does not have any of the patterns $R.1$, $R.2$, $R.3$, $R.4$, $R.5$, or $R.6$ depicted in Figure 2,

(2) G does not contain two vertices with the same neighborhood,

(3) each vertex of G has degree at least 2, and

(4) at least one vertex in B has degree at most 2,

then a uniquely restricted matching M of G of size at least $\frac{5}{9}\nu_{ur}(G)$ can be found in polynomial time.

Proof: If $n(G) \leq 10$, then we solve the problem optimally by brute force (we will see that the largest pattern without neighbors outside of it considered in the proof has 10 vertices). Therefore, we assume henceforth that G contains at least 11 vertices. In the following, we look for a uniquely restricted matching M of size at least $\frac{5}{9}|A|$. As $|A| \geq \nu_{ur}(G)$, this implies the desired result. We define two types of C_4, namely C_4^1 and C_4^2, as follows. A C_4^1 is a subgraph of G isomorphic to a C_4 such that if $V(C_4^1) \cap A = \{a_1, a_2\}$, then

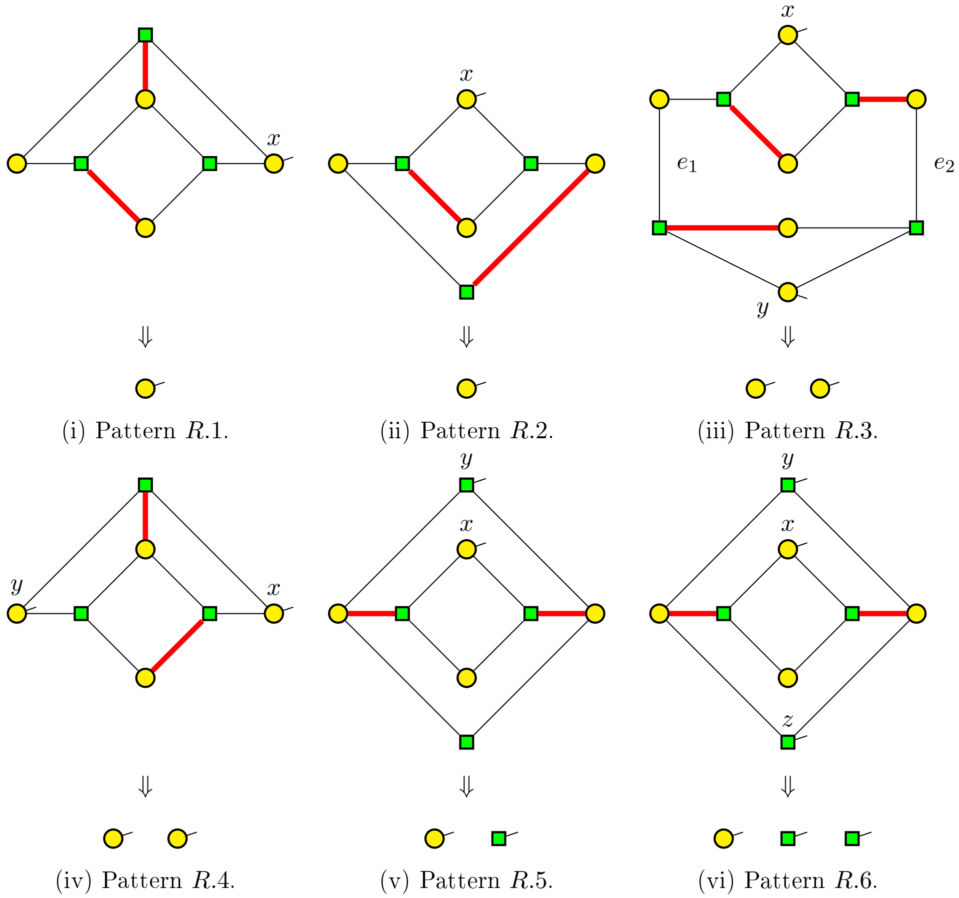


Figure 2: The six forbidden patterns of Lemma 2.

$d_G(a_1) = 3$ and $d_G(a_2) = 2$. A C_4^2 is a subgraph of G isomorphic to a C_4 such that if $V(C_4^2) \cap A = \{a_1, a_2\}$, then $d_G(a_1) = d_G(a_2) = 3$. Note that because of condition (2), there is no subgraph G' of G isomorphic to C_4 such that if $V(G') \cap A = \{a_1, a_2\}$, then $d_G(a_1) = d_G(a_2) = 2$, as it implies that a_1 and a_2 have the same neighborhood.

Similarly to the proof of Lemma 1, throughout the execution of our algorithm we maintain a triple (U, M, γ) that respects the following properties:

(a) $U \subseteq V(G)$,

(b) M is a uniquely restricted matching of G such that $V(M) \subseteq U$,

(c) $\gamma : U \cap A \to \{\top, \vdash, \bot\}$,

(d) for every $v \in A \cap U$, $\gamma(v) = \top$ if and only if $v \in V(M)$ and $\gamma(v) = \vdash$ only if that v has at least one neighbor in $B \setminus U$,

(e) there is no edge between vertices in $U \cap B$ and vertices in $A \setminus U$,

(f) every vertex in $B \setminus U$ has at least one neighbor in $A \setminus U$,

(g) if $s = |\{v \in A : \gamma(v) = \top\}|$, $d = |\{v \in A : \gamma(v) = \vdash\}|$, and $f = |\{v \in A : \gamma(v) = \bot\}|$, then
$$4\,(s - (d + f)) \geq f, \quad \text{and} \tag{2}$$

(h) for each C_4^1 in $G \setminus U$, that we name G', , there is no vertex v in $N_G(V(G')) \cap U \cap A$ such that $\gamma(v) = \vdash$.

