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Josefran de Oliveira Bastos, Fabrício Siqueira Benevides, Guilherme Oliveira Mota, Ignasi Sau Valls. Counting Gallai 3-colorings of complete graphs. *Discrete Mathematics*, Elsevier, 2019, 342 (9), pp.2618-2631. 10.1016/j.disc.2019.05.015 . lirmm-02410619

HAL Id: lirmm-02410619

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Submitted on 13 Dec 2019

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COUNTING GALLAI 3-COLORINGS OF COMPLETE GRAPHS

JOSEFRAN DE OLIVEIRA BASTOS, FABRÍCIO SIQUEIRA BENEVIDES,
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ABSTRACT. An edge coloring of the n -vertex complete graph K_n is a *Gallai coloring* if it does not contain any rainbow triangle, that is, a triangle whose edges are colored with three distinct colors. We prove that the number of Gallai colorings of K_n with at most three colors is at most $7(n+1)2^{\binom{n}{2}}$, which improves the best known upper bound of $\frac{3}{2}(n-1)! \cdot 2^{\binom{n-1}{2}}$ in [Discrete Mathematics, 2017].

§1. INTRODUCTION

An edge coloring of the complete graph K_n is a *Gallai coloring* if it contains no *rainbow* K_3 , that is, a copy of K_3 in which all edges have different colors. A t -coloring is a edge coloring that uses at most t colors. The term *Gallai coloring* was used by Gyárfás and Simonyi in [20], but those colorings have also been studied under the name *Gallai partitions* by Körner, Simonyi and Tuza in [28]. See also [19] for a generalization to non-complete graphs and [12] for hypergraphs. The nomenclature is due to a close relation to a result in the influential Gallai's original paper [17] – translated to English and endowed by comments in [18]. The above mentioned papers are mostly concerned with structural and Ramsey-type results about Gallai colorings. For example, in [20] it was proved that any Gallai coloring can be obtained by substituting complete graphs with Gallai colorings into vertices of 2-colored complete graphs ([20, Theorem A]), and that any Gallai coloring contains a monochromatic spanning tree ([20, Theorem 2.2]).

We are interested in the problem of counting the number of Gallai colorings of K_n with a fixed set of colors, and focus here on the case where we use at most three colors. Here, we consider that the vertices of K_n are labeled. This problem has been investigated in the recent literature by other authors (under the more descriptive name *rainbow triangle-free colorings*). Actually, in a more general setting, in the past years there has been a growth

Key words and phrases. Gallai colorings, rainbow triangles, complete graphs, counting.

The first and second author were supported by CAPES Probral (Proc. 88887.143992/2017-00); the second author by CNPQ (Proc. 310512/2015-8 and Proc. 425297/2016-0) and FUNCAP; the third author by FAPESP (Proc. 2013/11431-2); and the fourth author was by projects DEMOGRAPH (ANR-16-CE40-0028) and ESIGMA (ANR-17-CE40-0028).

on the number of results about counting the number of structures that do not contain a particular kind of substructure, due to the recent development of modern and classic methods such as the Containers Method [7, 31], Regularity Method [1, 23, 25, 26], and the Entropy Compression Method [4, 15]: for example, counting sum-free sets in Abelian groups [3, 6], graphs without given subgraphs [8, 9, 24, 29], sets of integers with no k -term arithmetic progression [5], and B_h -sets [13], to cite only a few.

In [10], Benevides, Hoppen and Sampaio, motivated by a question of Erdős and Rothschild (see, e.g., [14]), studied the general problem of counting the number of colorings of a graph that avoid a subgraph colored with a given pattern (see also [2, 21, 22, 30]). The problem of computing exactly the number of Gallai colorings of K_n , which we denote by $c(n)$, appears to be hard even when we restrict the number of colors to be at most three. Since the vertices of K_n are labeled, we have the trivial lower bound $c(n) \geq 3 \cdot 2^{\binom{n}{2}} - 3$, given by the colorings that use only one or two of the three colors. To the best of our knowledge, the best known upper bound for $c(n)$ is $\frac{3}{2}(n-1)! \cdot 2^{\binom{n-1}{2}}$ (see [10]). We remark that a previous work using entropy, graph limits and the Container Method, lead to a general result that in turn implies a weaker upper bound of the form $2^{(1+o(1))\binom{n}{2}}$ (see [16]).

Our main result (see Theorem 1 below) improves the best known upper bound on $c(n)$. While most recent related results only work for sufficiently large structures, our result holds for every value of $n \geq 2$. Furthermore, we provide an elementary proof of such result, which is also relatively short.

Theorem 1. *For all $n \geq 2$ we have*

$$c(n) \leq 7(n+1) 2^{\binom{n}{2}}.$$

Even though our proof is completely mathematical, we also used computer search to calculate the exact number of Gallai colorings of small complete graphs (see Table 1 in the appendix) and to list all Gallai colorings of K_n for n up to 5. This provided us an insight on how to use a simple induction to estimate $c(n)$ for large values of n and how organize some case analysis for small values of n . We believe our strategy could also give good bounds when more colors are allowed, or even improve the above bound with a finer analysis (but probably leading to a much longer proof than we would like to show here).

Regardless of the nomenclature, it is also worth mentioning that Gallai colorings appear naturally in other fields, such as Information Theory [27] (what motivates the use of entropy for solving this kind of problems), and the perfect graph theorem [11].

In Sections 2.1 and 2.2 we estimate the number of extensions of colorings of the complete graphs that use, respectively, exactly two colors and exactly three colors. Most proofs in

those sections are done by induction. The base cases are usually tedious, so we moved some of them to the appendix (Section 4). For those cases we only need to check some particular colorings of K_4 , K_5 and K_6 , but we also did an extensive computer search for all Gallai colorings of K_n for $n \leq 8$, and a computer checked our results up to this value. The appendix also contains a table with the exact values of $c(n)$ for $n \geq 8$. In Section 2.3 we give a proof of Theorem 1.

§2. COUNTING THE MAXIMUM NUMBER OF EXTENSIONS

Let Φ_n be the set of all Gallai colorings of K_n that use colors red, green, and blue. So, each element of Φ_n is a function $\varphi : E(K_n) \rightarrow \{\text{red, green, blue}\}$, and $c(n) = |\Phi_n|$. For a fixed $\varphi \in \Phi_n$, we denote by $w(\varphi)$ the number of ways to extend φ to a Gallai coloring of the complete graph K_{n+1} . We think of K_{n+1} as obtained from K_n by adding a new vertex u , and $w(\varphi)$ as the number of ways to color all edges incident to u without creating a rainbow triangle given that $E(K_n)$ had already been colored as in φ . We start with the following trivial fact that calculates the number of extensions of a monochromatic coloring.