We initialize the algorithm with $U = \emptyset$, $M = \emptyset$, and $\gamma : \emptyset \to \{\top, \vdash, \bot\}$ being equal to the empty function. Note that properties (a) to (h) are satisfied.

In the first part of the algorithm, we focus on removing the C_4's from $G \setminus U$. For this we first take care the C_4^1's in $G \setminus U$, and then we deal with the C_4^2's.

As long as $G \setminus U$ is not empty, we consider the first of the following cases such that the corresponding condition is fulfilled, where $d_{\bar{U}}(u) = |N_G(u) \setminus U|$:

- Case 1: there exists $u \in B \setminus U$ such that $d_{\bar{U}}(u) = 1$.
- Case 2: there exists G', a C_4^1 in $G \setminus U$.
- Case 3: there exists G', a C_4^2 in $G \setminus U$.
- Case 4: there exists $u \in B \setminus U$ such that $d_{\bar{U}}(u) = 2$.

Note that, by property (e) and the connectivity of G, we know that, as long as $G \setminus U$ is not the empty graph, at least one of these four cases should apply. We study each case and show that for each of them, we can find a new triple (U', M', γ') starting from (U, M, γ) such that U is a proper subset of U', M is a proper subset of M', γ is the restriction of γ' to U, and properties (a) to (h) hold for (U', M', γ'), where s', d', and f' denote the updated values considered in (g). Once $U = V(G)$, we have $s = |M|$, $d = 0$, and $f = |A| - |M|$, and (2) implies that $|M| \geq \frac{5}{9}|A|$.

In order to prove that such a triple can indeed be found in polynomial time, we distinguish four cases.

In the following, by *resolving* a pattern P we mean that from a triple (U, M, γ) that respects properties (a) to (h) such that $U \cap V(P) = \emptyset$, we exhibit a triple (U', M', γ') that also respects properties (a) to (h) and such that $U' = U \cup V(P)$.

Case 1: there exists $u \in B \setminus U$ such that $d_{\bar{U}}(u) = 1$.

Assume that $d_{\bar{U}}(u) = 1$ and let v be the only neighbor of u in $A \setminus U$. Let $\{u_i : i \in [k]\}$ be the set of all vertices u in $B \setminus U$ with $N_G(u) \setminus U = \{v\}$. Note that $1 \leq k \leq 3$. Let $U' = U \cup \{v\} \cup \{u_i : i \in [k]\}$. By construction of G, we know that every vertex u_i, $i \in [k]$, is of degree at least 2 in G, so it has at least one neighbor in $A \cap U$. For every $i \in [k]$, we define $W_i = N_G(u_i) \cap U$. Note that for $i, j \in [k]$ with $i \neq j$, W_i and W_j

can intersect. Let n_d be the number $|\{w \in \bigcup_{i\in[k]} W_i : \gamma(w) =\vdash\}|$. Note that for each u_i, $i \in [k]$, $v \in N_G(u_i)$ and $v \notin U$. This implies that for each $i \in [k]$, $|W_i| \leq 2$, and so, $n_d \leq 6$.

First, assume that $n_d \leq 4$. Let M' arise from M by adding the edge u_1v. Let γ' be obtained from γ where $\gamma'(v) = \top$, and for each $w \in \bigcup_{i\in[k]} W_i$, such that $\gamma(w) =\vdash$, then $\gamma'(w) = \perp$. Clearly, replacing (U, M, γ) with (U', M', γ'), we maintain properties (a) to (f). By construction of γ', we have that $s' = s + 1$, $d' = d - n_d$, and $f' = f + n_d$. As $n_d \leq 4$, property (g) is maintained. Note that $\gamma'^{-1}(\vdash) \subseteq \gamma^{-1}(\vdash)$. This implies that property (h) is maintained.

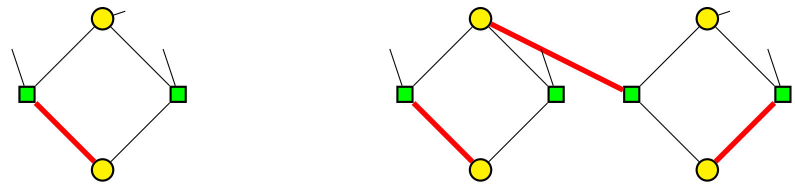
Assume now that $n_d \geq 5$. This implies that $k = 3$, and two sets of $\{W_i : i \in [k]\}$, say W_1 and W_2, are such that $\{w \in W_1 \cup W_2 : \gamma(w) \neq\vdash\} = \emptyset$ and $|W_1| = |W_2| = 2$. Because of condition (2), we know that we can find $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. Let M' arise from M by adding the edges u_1w_1 and u_2w_2. Let γ' be obtained from γ where $\gamma'(v) = \perp$, for each $w \in (\bigcup_{i\in[k]} W_i) \setminus \{w_1, w_2\}$, such that $\gamma(w) =\vdash$, then $\gamma'(w) = \perp$ and $\gamma'(w_1) = \gamma'(w_2) = \top$. Note that M' is a uniquely restricted matching. Clearly, replacing (U, M, γ) with (U', M', γ'), we maintain properties (a) to (f). Then by construction of γ' we obtain that $s' = s + 2$, $d' = d - n_d$, and $f' = f + n_d - 2 + 1$. We obtain that $s' - (d' + f') = s - (d + f) + 3$. As, $n_d \leq 6$, property (g) is maintained. Note that $\gamma'^{-1}(\vdash) \subseteq \gamma^{-1}(\vdash)$. This implies that property (h) is maintained.

Case 2: there exists a C_4^1, that we name G', in $G \setminus U$.

We assume in this case that for every $u \in B \setminus U$, $d_{\bar{U}}(u) \geq 2$. Let $V(G') \cap A = \{a_1, a_2\}$ such that $d_G(a_1) = 3$ and $V(G') \cap B = \{b_1, b_2\}$. For this case, when we say that we define M' and γ' by updating the values of M and γ according to a figure, it means that we add to M every red edge of the figure and for every v of A that is depicted in the figure, then $\gamma'(v) = \top$ if v is the endpoint of a red edge, and $\gamma'(v) = \perp$ otherwise.

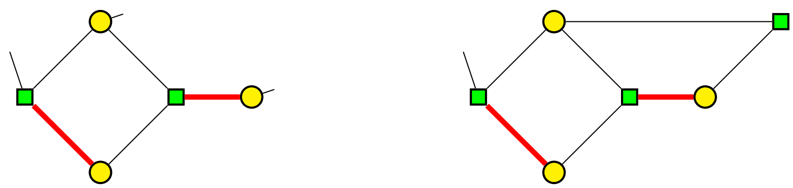
First, assume that $N_G(b_1) \setminus U = N_G(b_2) \setminus U = \{a_1, a_2\}$. If the only vertex of $N_G(a_1) \setminus V(G')$ is inside another C_4, then we are in the situation depicted in Figure 3(ii) and we define U' to be the union of U and the vertices of the two C_4's, and we define M' and γ' by updating the values of M and γ according to Figure 3(ii). Otherwise, we are in the situation depicted in Figure 3(i), we define U' to be $U \cup V(G')$, and we define M' and γ' by updating the values of M and γ according to Figure 3(i). In both cases, one can see that properties (a) to (h) are maintained.

Secondly, assume that there exists $a_3 \in A\setminus U$ such that $a_3 \in N_G(b_2)$ and $a_3 \notin N_G(b_1)$. If there exists b_3 with neighbors only in $U \cup V(G) \cup \{a_3\}$, then we define U' to be the union of U and $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and we define M' and γ' by updating the values of M and γ according to Figure 4(ii). Otherwise, we are in the situation depicted in Figure 4(i), we define U' to be $U \cup V(G') \cup \{a_3\}$, and we define M' and γ' by updating the values of M and γ according to Figure 4(i). In both cases we can see that properties (a) to (h) are maintained.



(i) Not attached to a C_4. (ii) Attached to a C_4.

Figure 3: First case of C_4^1.



(i) If b_3 does not exist. (ii) If b_3 exists.

Figure 4: Second case of C_4^1.

Third, assume that there exists a_3 and a_4 in $A \backslash U$ such that $a_3 \in N_G(b_2)$, $a_3 \notin N_G(b_1)$, $a_4 \in N_G(b_1)$, and $a_4 \notin N_G(b_2)$. This case is the most involved one, as many subcases have to be considered. Namely, we need to take care that there is no vertex in $B \setminus U'$ of degree 0 in $G - U'$ and to make sure that property (h) is maintained. In order to reduce the number of subcases, we sometimes do not take into consideration some vertex u of B that became of degree 0 in $G - U$ after the update of U' in the cases where property (g) is still maintained with the value of s'', f'', and d'' such that $d'' = d' - 2$ and $f'' = f' + 2$. This excludes the case where u is connected to $\{v_1, v_2, v_3\}$ such that $\gamma'(v_i) = \vdash$ for every $i \in [3]$ and then we can safely define $U'' = U' \cup \{u\}$, $M'' = M' \cup \{\{u, v_1\}\}$, and $\gamma''(v) = \gamma'(v)$ for every $v \in (A \cap U') \setminus \{v_1, v_2, v_3\}$, $\gamma''(v_1) = \top$, and $\gamma''(v_2) = \gamma''(v_3) = \bot$. The same applies to the case where u is connected to $\{v_1, v_2, v_3\}$ such that there exists $i \in [3]$ such that $\gamma'(v_i) \neq \vdash$. In this case, we only add u to U' without adding edges to M', but for every $i \in [3]$ such that $\gamma'(v_i) = \vdash$, then $\gamma'(v_i) = \bot$. This last condition is necessary in order to make sure that property (d) is maintained. In both cases, we will ignore these vertices of B in the analysis, but we need to keep in mind that we need to add them each time one of these cases appears. We also sometime forget the third neighbor of a vertex of A whenever its existence does not change how to resolve the pattern.