Fact 2. *Let $n \geq 2$ be an integer and let $\varphi \in \Phi_n$ be a monochromatic coloring of the edges of K_n . Then,*

$$w(\varphi) = 2^{n+1} - 1.$$

Proof. Without loss of generality assume that all edges of K_n are colored blue. Let $\{u\} = V(K_{n+1}) \setminus V(K_n)$. Notice that in any extension we cannot use colors red and green on two different edges between u and K_n . So, all extensions either use only colors blue and red, or only blue and green. As the extension in which all edges are blue is counted in both cases, there are $2^n + 2^n - 1$ extensions, as claimed. \square

An straightforward approach to estimate $c(n)$ is to use bounds on the parameter $w(\cdot)$ to bound $c(\cdot)$, as we have the trivial relation $c(n) = \sum_{\varphi \in \Phi_{n-1}} w(\varphi)$. Unfortunately, computing $w(\varphi)$ for each $\varphi \in \Phi_{n-1}$ may be as hard as the original problem. Also, a trivial but tight general upper bound on $w(\varphi)$, for $\varphi \in \Phi_{n-1}$, leads to a very weak upper bound on $c(n)$. But we can partition Φ_{n-1} into classes in a way that we know how to estimate the maximum value of $w(\cdot)$ in each of those classes and use this fact to get a better upper bound on $c(n)$. So, our strategy is to partition Φ_{n-1} into three classes: the monochromatic colorings, the colorings that use exactly two colors, and those that use exactly three colors. We denote those classes, $\Phi_n(1)$, $\Phi_n(2)$, and $\Phi_n(3)$ respectively, so that $\Phi_n = \Phi_n(1) \cup \Phi_n(2) \cup \Phi_n(3)$ is a disjoint union. We compute the maximum possible value of $w(\varphi)$ for φ in each of those classes and determine for which colorings this maximum is achieved. A coloring that has

the maximum number of extensions among colorings in its class will be called *extremal*. The underlying reasoning for this method to work is that there is a large gap between the maximums of those classes. Before that, we state a general result about the (maximum) number of extensions of an extension.

Lemma 3. *Let $\varphi \in \Phi_n$ be a Gallai coloring. If φ' is an extension of φ to $E(K_{n+1})$, then*

$$w(\varphi') \leq 2w(\varphi) + 1.$$

Proof. Let K_n be the complete graph on n vertices, V be its vertex set, and $\varphi \in \Phi_n$ be a Gallai coloring. Let φ' be any extension of φ to $E(K_{n+1})$ and $u \notin V$ be the new vertex (added to obtain K_{n+1}). To count the number of Gallai extensions of φ' to $E(K_{n+2})$, we will add a new vertex x and all edges from x to $V \cup \{u\}$. We first color the edges from x to V . If we let $t = w(\varphi)$, there are t colorings, say $\varphi_1, \dots, \varphi_t$, of the edges from x to V . For each $i \in \{1, \dots, t\}$ we let m_i be number of ways we can color the edge ux given that we have colored the edges from x to V as in φ_i . Clearly, $m_i \in \{0, 1, 2, 3\}$ and $w(\varphi') = \sum_{i=1}^t m_i$.

Fix any i , with $1 \leq i \leq t$. Recall that the edges from u to V are already colored (in φ'). If there is any vertex $v \in V$ such that $\varphi'(xv) \neq \varphi_i(uv)$, then there is a forbidden color for xu and $m_i \leq 2$. Therefore, the only way to have $m_i = 3$ is when the coloring φ_i is such that $\varphi_i(xy) = \varphi_i(uy)$ for every y in V (and for such coloring we have, indeed, $m_i = 3$). This implies that $\sum_{i=1}^t m_i \leq 2t + 1$. \square

We remark that when φ is a monochromatic coloring and φ' is its monochromatic extension, by Fact 2, we have $w(\varphi) = 2^{n+1} - 1$ and $w(\varphi') = 2^{n+2} - 1$. Therefore, $w(\varphi') = 2w(\varphi) + 1$, which implies that Lemma 3 is best possible.

In what follows we introduce a few definitions that play an important role in our proofs. Given a coloring of the edges of K_n and a vertex $v \in K_n$, we say that v is *monochromatic* if all edges incident to v have the same color. We also say that v is a *red vertex* (resp. *blue vertex*, *green vertex*) if all edges incident to v are red (resp. blue, green).

Given a vertex $v \in V(K_n)$, we denote by $\varphi^{\bar{v}}$ the restriction of the coloring φ to the complete graph K_{n-1} obtained by removing v from K_n . We will define some *special* colorings of $E(K_n)$ and later we will prove that those are the unique extremal colorings in each of their classes (see Figure 1).

First, any monochromatic coloring is considered special. Next, consider the case where $E(K_n)$ is colored with exactly two colors. We say that $\varphi \in \Phi_n(2)$ is *vertex-special* if there is a monochromatic vertex v , say of color c_1 , and all edges not incident to v have the same color, say c_2 , where $c_1 \neq c_2$. We say that φ is *edge-special* if all edges have the same color with the exception of exactly one edge.

Now assume that $\varphi \in \Phi_n(3)$. In this case, we say that φ is *vertex-special* if there is a monochromatic vertex v in a color $c \in \{\text{red, blue, green}\}$, and $\varphi^{\bar{v}}$ is an edge-special coloring with colors $\{\text{red, blue, green}\} \setminus \{c\}$. Furthermore, we say that φ is *edge-special* if there are two non-adjacent edges with different colors c_1 and c_2 in $\{\text{red, blue, green}\}$ and all other edges are colored with color $\{\text{red, blue, green}\} \setminus \{c_1, c_2\}$.

Finally, we say that φ is *special* to mean that φ is vertex-special or edge-special, and it is *non-special* otherwise. See Figure 1 for the non-monochromatic special 3-colorings of K_7 .

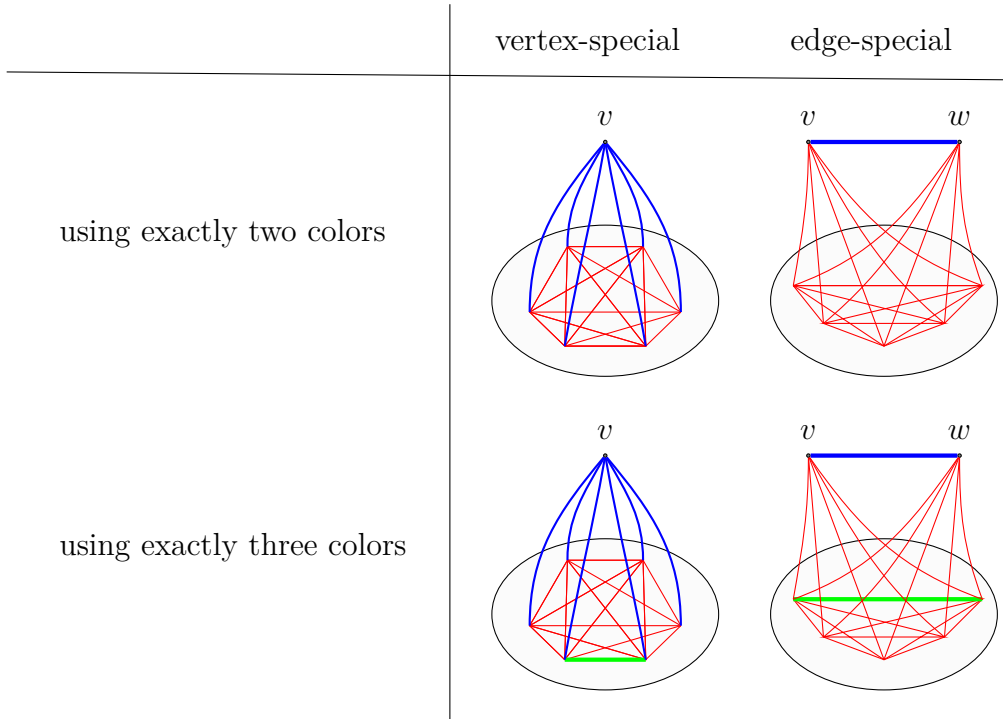


FIGURE 1. All special non-monochromatic colorings (up to isomorphism).

Many of the lemmas presented here are dedicated to compute the number of extensions of some particular colorings. In order to keep the notation simple, whenever it is clear from the context, we keep the same name, φ , for the coloring of $E(K_n)$ and a particular extension $\varphi : E(K_{n+1}) \rightarrow \{\text{red, green, blue}\}$ of it.

2.1. Graphs colored with exactly two colors. In the main result of this section (Lemma 6) we calculate a tight upper bound to the number of extensions of complete graphs that use exactly two colors. Before that, we compute the number of extensions of the special non-monochromatic 2-colorings.