Let $K = \{a_1, a_2, a_3, a_4, b_1, b_2\}$. As there are at most five edges between K and $V(G) \setminus (U \cup K)$, at most two vertices of B can become of degree 0. We depict in Figure 5 every possible way these vertices can be connected to K together with the possible extra edge from K to $V(G) \setminus (U \cup K)$. One can check that there is no other way to connect at most 2 vertices of B with at least two neighbors in K to Pattern (i). First note that Patterns (ii), (iii), and (x) have no neighbor outside of the pattern and contain less than

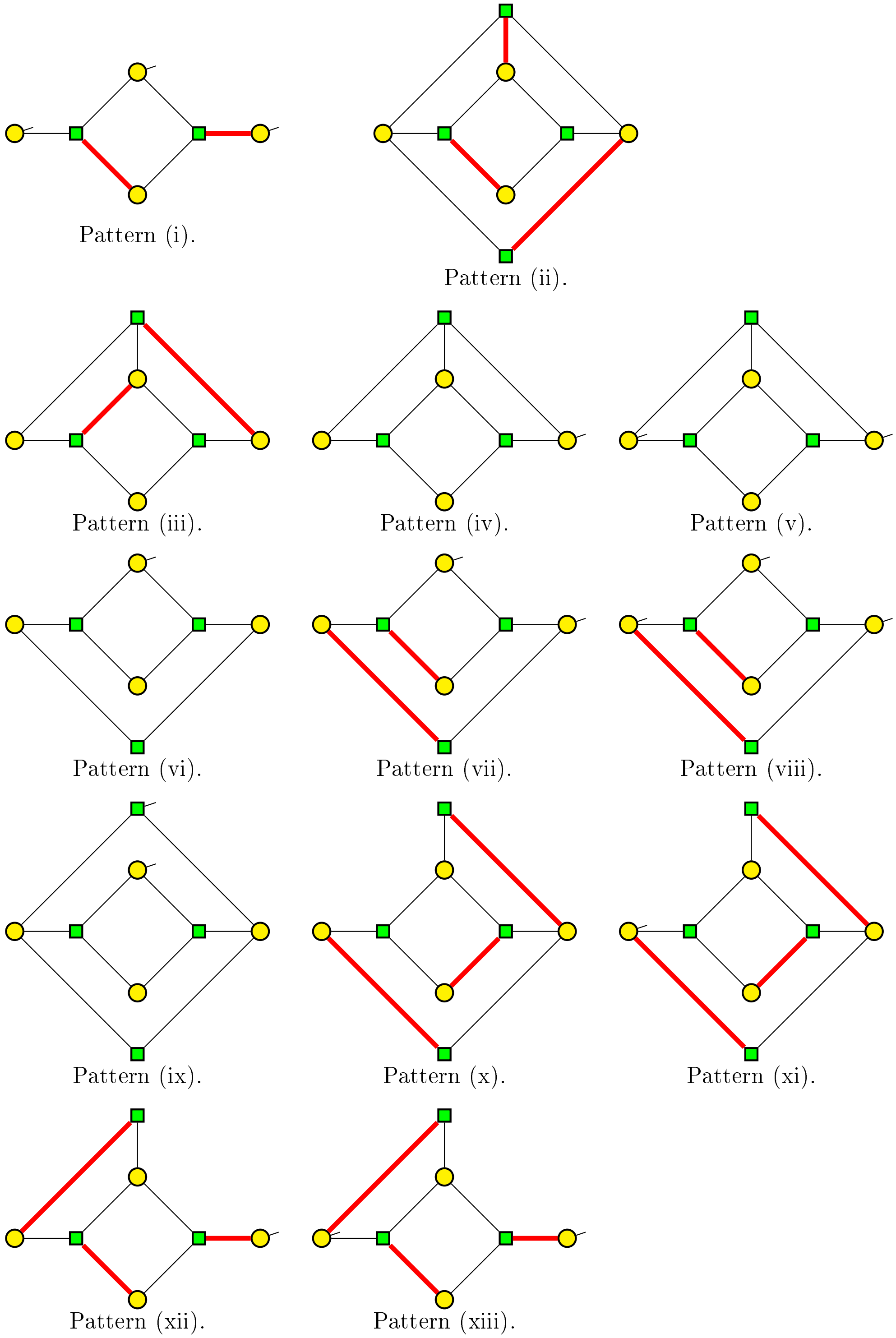


Figure 5: Third case of C_4^1.