Lemma 4. *For all $n \geq 3$, if $\varphi \in \Phi_n(2)$ is a special coloring of K_n using exactly two colors, then*

$$w(\varphi) = 3 \cdot 2^{n-1} + 1.$$

Proof. Let $n \geq 3$ be a fixed integer. Suppose first that φ is the vertex-special coloring of K_n with colors red and blue (see Figure 1) and $v \in K_n$ is its monochromatic vertex, say blue. Let u be the new vertex added to K_n to obtain K_{n+1} . We count the number of ways to color the edges from u to $V(K_n)$, considering three cases according to the color of uv . (As mentioned earlier, we will also call φ the extension).

Case $\varphi(uv) = \text{blue}$. Since we also have $\varphi(vx) = \text{blue}$ for every $x \in V(K_n) \setminus \{v\}$, there is no chance of having a rainbow triangle that uses v . And since $K_n - v$ is a monochromatic red graph, by Fact 2, we have $2^n - 1$ ways of coloring the edges from u to $V(K_n) \setminus \{v\}$.

Case $\varphi(uv) = \text{red}$. Since $\varphi(vx) = \text{blue}$ for every $x \in V(K_n) \setminus \{v\}$, then we cannot have any green edge from u to $V(K_n) \setminus \{v\}$. Furthermore, there is no restriction about using colors red or blue from u to $V(K_n) \setminus \{v\}$. Then, we have a total of 2^{n-1} ways of coloring the edges from u to $V(K_n) \setminus \{v\}$.

Case $\varphi(uv) = \text{green}$. For this case note that we cannot use color red on the edges between u and $V(K_n) \setminus \{v\}$, so they are all blue or green. Recall that $K_n - v$ is a monochromatic red graph, and since $n \geq 3$, we have that $|V(K_n) \setminus \{v\}| \geq 2$. Therefore, for any two distinct vertices $y_1, y_2 \in V(K_n) \setminus \{v\}$, we must have $\varphi(uy_1) = \varphi(uy_2)$ as otherwise we would have a rainbow triangle. Thus all edges from u to $V(K_n) \setminus \{v\}$ should have the same color (green or blue).

Then, we conclude that there are $w(\varphi) = (2^n - 1) + 2^{n-1} + 2 = 3 \cdot 2^{n-1} + 1$ ways to extend the coloring φ .

Now, suppose φ is the edge-special coloring of K_n with the colors red and blue, and $b_1b_2 \in E(K_n)$ is its only blue edge. Let u be the new vertex added to K_n to obtain K_{n+1} . Similar to the vertex-special coloring, we consider cases according to the colors of the edges ub_1 and ub_2 .

Case $\varphi(ub_1) = \varphi(ub_2) = \text{red}$. Note for this case that we do not have any restriction to the colors of the remaining edges incident to u , and as $K_n - b_1 - b_2$ is monochromatic, by Fact 2, there are $2^{n-1} - 1$ ways to color those remaining edges.

Cases ($\varphi(ub_1) = \text{blue}$ **and** $\varphi(ub_2) = \text{green}$) **or** ($\varphi(ub_1) = \text{green}$ **and** $\varphi(ub_2) = \text{blue}$). Note for each of these cases that there is only one way to color the edges from u to $V(K_n) \setminus \{b_1, b_2\}$ because all those edges must be red. So, this gives us two other extensions.

Other cases. There are other four cases for the colors of ub_1 and ub_2 (as the number of 3-colored Gallai extensions of a single colored edge is seven). In each of those, we

forbid the use of exactly one color to be used on the edges between u and $V(K_n) \setminus \{b_1, b_2\}$. Furthermore, as opposed to the previously analyzed case, we can color freely such edges with the available two colors. Thus, for each of the four remaining cases, we have exactly 2^{n-2} ways to color the remaining edges.

In total we have $w(\varphi) = 2^{n-1} - 1 + 2 + 4 \cdot 2^{n-2} = 3 \cdot 2^{n-1} + 1$ ways to extend φ . \square

We will also use the following fact.

Fact 5. *Let $k \geq 1$ and $n \geq \max\{4, 2k - 1\}$ be fixed integers. For any coloring φ of $E(K_n)$ with exactly k colors, there is a vertex v such that $\varphi^{\bar{v}}$ uses exactly k colors.*

Proof. For $k = 1$ the result is trivial. For $k = 2$ and $n \geq 4$, clearly there must be a vertex x that is incident to edges of both colors; say xa and xb have different colors. Letting v be any vertex in $V(K_n) - \{x, a, b\}$, we have that $\varphi^{\bar{v}}$ uses both colors.

So we may assume that $k \geq 3$ and $n \geq 2k - 1$. Let φ be an arbitrarily coloring of $E(K_n)$ with k colors. For a given $j \in \{1, \dots, k\}$, we denote by I_j the set of vertices v of $V(K_n)$ such $\varphi^{\bar{v}}$ does not use color j . Thus we want to show that $\bigcup_{j=1}^k I_j \neq V(K_n)$. For each $u \in I_j$, all edges with color j must be incident to u . Therefore, $|I_j| \leq 2$ and the only way to have $|I_j| = 2$ is when there is only one edge of color j .

Suppose for a contradiction that $\bigcup_{i=1}^k I_i = V(K_n)$. Since we have at least $2k - 1$ vertices and $|I_j| \leq 2$ for all $j \in \{1, \dots, k\}$, at least $k - 1$ of those sets must have size 2. Assume without loss of generality that $|I_1| = \dots = |I_{k-1}| = 2$. Then, the set $I_1 \cup \dots \cup I_{k-1}$ has at most $2k - 2 \leq n - 1$ vertices and contains exactly one edge for each color in $\{1, \dots, k - 1\}$. Therefore, all other edges induced by $I_1 \cup \dots \cup I_{k-1}$ must have color k (and as $k \geq 3$, this induced graph has more than $k - 1$ edges). So we can take v to be any vertex not in $I_1 \cup \dots \cup I_{k-1}$. \square

Lemma 6 below shows that the only extremal Gallai colorings that use exactly two colors are the special ones. In the proof of our main theorem (Theorem 1), for $\varphi \in \Phi_n(2)$ we will only need to use that $w(\varphi) \leq 3 \cdot 2^{n-1} + 1$. But we note that, instead of this, it is easier to prove that $w(\varphi) < 3 \cdot 2^{n-1}$ for non-special 2-colorings, because of the way we apply induction.

Lemma 6. *Given $n \geq 3$, let $\varphi: E(K_n) \rightarrow \{\text{red}, \text{blue}\}$ be a non-monochromatic coloring. Then, the following hold:*

- (1) *If φ is special, then $w(\varphi) = 3 \cdot 2^{n-1} + 1$.*
- (2) *If φ is non-special, then $w(\varphi) < 3 \cdot 2^{n-1}$.*

Proof. If φ is special, then we are done by Lemma 4. For the non-special colorings, we proceed by induction on n .

In what follows consider a non-special (in particular, non-monochromatic) 2-coloring $\varphi: E(K_n) \rightarrow \{\text{red, blue}\}$. Since every Gallai 2-coloring of K_3 is special, we may assume that $n \geq 4$. Our inductive step will only work for $n \geq 5$, so we also need to treat $n = 4$ as a base case.

Let $n = 4$ and φ be a non-special coloring of K_4 that uses colors red and blue. We will show that φ has to be isomorphic to one of the colorings depicted in Figure 2. Assuming this, we have that $w(\varphi) \leq 23 < 3 \cdot 2^3$ (see Lemma 11 in the appendix).