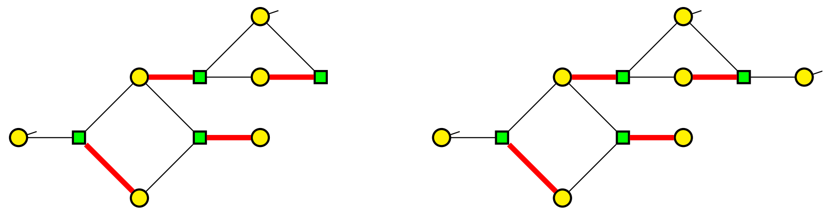


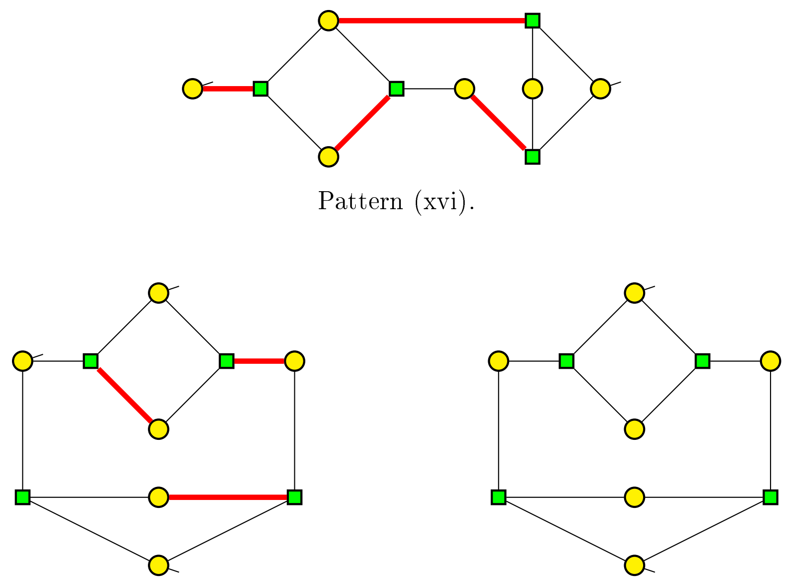
Figure 6: Fourth case of C_4^1.

10 vertices, so we have already resolved these patterns. Note also that Patterns (iv), (v), (vi), and (ix) correspond to patterns of condition (1) and so are not in G. Moreover, Patterns (xi), (xii), and (xiii) do not need to have vertices labeled $\vdash$ in order to define the triple (U', M', γ') that incorporates the corresponding pattern to the part already treated and that respects properties (a) to (h). Therefore, there are only three remaining patterns to resolve, namely (i), (vii), and (viii). Note that in these three cases, there is at most one extra vertex of B.

In the following, we focus our attention on Pattern (i) but the same arguments apply to Patterns (vii) and (viii). As discussed above, we need to take care of property (h). If the pattern has no neighbor inside a C_4, then we can extend the triple (U, M, γ) where the vertices of A of this pattern that will not be labeled $\top$ are labeled $\vdash$. Note that in this case, properties (a) to (h) are maintained. Assume that there is a C_4 such that exactly one vertex of this C_4 is a neighbor of a vertex of K. This mean that this new C_4 as at least one vertex in B that is connected to K and so this vertex is of degree 3. Then we are in one of the cases depicted in Figure 6 (either the new C_4 is a C_4^1 or a C_4^2, and we can extend the triple (U, M, γ). Note that in Figure 6, we connected the new C_4 with the vertex on top of Pattern (i) but the same argument works if itis connected to the vertices on the left or the right. Assume now that it is not the case and there is a C_4 with two vertices of this C_4 that have a neighbor in K. Then we are in one of the cases depicted in Figure 7 corresponding to the way to select two vertices up to three. For Pattern (xviii), it cannot exist because of condition (1). For Pattern (xvi), we can extend the triple (U, M, γ) according to the figure. For Pattern (xvii), we assume that for both C_4's of this pattern, we cannot have either Pattern (xvi) or any of the patterns of Fig 6. Otherwise, we start by solving one of these patterns. This implies that we can extend the triple (U, M, γ) where the vertices of A of the pattern that will not be labeled $\top$ are labeled $\vdash$, and still respect property (h).

Case 3: there exists a C_4^2, that we name G', in $G \setminus U$.

We assume in this case that for every $u \in B \setminus U$, $d_{\bar{U}}(u) \geq 1$ and there is no C_4^1 in $G - U$. In particular, this implies that Case 2 will not occur anymore, and therefore, in the following property (h) will always be maintained. Thus, now we only focus on properties (a) to (g). Let $V(G') \cap A = \{a_1, a_2\}$ and $V(G') \cap B = \{b_1, b_2\}$. For this case, when we say that we define M' and γ' by updating the values of M and γ according to



Pattern (xvi).

Pattern (xvii).

Pattern (xviii).

Figure 7: Fifth case of C_4^1.

a figure, it means that we add to M every red edge of the figure and for every v of A that is depicted in the figure, then $\gamma'(v) = \top$ if v is either the endpoint of a red edge or $v \in U$ and $\gamma(v) = \top$, otherwise either v has a neighbor outside of the pattern depicted in the figure, and thus $\gamma'(v) = \vdash$, or it does not and thus $\gamma'(v) = \bot$.

First, assume that $N_G(b_1) = N_G(b_2) = \{a_1, a_2\}$. In this case, we need to distinguish between the labels of the vertices of $(N_G(b_1) \cup N_G(b_2)) \cap U$. There are four possibilities depicted in Figure 8. We define U' to be the union of U and the vertices of G', and we define M' and γ' by updating the values of M and γ according to Figure 8. One can check that, in each case, properties (a) to (g) are maintained.

Secondly, assume that there exists $a_3 \in A \backslash U$ such that $a_3 \in N_G(b_2)$ and $a_3 \notin N_G(b_1)$. Then we are in the case depicted in Figure 9. We define U' to be the union of U and the vertices of G', and we define M' and γ' by updating the values of M and γ according to Figure 9. One can check that properties (a) to (g) are maintained.

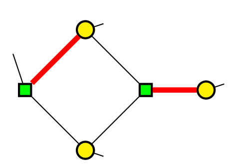


Figure 9: Second case of C_4^2.

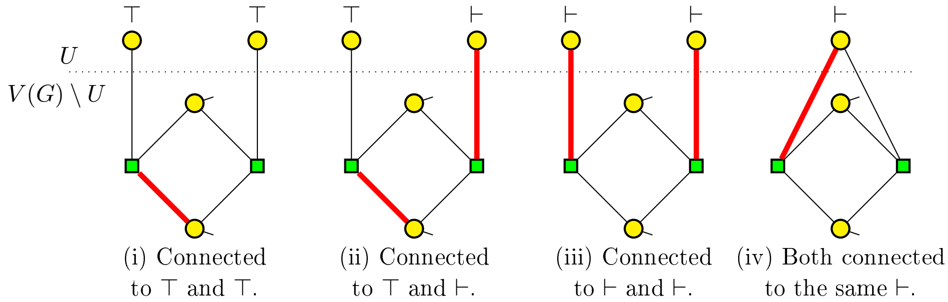


Figure 8: First cases of C_4^2. The vertices of $A \cap U$ labeled $\perp$ are treated in the same way than vertices labeled $\vdash$. If both are connected to the same vertex labeled $\top$, the case depicted in subfigure (i) applies.

Thirdly, assume that there exist a_3 and a_4 in $A \setminus U$ such that $a_3 \in N_G(b_2)$, $a_3 \notin N_G(b_1)$, $a_4 \in N_G(b_1)$, and $a_4 \notin N_G(b_2)$. As in Case 2, this situation is a bit more complicated to handle, but now we do not need to take care of property (h), which simplifies the case analysis compared to Case 2. Again, we shall ignore the vertices of B that became of degree 0 after the removal of the new U' when the situation is favorable to us, i.e., in exactly the same situations as in Case 2, and thus it does not interfere with the fact that the new triple (U', M', γ') respects property (g). We also sometimes forget the third neighbor of a vertex of A when its existence does not change how to resolve the pattern. Using the fact that, by condition (2), two vertices cannot have the same neighborhood, it follows that the possible patterns are those depicted in Figure 10. As in Case 2, these patterns corresponds to every way to connect at most 2 vertices of B with at least two neighbors in K to Pattern (i). As Pattern (ii) is not connected to the rest of the graph and has less than 10 neighbors, it has already been treated by the algorithm. Note that Pattern (iii) cannot exist because of condition (1). For all other patterns, namely Pattern (i) and Patterns (iv) to (ix), we define U' to be the union of U and the vertices of the given pattern, and we define M' and γ' by updating the values of M and γ according to Figure 9. One can check that properties (a) to (g) are maintained.

Case 4: there exists $u \in B \setminus U$ such that $d_{\bar{U}}(u) = 2$.

We assume in this case that for every $u \in B \setminus U$, $d_{\bar{U}}(u) \geq 2$, and that there is no C_4 in $G - U$.

Let $u \in B \setminus U$ such that $d_{\bar{U}}(u) = 2$. Let $\{v_1, v_2\} = N_G(u) \setminus U$ and $W = N_G(u) \cap U$, and $U' = U \cup \{u, v_1, v_2\}$. Note that both v_1 and v_2 have at least one neighbor that is not in U. Note also that $|W| \leq 1$.

Assume first that $W = \emptyset$ or $W = \{w\}$ and $\gamma(w) \neq \vdash$. Let M' arise from M by adding the edge uv_1. Let γ' be obtained from γ where $\gamma'(v_1) = \top$ and $\gamma'(v_2) = \vdash$. Clearly, replacing (U, M, γ) with (U', M', γ'), properties (a) to (f) are maintained. We obtain that $s' = s + 1$, $d' = d + 1$, and $f' = f$. These inequalities directly imply that property (g) is maintained.