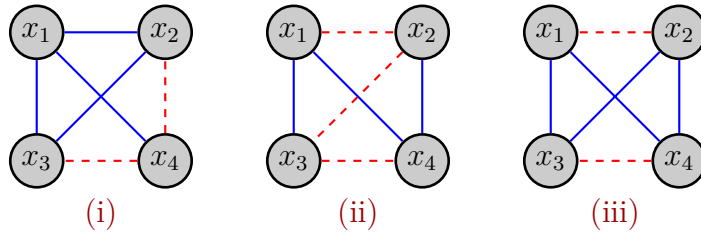


FIGURE 2. All non-special 2-colorings of K_4 (up to isomorphism). The number extension of each of them is computed in Lemma 11.

Let $V(K_4) = \{x_1, x_2, x_3, x_4\}$. If there is a monochromatic vertex, say blue, then the other three vertices form a triangle with two red edges and one blue edge, as otherwise φ would be special. So we have the coloring depicted in Figure 2-(i). Thus, we may assume that there is no monochromatic vertex. Since x_1 is not monochromatic, we may assume without loss of generality that x_1x_2 is red, and x_1x_3 and x_1x_4 are blue. Now, x_2 must have at least one blue neighbor and, by symmetry, we may assume that x_2x_4 is blue. Now x_4 must have a red neighbor and the only option is x_3 . Finally, depending on the color of x_2x_3 we either have the coloring depicted in Figure 2-(ii) or the one in Figure 2-(iii).

For the inductive step, suppose that $n \geq 5$ and that the result holds for any non-monochromatic 2-coloring of the edges of K_{n-1} . Recall that φ is a non-monochromatic coloring of $E(K_n)$ with colors red and blue. By Fact 5, there exists a vertex v such that $\varphi^{\bar{v}}$ is not monochromatic. If $w(\varphi^{\bar{v}}) < 3 \cdot 2^{n-2}$ then, by Lemma 3, we have $w(\varphi) \leq 2(3 \cdot 2^{n-2} - 1) + 1 = 3 \cdot 2^{n-1} - 1$. Thus, we may assume that $w(\varphi^{\bar{v}}) \geq 3 \cdot 2^{n-2}$. Then, from the inductive hypothesis, we conclude that $\varphi^{\bar{v}}$ is special. We will consider separately the cases where $\varphi^{\bar{v}}$ is vertex-special and edge-special.

Suppose first $\varphi^{\bar{v}}$ is vertex-special and, assume without loss of generality, that there is a blue vertex w in $\varphi^{\bar{v}}$. Let $\{x_1, \dots, x_{n-2}\} = V(K_n) \setminus \{v, w\}$. As $n \geq 5$, the coloring $\varphi^{\bar{v}_i}$ is not

monochromatic for any $1 \leq i \leq n - 2$ (even in $K_n - \{v, x_i\}$ there are two edges of different colors). As before, by Lemma 3, we conclude that $\varphi^{\overline{x_i}}$ is special, for any $1 \leq i \leq n - 2$. This implies that $\varphi(vw) = \text{blue}$, and $\varphi(vx_i) = \varphi(vx_j)$ for any $1 \leq i < j \leq n - 2$. If $\varphi(vx_i) = \text{red}$ for every i , then φ would be vertex-special (w would be a blue vertex and every other edge would be red), a contradiction. Thus, we may assume that $\varphi(vx_i) = \text{blue}$ for every i . Note that, since $\varphi^{\overline{x_i}}$ is extremal, this is possible only if $n = 5$, and the coloring is the one depicted in Figure 3 (with $\{v, w\} = \{y_1, y_2\}$). But for such coloring, we conclude that $w(\varphi) = 45 < 3 \cdot 2^4$ (see Lemma 12 in the appendix).

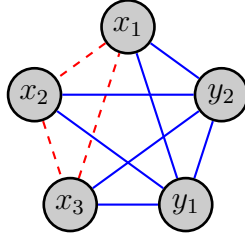


FIGURE 3. Coloring of K_5 isomorphic to the one of Lemma 12.

At last, suppose $\varphi^{\overline{v}}$ is edge-special and assume without loss of generality that there are vertices y and z with $\varphi^{\overline{v}}(yz) = \text{blue}$ and all other edges of $K_n - v$ are colored red. Let x_1, \dots, x_{n-3} be the other vertices of K_n . Similarly as before, we can easily conclude that $\varphi^{\overline{x_i}}$ is not monochromatic and, therefore, is special for any $1 \leq i \leq n - 3$. Note that since $\varphi^{\overline{x_i}}$ is special, we have $\varphi(vx_i) = \text{red}$, and $\varphi(vy) = \varphi(vz)$. If $\varphi(vy) = \varphi(vz) = \text{red}$, then φ would be edge-special (yz would be the only blue edge of φ), a contradiction. Thus, we may assume that $\varphi(vy) = \varphi(vz) = \text{blue}$. Note that, since $\varphi^{\overline{x_i}}$ is extremal, this is possible only if $n = 5$. Again, for such coloring, from Lemma 12, we conclude that $w(\varphi) = 45 < 3 \cdot 2^4$. \square

2.2. Graphs colored with exactly three colors. In the main result of this section (Lemma 9) we calculate a tight upper bound to the number of extensions of complete graphs that use exactly three colors. As in Section 2.1, we start by computing the number of extensions of special colorings on three colors.

Lemma 7. *For all $n \geq 4$, if $\varphi \in \Phi_n(3)$ is a special coloring of K_n using exactly three colors, then*

$$w(\varphi) = 2^n + 3.$$

Proof. Let $n \geq 4$ be a fixed integer. Suppose first that $\varphi \in \Phi_n(3)$ is a vertex-special coloring, $v \in V(K_n)$ is its monochromatic vertex, say blue, and $g_1g_2 \in E(K_n)$ is a green edge such that all the remaining edges are red. Let u be the new vertex added to K_n to obtain K_{n+1} . Similarly to the proof of Lemma 4, we count the number of ways to extend φ by considering cases according to the color of uv .

Case $\varphi(uv) = \text{blue}$. Since the vertex v is a monochromatic blue vertex, the fact that $\varphi(uv) = \text{blue}$ does not restrict the choices of the colors for the remaining edges. On the other hand, $\varphi^{\bar{v}}$ is the edge-special coloring for two colors. Thus, by Lemma 4, there are $3 \cdot 2^{n-2} + 1$ ways to color the remaining edges.

Case $\varphi(uv) = \text{red}$. Note that we will be able to use only colors red and blue on the edges from u to $V(K_n) \setminus \{v\}$. This already implies that no rainbow triangle contains v . Note that ug_1 and ug_2 must have the same color, but there are no extra restrictions to color the edges between u and $V(K_n) - v - g_1 - g_2$. Therefore, there are $2 \cdot 2^{n-3}$ colorings for this case.

Case $\varphi(uv) = \text{green}$. All edges between u and $V(K_n) \setminus \{v\}$ must be blue or green. Note that, as all edges between $\{g_1, g_2\}$ and $V(K_n) \setminus \{g_1, g_2, v\}$ are red, $\varphi(ug_1) = \varphi(ug_2)$, as otherwise we would have no color choice for the edge uy , for any $y \in V(K_n) \setminus \{g_1, g_2, v\}$ (recall that $n \geq 4$). On the other hand, once we choose the color of $\varphi(ug_1)$ the color of uy has to be the same for every $y \in V(K_n) \setminus \{g_1, g_2, v\}$. Thus, there are only two ways of coloring the remaining edges. Therefore, we conclude that $w(\varphi) = (3 \cdot 2^{n-2} + 1) + 2^{n-2} + 2 = 2^n + 3$.