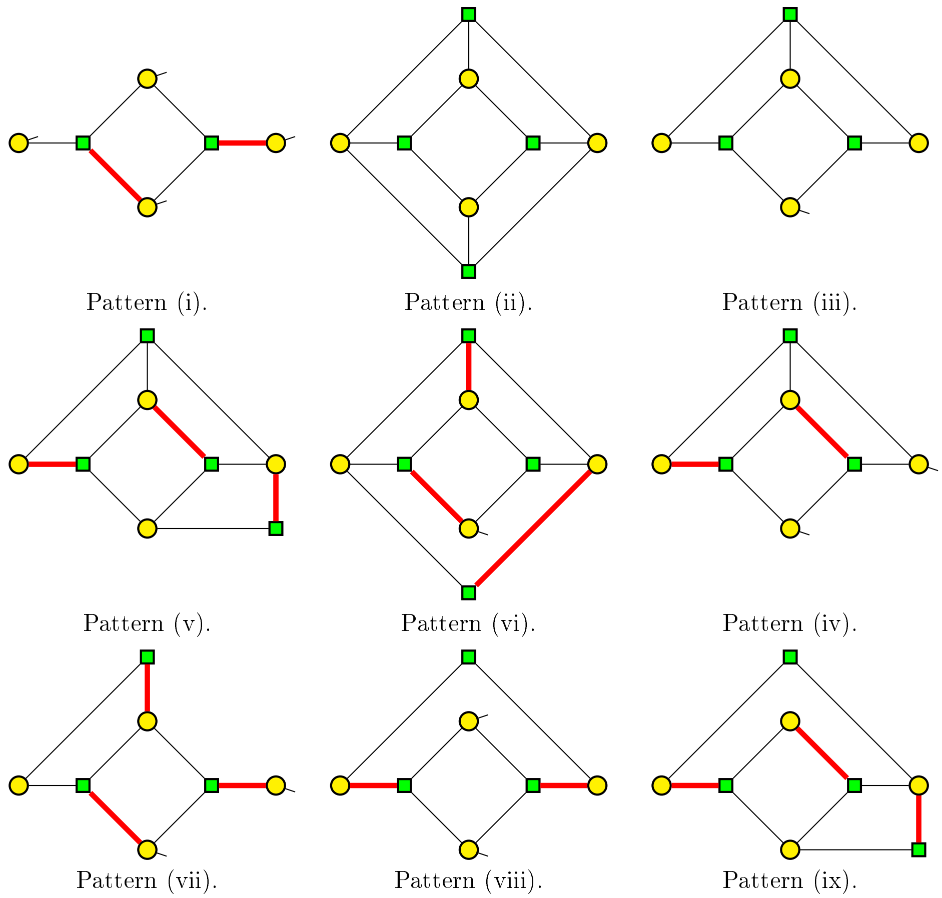


Figure 10: Third case of C_4^2.

Assume now that $W = \{w\}$ and $\gamma(w) =\vdash$. Let M' arise from M by adding the edge uw. Let γ' be obtained from γ where $\gamma'(v_1) = \gamma'(v_2) =\vdash$ and $\gamma'(w) = \top$. Clearly, replacing (U, M, γ) with (U', M', γ'), properties (a) to (f) are maintained. We obtain anew that $s' = s + 1$, $d' = d + 1$, and $f' = f$. These inequalities imply again that property (g) is maintained.

Since the considered cases exhaust all possibilities, and in each case we described an extension that maintains the relevant properties, the proof is complete up to the running time of the algorithm, which we proceed to analyze. One can easily check whether an extension operation can be realized in time $O(n)$, where $n = n(G)$. Indeed, we consider a constant number of patterns, and in each of them we fix a specific vertex. Then, for each vertex in $V(G) \setminus U$, we can check whether this vertex corresponds to a specific vertex of one of the patterns in constant time, by exploring the neighborhood at distance at most $p - 1$ from this vertex, where $p = 11$ is the size of the largest pattern (cf. Figure 6). As in each extension operation the size of U is incremented by at least one, it follows that the overall running time of the algorithm is $O(n^2)$. $\square$

Equipped with Lemma 2, we are now ready to prove Theorem 2.

Proof of Theorem 2: Again, we give an algorithmic proof such that the running time of the corresponding algorithm is polynomial in $n(G)$. Let $\alpha = \frac{5}{9}$ and let $\mathcal{G}$ be the set of all bipartite graphs G of maximum degree at most 3 such that every component of G has a vertex of degree at most 2. First, we prove that, for every given graph G in $\mathcal{G}$, one can find in polynomial time a uniquely restricted matching M of size at least $\alpha\nu_{ur}(G)$. Therefore, let G be in $\mathcal{G}$.

In order to be able to apply Lemma 2, we apply some reductions. Namely, as long as at least one of the following conditions is fulfilled in G, we apply the corresponding reduction, which is described and analyzed below:

- *Condition (1)*: G contains one of the patterns $R.1$, $R.2$, $R.3$, $R.4$, $R.5$, or $R.6$ depicted in Figure 2.
- *Condition (2)*: There exist two vertices in G with the same neighborhood.
- *Condition (3)*: There exists a vertex u in G of degree 1.
- *Condition (4)*: There exists a vertex u in G of degree 0.

Reduction (1). First, if there is in G a subgraph P isomorphic to the pattern $R.1$ or the pattern $R.2$ depicted in Figure 2(i) and Figure 2(ii), respectively, such that only the vertex x has a neighbor outside of P in G, then we define G' to be the graph obtained from G by removing every vertex of P except vertex x as depicted in Figure 2(i) and Figure 2(ii). Then, if there is in G a subgraph P isomorphic to the pattern $R.3$ or the pattern $R.4$ depicted in Figure 2(iii) and Figure 2(iv), respectively, such that only the vertices x and y have a neighbor outside of P in G, then we define G' to be the graph obtained from G by removing every vertex of P except vertices x and y as depicted in Figure 2(iii) ad Figure 2(iv). Finally, if there is in G a subgraph P isomorphic to the pattern $R.5$ or the pattern $R.6$ depicted in Figure 2(v) and Figure 2(vi), respectively, such that only the vertices x, y, and z have a neighbor outside of P in G, then we define G' to be the graph obtained from G by removing every vertex of P except vertices x, y, and z as depicted in Figure 2(v) and Figure 2(vi).