Suppose that $\varphi \in \Phi_n(3)$ is the edge-special coloring. Let $b_1b_2 \in E(K_n)$ be the blue edge and $g_1g_2 \in E(K_n)$ be the green edge such that all the remaining edges are red. Let u be the new vertex added to K_n to obtain K_{n+1} . We consider the (seven) possible ways to color the edges ub_1 and ub_2 (so that the triangle ub_1b_2 is not rainbow). On the description of each case we list the values of $(\varphi(ub_1), \varphi(ub_2))$ considered.

Case (red, red): No matter how we color the edges between u and $K_n - b_1 - b_2$, there will be no rainbow triangle that contains the vertices b_1 or b_2 , so there is no extra restriction on the choices of the colors of those edges. Since $K_n - b_1 - b_2$ is the edge-special coloring on two colors, there are $3 \cdot 2^{n-3} + 1$ ways to color the edges between u and $K_n - b_1 - b_2$.

Cases (blue, blue) or (blue, red) or (red, blue): Each of these configurations only forbids us to use the color green on the remaining edges. On the other hand, for all possible ways to color the remaining edges using colors red and blue, the restriction is that ug_1 and ug_2 must have the same color. Therefore, there are $2 \cdot 2^{n-2} = 2^{n-1}$ ways to color them.

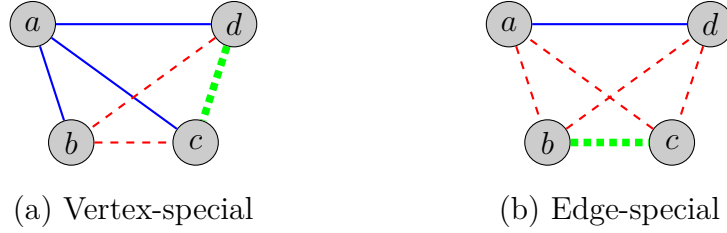


FIGURE 4. All special colorings of K_4 that use exactly three colors (up to isomorphism). These are particular cases of the second row of Figure 1.

Case (green, green): This configuration forbids us to choose the color blue for any remaining edges. But the coloring φ restricted to the graph $K_n - b_1 - b_2$ uses only colors red and green, so using red and green for edges between u and $V(K_n) - b_1 - b_2$ does not create any rainbow triangle. Thus, there are 2^{n-2} ways to color the remaining edges.

Cases (blue, green) or (green, blue): Each of these configurations forbids us to choose both green and blue for the remaining edges. Thus, there is only one way to color the remaining edges.

In total, we obtain $w(\varphi) = (3 \cdot 2^{n-3} + 1) + 3 \cdot 2^{n-3} + 2^{n-2} + 2 = 2^n + 3$. \square

Lemma 9 below is the analogue of Lemma 6 for three colors, and its proof is also based on Lemma 3 and induction on n . It shows that the special colorings are the only extremal ones in $\Phi_n(3)$. As with Lemma 6, in the proof of Theorem 1 we will only need that $w(\varphi) \leq 2^n + 3$ for every $\varphi \in \Phi_n(3)$. However, using induction, it is easier to prove that $w(\varphi) < 2^n$ for non-special colorings in $\Phi_n(3)$.

We will use the following simple fact.

Fact 8. *Every Gallai coloring of K_4 that uses exactly three colors is isomorphic to a special coloring. The special colorings of K_4 are depicted in Figure 4.*

Lemma 9. *Given $n \geq 4$, let $\varphi: E(K_n) \rightarrow \{\text{red, blue, green}\}$ be a Gallai coloring with exactly three colors. Then, the following hold:*

- (1) *If φ is special, then $w(\varphi) = 2^n + 3$.*
- (2) *If φ is non-special, then $w(\varphi) < 2^n$.*

Proof. If φ is special, then we are done by Lemma 7. For the rest we use induction on n . So, assume that $\varphi \in \Phi_n(3)$ is a non-special Gallai coloring.

By Fact 8, we may assume $n \geq 5$. For the inductive step to work we will need to have $n \geq 6$. The proof of the base case $n = 5$ is tedious (but not so long), and we postpone it

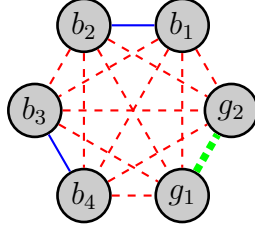
to the appendix. Lemma 15 shows that, for every non-special coloring $\varphi \in \Phi_5(3)$, we have $w(\varphi) \leq 31 < 2^5$.

Now, suppose that $n \geq 6$ and that Lemma 9 holds for any Gallai colorings in $\Phi_{n-1}(3)$. By Fact 5, there exists a vertex v such that $\varphi^{\bar{v}}$ still uses exactly three colors. If $w(\varphi^{\bar{v}}) < 2^{n-1}$ then, by Lemma 3, we have $w(\varphi) \leq 2(2^{n-1} - 1) + 1 = 2^n - 1$. Thus, we may assume that $w(\varphi^{\bar{v}}) \geq 2^{n-1}$. Then, by the inductive hypothesis, as $\varphi^{\bar{v}}$ uses all three colors, we conclude that $\varphi^{\bar{v}}$ is special. We will consider separately the cases where $\varphi^{\bar{v}}$ is vertex-special and edge-special.

Suppose first $\varphi^{\bar{v}}$ is vertex-special and assume without loss of generality that there is a blue vertex w in $\varphi^{\bar{v}}$, a unique green edge g_1g_2 , and let x_1, \dots, x_{n-4} be the other vertices of K_n (note that since $n \geq 6$, the number of vertices in x_1, \dots, x_{n-4} is at least two). Since $\varphi^{\bar{x}_i}$ uses all three colors for any $1 \leq i \leq n-4$, we know that $\varphi^{\bar{x}_i}$ is special, as otherwise we would have $w(\varphi^{\bar{x}_i}) < 2^{n-1}$ and Lemma 3 would finish the proof. But for $\varphi^{\bar{x}_i}$ to be edge-special, there should be at least two colors that appear each only once in $\varphi^{\bar{x}_i}$, and this is not the case for colors red and blue (as $n \geq 6$). Thus we may assume that $\varphi^{\bar{x}_i}$ is vertex-special for any $1 \leq i \leq n-4$. But note that then w should be the monochromatic (blue) vertex of $\varphi^{\bar{x}_i}$ for any $1 \leq i \leq n-4$, from where we conclude that $\varphi(vx_i) = \text{red}$ for any $1 \leq i \leq n-4$. Therefore, φ is vertex-special (with blue vertex w), a contradiction with the fact that φ is non-special.

At last, suppose $\varphi^{\bar{v}}$ is edge-special and assume without loss of generality that there are vertices b_1 and b_2 joined by the only blue edge, vertices g_1 and g_2 joined by the only green edge, and let x_1, \dots, x_{n-5} be the other vertices of K_n (note that since $n \geq 6$ there is at least one vertex in x_1, \dots, x_{n-5}). Similarly as before (using Lemma 3) we can conclude that $\varphi^{\bar{x}_i}$ is special for any $1 \leq i \leq n-5$. For $\varphi^{\bar{x}_i}$ to be vertex-special, there should be a monochromatic vertex w in $\varphi^{\bar{x}_i}$ with color $c \in \{\text{red}, \text{blue}, \text{green}\}$ such that c does not appear in $\varphi^{\bar{v}}$, but this is not possible. Thus we may assume that $\varphi^{\bar{x}_i}$ is edge-special for $1 \leq i \leq n-5$. But note that then v should be monochromatic in red in the coloring $\varphi^{\bar{x}_i}$. If $n \geq 7$, then there are at least two vertices in x_1, \dots, x_{n-5} and we know that $\varphi(vx_1) = \dots = \varphi(vx_{n-5}) = \text{red}$, from where we conclude that φ is edge-special, a contradiction. On the other hand, if $n = 6$, then it is not clear what is the color of the edge vx_1 . Clearly, if $\varphi(vx_1) = \text{red}$, then we get that φ is edge-special, obtaining the desired contradiction. Thus, we may assume that $\varphi(vx_1) \in \{\text{green}, \text{blue}\}$, in which case we obtain the coloring depicted in Figure 5. Therefore, $w(\varphi) = 53 < 2^6 + 3$ as necessary (by Lemma 16 in the appendix).

□

FIGURE 5. Coloring of K_6 isomorphic to the one of Lemma 16.

2.3. Proof of Theorem 1. Let $c_i(n) = |\Phi_n(i)|$, for $i \in \{1, 2, 3\}$, be the number of Gallai colorings of K_n that use exactly i colors. Then,

$$c(n) = c_1(n) + c_2(n) + c_3(n). \quad (1)$$

Clearly, $c_1(n) = 3$ for every $n \geq 2$, and $c_2(n) = 3 \cdot (2^{\binom{n}{2}} - 2)$ for every $n \geq 3$. Also notice that $c_3(1) = c_3(2) = c_3(3) = 0$. In particular, $c(2) = 3$ and $c(3) = 21$.

So, our aim is to estimate $c_3(n)$ for $n \geq 4$. Fix $i \in \{1, 2, 3\}$. For each coloring φ in $\Phi_n(i)$ count the number of extensions of φ such that the resulting coloring of $E(K_{n+1})$ uses all three colors, and let $w_i(n)$ be the maximum of those numbers.

Since there is no way to extend a monochromatic coloring of K_n to a Gallai coloring of K_{n+1} that uses all three colors, $w_1(n) = 0$ for every n . Using Lemma 6, for $n \geq 3$, we obtain $w_2(n) \leq (3 \cdot 2^{n-1} + 1) - 2^n = 2^{n-1} + 1$. In fact, for any non-monochromatic 2-coloring of K_n , say with red and blue, its number of extensions is at most $3 \cdot 2^{n-1} + 1$ and among those there are at least 2^n extensions that are valid extensions but use only colors red and blue. Using Lemma 9, for $n \geq 4$, we have $w_3(n) \leq 2^n + 3$. And because $c_3(3) = 0$, the following holds for every $n \geq 3$:

$$\begin{aligned} c_3(n+1) &\leq w_1(n)c_1(n) + w_2(n)c_2(n) + w_3(n)c_3(n) \\ &\leq 0 + (2^{n-1} + 1) \cdot (3 \cdot 2^{\binom{n}{2}} - 6) + (2^n + 3) \cdot c_3(n) \end{aligned} \quad (2)$$

$$\leq 3 \cdot (2^{n-1} + 1) \cdot 2^{\binom{n}{2}} + (2^n + 3) \cdot c_3(n). \quad (3)$$

We claim that $c_3(n) \leq 7n 2^{\binom{n}{2}}$ for every n . Setting $n = 3$ at inequality (2) gives $c_3(4) \leq 18 \cdot 5 = 90 < 7 \cdot 4 \cdot 2^{\binom{4}{2}}$. So, the claim holds for every $n \leq 4$. Now assume that it holds for some particular $n \geq 4$ and let us show that it holds for $n + 1$.

From inequality (3), we have

$$\begin{aligned} c_3(n+1) &\leq 3 \cdot (2^{n-1} + 1) \cdot 2^{\binom{n}{2}} + (2^n + 3) \cdot 7n \cdot 2^{\binom{n}{2}} \\ &\leq 7(n+1) 2^{\binom{n+1}{2}}. \end{aligned} \tag{4}$$

To see that (4) holds, note that $2^{\binom{n+1}{2}} = 2^n 2^{\binom{n}{2}}$. Finally, using (1), we have

$$\begin{aligned} c(n) &\leq 3 + \left(3 \cdot 2^{\binom{n}{2}} - 6\right) + 7n 2^{\binom{n}{2}} \\ &\leq 3 \cdot 2^{\binom{n}{2}} + 7n 2^{\binom{n}{2}} \\ &\leq 7(n+1) 2^{\binom{n}{2}}. \end{aligned}$$

Remark. To keep the proof simple, we did not optimize the constant multiplying $(n+1)$ above. However, this is not hard to do. If we consider a function $f(n)$ such that $f(1) = f(2) = f(3) = 0$ and that satisfies the analogous of (3) with equality, that is,

$$f(n+1) = 3 \cdot 2^n \cdot 2^{\binom{n}{2}} + (2^n + 3)f(n),$$

it is trivial to see (by induction) that $c_3(n) \leq f(n)$ for every n . To solve such recursion we can use the substitution $f(n) = k(n) 2^{\binom{n}{2}}$. We get the linear recursion

$$k(n+1) = 3 + \left(1 + \frac{3}{2^n}\right) k(n),$$

that can be solved exactly with standard methods, noting also that the product $\prod_{n=3}^{\infty} \left(1 + \frac{3}{2^n}\right)$ is convergent. It turns out that $k(n)$ is asymptotically approximately tn , for some constant $t < 7$. Similarly, we could also define $f(n)$ using inequality (2) instead of (3), but again, this would one would only improve this bound slightly.

§3. CONCLUDING REMARKS

Recall that $c_i(n) = |\Phi_n(i)|$, for $i \in \{1, 2, 3\}$, is the number of Gallai colorings of K_n that use exactly i colors from a total of 3 colors. In Table 1 below we show the exact values of $c_1(n)$, $c_2(n)$, $c_3(n)$, and $c(n)$ for $n \leq 8$. These values were obtained by enumerating (with a computer program) all Gallai colorings of K_n .

The previous best known upper bound on the number of Gallai colorings of K_n was asymptotically $(n-1)! 2^{\binom{n}{2}}$. In Theorem 1 we improve this substantially showing that the number of Gallai 3-colorings of K_n is at most $7(n+1) 2^{\binom{n}{2}}$. But since the lower bound is $3 \cdot 2^{\binom{n}{2}} - 3$, there is still a large gap between the lower and upper bounds. Even though Table 2 considers only very small values of n , it may indicate that $c(n)$ is asymptotically closer to the lower bound. However, in Table 2, we compute the ratios of our upper bound to $c(n)$ and of $c(n)$ to that lower bound. We found, surprisingly, that neither of those are

monotone. Again, this could simply be some artifact due to the fact that n is very small, but it could also indicate that there is a term multiplying $2^{\binom{n}{2}}$ that is not a constant. It would be interesting to fully understand the behavior of $c(n)$ (even only for large n). It would also be interesting to study what happens when we consider more colors.

n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c(n)$
2	3	-	-	3
3	3	18	-	21
4	3	186	90	279
5	3	3,066	3,060	6,129
6	3	98,298	112,686	210,987
7	3	6,291,450	5,522,496	11,813,949
8	3	805,306,362	407,207,826	1,212,514,191

TABLE 1. Values of $c_1(n)$, $c_2(n)$, $c_3(n)$, and $c(n)$.

n	$c(n)$	$3 \cdot 2^{\binom{n}{2}} - 3$	$7(n+1)2^{\binom{n}{2}}$	$7(n+1)2^{\binom{n}{2}}/c(n)$	$c(n)/(3 \cdot 2^{\binom{n}{2}} - 3)$
2	3	3	42	14.00	1.00
3	21	21	224	10.66	1.00
4	279	189	2,240	8.02	1.47
5	6,129	3,069	43008	7.01	1.99
6	210,987	98,301	1,605,632	7.61	2.14
7	11,813,949	6,291,453	117,440,512	9.94	1.87
8	1,212,514,191	805,306,365	16,911,433,728	13.94	1.50

TABLE 2. Comparison between $c(n)$ and the lower and upper bounds.