Let M^* be the set of red edges depicted in the corresponding figures. Then $\nu_{ur}(G') = \nu_{ur}(G) - |M^*|$. Indeed, in each case, we can exhaustively check that we cannot select more edges inside the pattern P, and in each configuration we provide, in the figures, a solution that leaves vertex x (and vertices y and z, if they exist) free to be taken from outside of P. Moreover, the choice of the red edges is such that they cannot be inside any alternating cycle, whatever the edges we select outside of P. If M' is a uniquely restricted matching of G', then $M' \cup M^*$ is a uniquely restricted matching of G.

Reduction (2). Assume that v and v' are two vertices having exactly the same neighborhood in G. We define $G' = G - \{v'\}$. Indeed, v and v' cannot be inside the same uniquely restricted matching, as otherwise there would exist an alternating cycle. This implies that $\nu_{ur}(G') = \nu_{ur}(G)$. Hence, we can safely remove v' from G. If M' is a uniquely restricted matching of G', then M' is a uniquely restricted matching of G.

Reduction (3). If G has a vertex u of degree 1, and v is the neighbor of u, then let $G' = G - \{u, v\}$. Clearly, $\nu_{ur}(G') = \nu_{ur}(G) - 1$, and if M' is a uniquely restricted matching of G', then $M' \cup \{uv\}$ is a uniquely restricted matching of G.

Reduction (4). If G has a vertex u of degree 0, then let $G' = G - \{u\}$. Clearly, $\nu_{ur}(G') = \nu_{ur}(G)$, and if M' is a uniquely restricted matching of G', then M' is a uniquely restricted matching of G.

In each of the four reductions defined above, note that the graph G' belongs to $\mathcal{G}$. Similarly to in the proof of Lemma 2, it can be checked whether a reduction can be realized in time $O(n)$, where $n = n(G)$.

By iteratively repeating these reductions, we eventually obtain a set M_1 of edges of G as well as a subgraph G_2 of G such that $G_2 \in \mathcal{G}$, G_2 does not contain a subgraph P isomorphic to one of the patterns $R.1$, $R.2$, $R.3$, $R.4$, $R.5$, or $R.6$ depicted in Figure 2, G_2 does not contain two vertices with the same neighborhood, $\nu_{ur}(G_2) = \nu_{ur}(G) - |M_1|$, $M_1 \cup M_2$ is a uniquely restricted matching of G for every uniquely restricted matching M_2 of G_2, and either $n(G_2) = 0$ or $\delta(G_2) \geq 2$. Now, by suitably choosing the bipartition of each component K of G_2, and applying Lemma 2 to K, one can determine in polynomial time a uniquely restricted matching M_2 of G_2 with $|M_2| \geq \alpha\nu_{ur}(G_2)$. Since the set $M_1 \cup M_2$ is a uniquely restricted matching of G of size at least $|M_1| + \alpha\nu_{ur}(G_2) \geq |M_1| + \alpha(\nu_{ur}(G) - |M_1|) \geq \alpha\nu_{ur}(G)$, the proof of our claim about $\mathcal{G}$ is complete. Note that the overall running time of the algorithm for graphs in $\mathcal{G}$ is $O(n^2)$, since each reduction strictly decreases the size of the graph, and the algorithm of Lemma 2 also runs in time $O(n^2)$.

Now, let G be a given connected bipartite graph G of maximum degree at most 3. If G is not 3-regular, then $G \in \mathcal{G}$, and the desired statement already follows. Hence, we may assume that G is 3-regular, which implies that its two partite sets A and B are of the same order. By [19], we can efficiently decide whether $\nu_{ur}(G) = \nu(G)$. Furthermore, if $\nu_{ur}(G) = \nu(G)$, then, again by [19], we can efficiently determine a maximum matching that is uniquely restricted. Hence, we may assume that $\nu_{ur}(G) < \nu(G)$. This implies that $\nu_{ur}(G) < |A|$, and, hence, there is some vertex u in $V(G)$ such that $\nu_{ur}(G-u) = \nu_{ur}(G)$. Since $G - u \in \mathcal{G}$ for every vertex u of G, considering the $n(G)$ induced subgraphs $G - u$ for $u \in V(G)$, one can determine in polynomial time a uniquely restricted matching M of G with $|M| \geq \max\{\alpha\nu_{ur}(G - u) : u \in V(G)\} = \alpha\nu_{ur}(G)$. Thus, we obtain an algorithm that finds the desired uniquely restricted matching. Note that since we make $O(n)$ calls to the algorithm for graphs in $\mathcal{G}$, the overall running time is $O(n^3)$. $\square$

4 Concluding remarks

Our results motivate several open problems. First of all, we believe that Theorem 2 extends to larger maximum degrees, that is, the conclusion of Theorem 1 should hold without the assumption of C_4-freeness. We also believe that better approximation factors are possible, and that approximation lower bounds in terms of the maximum degree could be proved. Finally, one could study the approximability of the uniquely restricted matching number in other classes of graphs.

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