REFERENCES

- [1] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Blow-up lemmas for sparse graphs*, ArXiv e-prints (Dec. 2016), available at [arXiv:1612.00622](https://arxiv.org/abs/1612.00622). [↑2](#)
- [2] N. Alon, J. Balogh, P. Keevash, and B. Sudakov, *The number of edge colorings with no monochromatic cliques*, *Journal of the London Mathematical Society* **70** (2004), no. 2, 273–288. [↑2](#)
- [3] N. Alon, J. Balogh, R. Morris, and W. Samotij, *Counting sum-free sets in abelian groups*, *Israel Journal of mathematics* **199** (2014), no. 1, 309–344. [↑2](#)

- [4] R. G. Alves, A. Procacci, and R. Sanchis, *A local lemma via entropy compression*, ArXiv e-prints (Sep. 2015), available at [arXiv:1509.04638](https://arxiv.org/abs/1509.04638). [↑2](#)
- [5] J. Balogh, H. Liu, and M. Sharifzadeh, *The number of subsets of integers with no k -term arithmetic progression*, International Mathematics Research Notices **2017** (2016), no. 20, 6168–6186. [↑2](#)
- [6] J. Balogh, H. Liu, M. Sharifzadeh, and A. Treglown, *The number of maximal sum-free subsets of integers*, Proceedings of the American Mathematical Society **143** (2015), no. 11, 4713–4721. [↑2](#)
- [7] J. Balogh, R. Morris, and W. Samotij, *Independent sets in hypergraphs*, Journal of the American Mathematical Society **28** (2015), no. 3, 669–709. [↑2](#)
- [8] J. Balogh and Š. Petříčková, *The number of the maximal triangle-free graphs*, Bulletin of the London Mathematical Society **46** (2014), no. 5, 1003–1006. [↑2](#)
- [9] J. Balogh and W. Samotij, *The number of k m , m -free graphs*, Combinatorica **31** (2011), no. 2, 131. [↑2](#)
- [10] F. S. Benevides, C. Hoppen, and R. M. Sampaio, *Edge-colorings of graphs avoiding complete graphs with a prescribed coloring*, Discrete Mathematics **340** (2017), no. 9, 2143–2160. [↑2](#)
- [11] K. Cameron, J. Edmonds, and L. Lovász, *A note on perfect graphs*, Periodica Mathematica Hungarica **17** (1986), no. 3, 173–175. [↑2](#)
- [12] L. Chua, A. Gyárfás, and C. Hossain, *Gallai-colorings of triples and 2-factors of*, International Journal of Combinatorics **2013** (2013). [↑1](#)
- [13] D. Dellamonica, Y. Kohayakawa, S. J. Lee, V. Rödl, and W. Samotij, *On the number of b h -sets*, Combinatorics, Probability and Computing **25** (2016), no. 1, 108–129. [↑2](#)
- [14] P. Erdős, *Some new applications of probability methods to combinatorial analysis and graph theory*, University of Calgary, Department of Mathematics, Statistics and Computing Science, 1974. [↑2](#)
- [15] L. Esperet and A. Parreau, *Acyclic edge-coloring using entropy compression*, European J. Combin. **34** (2013), no. 6, 1019–1027. [↑2](#)
- [16] V. Falgas-Ravry, K. O’Connell, J. Strömberg, and A. Uzzell, *Multicolour containers and the entropy of decorated graph limits*, preprint arXiv:1607.08152 (2016). [↑2](#)
- [17] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar **18** (1967), 25–66. [↑1](#)
- [18] T. Gallai, *A translation of T. Gallai’s paper: “Transitiv orientierbare Graphen” [Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66; MR0221974 (36 #5026)]*, Perfect graphs, 2001, pp. 25–66. Translated from the German and with a foreword by Frédéric Maffray and Myriam Preissmann. [↑1](#)
- [19] A. Gyárfás and G. N. Sárközy, *Gallai colorings of non-complete graphs*, Discrete Mathematics **310** (2010), no. 5, 977–980. [↑1](#)
- [20] A. Gyárfás and G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, Journal of Graph Theory **46** (2004), no. 3, 211–216. [↑1](#)
- [21] C. Hoppen, H. Lefmann, and K. Odermann, *A rainbow erdős-rothschild problem*, Electronic Notes in Discrete Mathematics **49** (2015), 473–480. [↑2](#)
- [22] ———, *On graphs with a large number of edge-colorings avoiding a rainbow triangle*, European Journal of Combinatorics **66** (2017), 168–190. [↑2](#)
- [23] Y. Kohayakawa, *Szemerédi’s regularity lemma for sparse graphs*, Foundations of computational mathematics (Rio de Janeiro, 1997), 1997, pp. 216–230. [↑2](#)

- [24] Y. Kohayakawa, B. Kreuter, and A. Steger, *An extremal problem for random graphs and the number of graphs with large even-girth*, *Combinatorica* **18** (1998), no. 1, 101–120. [↑2](#)
- [25] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, *The regularity lemma and its applications in graph theory*, *Theoretical aspects of computer science (Tehran, 2000)*, 2002, pp. 84–112. [↑2](#)
- [26] J. Komlós and M. Simonovits, *Szemerédi’s regularity lemma and its applications in graph theory* (1996). [↑2](#)
- [27] J. Körner and G. Simonyi, *Graph pairs and their entropies: modularity problems*, *Combinatorica* **20** (2000), no. 2, 227–240. [↑2](#)
- [28] J. Körner, G. Simonyi, and Z. Tuza, *Perfect couples of graphs*, *Combinatorica* **12** (1992), no. 2, 179–192. [↑1](#)
- [29] R. Morris and D. Saxton, *The number of $C_{2\ell}$ -free graphs*, *Advances in Mathematics* **298** (2016), 534–580. [↑2](#)
- [30] O. Pikhurko, K. Staden, and Z. B. Yilma, *The erdős–rothschild problem on edge-colourings with forbidden monochromatic cliques*, *Mathematical proceedings of the cambridge philosophical society*, 2017, pp. 341–356. [↑2](#)
- [31] D. Saxton and A. Thomason, *Hypergraph containers*, *Inventiones mathematicae* **201** (2015), no. 3, 925–992. [↑2](#)

§4. APPENDIX

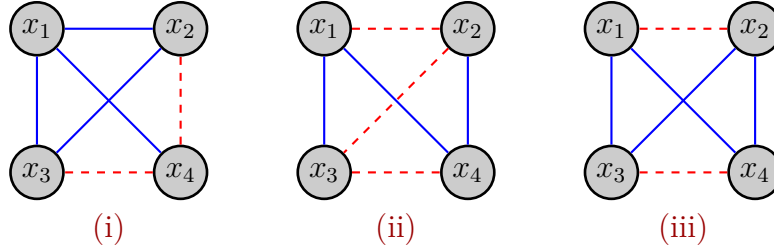
Here, we compute the number of Gallai extensions of some particular colorings of K_4 and K_5 . We also describe all Gallai 3-colorings of K_5 that do not have a monochromatic vertex (up to isomorphism). In most proofs of this appendix we use repeatedly, and sometimes implicitly, the following trivial fact.

Fact 10. *Let $c \in \{\text{red, blue}\}$ and $\varphi: E(K_n) \rightarrow \{\text{red, blue}\}$. Consider an extension of φ to a Gallai coloring of $E(K_{n+1})$ with colors red, green, or blue. Let u be the vertex added to K_n to obtain K_{n+1} . If vw is an edge of the initial K_n , say with color blue, and $\varphi(uv) = \text{green}$, then $\varphi(uw)$ cannot be red. In particular, if there are vertices v and w in the initial K_n such that all edges between v and $K_n - v$ are blue and $\varphi(uv) = \text{green}$, then there are no red edges between u and $K_n - w$.*

Let us start with the colorings of $E(K_4)$ isomorphic to those depicted in Figure 2. For convenience, we show Figure 2 again below.

Lemma 11. *Let $c \in \{\text{red, blue}\}$ and consider a coloring $\varphi: E(K_4) \rightarrow \{\text{red, blue}\}$. Then the following hold:*

- (i) *If there are only two edges with color c and they form a path, then $w(\varphi) = 23$ (see Figure 2-(i)).*



(ii) If there are only three edges with color c and they form a path, then $w(\varphi) = 21$ (see Figure 2-(ii)).

(iii) If there are only two edges with color c and they form a matching, then $w(\varphi) = 23$ (see Figure 2-(iii)).

Proof. The proofs of all items are similar and simple, but we show them here for completeness. Let $\varphi: E(K_4) \rightarrow \{\text{red, blue}\}$ and without loss of generality assume $c = \text{red}$. Let v be a new vertex and add all edges between v and the four vertices of K_4 . Let us count in how many ways we can color the edges incident to v , with colors red, blue, and green without creating rainbow triangles.

Clearly, there are exactly 2^4 possible ways to color the edges incident to v without using color green. By Fact 10, there are exactly four ways in which we use exactly one green edge (we can only choose which edge receives color green, and the color of the remaining edges are forced). Furthermore, clearly there is only one way in which all four edges are green. This adds up to 21 possible extensions. It remains to count the number of extensions such that there are two or three green edges incident to v .

Proof of item (i). Let $\{x_1, x_2, x_3\}$ be a blue triangle where x_1 is a blue vertex, and let x_4 be the other vertex of K_4 , so $\varphi(x_2x_4) = \varphi(x_4x_3) = \text{red}$. By Fact 10, there is only one way to have exactly three green edges incident to v , which is with $\varphi(vx_2) = \varphi(vx_3) = \varphi(x_4v) = \text{green}$ (and then $\varphi(vx_1)$ is forced to be blue). Moreover, the only way to have exactly two green edges incident to v is with $\varphi(vx_2) = \varphi(vx_3) = \text{green}$ (the colors of the other edges are forcibly determined and valid). This gives, in total, $2 + 21 = 23$ extensions of φ to a Gallai coloring.

Proof of item (ii). Let $W \subset V(K_4)$ be any set with two or three vertices. Note that there is always a vertex $y \in V(K_4) \setminus W$ such that y has a blue and a red neighbor in W . So, if the green edges incident to v are exactly those with an endpoint in W , by Fact 10, there is no color available for vy . Therefore, there is no way of extending φ to a Gallai coloring using exactly two or three green edges incident to v . So there are only the 21 previous

extensions to a Gallai coloring.

Proof of item (iii). Let $\{x_1, x_2, x_3, x_4\}$ be the set of vertices of K_4 such that x_1x_2 and x_3x_4 are the red edges, and all other edges are blue. Let $W \subset V(K_4)$ be any set with exactly three vertices. As in the previous item, the vertex $y \in V(K_4) \setminus W$ has a red and a blue neighbor in W , so there is no extension of φ that uses exactly three green edges incident to v . On the other hand, if we want exactly two green edges incident to v , then there are exactly two possible ways: $\varphi(vx_1) = \varphi(vx_2) = \text{green}$ or $\varphi(vx_3) = \varphi(vx_4) = \text{green}$ (the colors of the other edges are forcibly determined). This gives, in total, $2 + 21 = 23$ extensions of φ to a Gallai coloring. \square

Lemma 12 provides the number of extensions of colorings isomorphic the one in Figures 3.

Lemma 12. *Let $c \in \{\text{red}, \text{blue}\}$ and consider a coloring $\varphi: E(K_5) \rightarrow \{\text{red}, \text{blue}\}$. If there are only three edges with color c and they form a triangle, then $w(\varphi) = 45$.*

Proof. Let $\varphi: E(K_5) \rightarrow \{\text{red}, \text{blue}\}$ be a coloring as in the statement of this lemma and assume without loss of generality that $c = \text{red}$. Let $X = \{x_1, x_2, x_3\}$ be the vertices that form the red triangle, $Y = \{y_1, y_2\}$ the other two vertices of K_5 , and let v be a new vertex adjacent to all other vertices. Let us show that there are 45 ways to color the edges incident to v , with colors red, blue, and green without creating rainbow triangles.

First of all notice that there are exactly 32 possible ways to color the edges incident to v without using green edges. Now assume that there is at least one green edge incident to v . If there is a green edge between v and Y , then there are no red edges between v and X , by Fact 10. But since the edges inside X are red, we conclude that $\varphi(vx_1) = \varphi(vx_2) = \varphi(vx_3)$. Thus, there are only 2 possibilities of colors for the edges between v and X (they are all blue or all green). Since there are 3 ways to color the edges between v and Y using color green at least once, this yields 6 extensions of φ . One may check that all 6 extensions are valid.

It remains to count the number of extensions with no green edges between v and Y and at least one green edge between v and X . In this case the edges between v and Y must be blue, and those between v and X must be red or green (with at least one green edge). Then, there are 7 possible extensions of φ . Therefore, in total there are $32 + 6 + 7 = 45$ extensions of φ to a Gallai coloring. \square

The structure of the proof of the next result, Lemma 13, is very similar to the one of Lemma 7, but we include its proof here for completeness.

Lemma 13. *Let φ be a non-special Gallai coloring of K_5 that uses exactly three colors and has a monochromatic vertex v . Then, $w(\varphi) \leq 31$.*

Proof. Let φ be a non-special Gallai coloring of K_5 that uses exactly three colors and let $v \in V(K_n)$ be a monochromatic vertex, say blue. Suppose first that $\varphi^{\bar{v}}$ uses all three colors. Recall that the only two colorings of $K_4 = K_5 - v$ that use three colors are the special colorings. So, by Lemma 7, we have $w(\varphi^{\bar{v}}) = 2^4 + 3 = 19$.

Now, we follow the same steps as in the proof for vertex-special colorings in Lemma 7. We add a new vertex u and we want to count in how many way we can color all edges between u and $V(K_n)$. As before, we consider the same cases according to the color of uv .

Case $\varphi(uv) = \text{blue}$. This imposes no restriction on the color of the edges between v and $V(K_5) \setminus \{v\}$. Therefore, the number of extensions in this case is simply $w(\varphi^{\bar{v}}) = 2^4 + 3 = 19$.

Case $\varphi(uv) = \text{red}$. We are able to use only colors red and blue on the edges between u and $V(K_n) \setminus \{v\}$. This would give at most 2^4 extensions. However, for each green edge, say g_1g_2 , induced by $V(K_n) \setminus \{v\}$, the colors of ug_1 and ug_2 must be the same (otherwise u, g_1, g_2 would be rainbow). As we have at least one green edge, $w(\varphi^{\bar{v}}) \leq 2^3$, with equality if and only if there is only one green edge.

Case $\varphi(uv) = \text{green}$. Similar to the previous case, we obtain $w(\varphi^{\bar{v}}) \leq 2^3$, with equality only if there is only one red edge induced by $V(K_n) \setminus \{v\}$.

This would give a total of $19 + 8 + 8 = 35$ extensions, but to have exactly this amount, it must be the case where there is only one red and only one green edge. In this case, we would have an edge-special coloring, that is not possible. Therefore, either there are two green or two red edges, in which case the number of extensions of φ is at most $19 + 8 + 4 = 31$.

Now, assume that $\varphi^{\bar{v}}$ uses only two colors. As all edges incident to v are blue, and we must use all three colors, we must use colors red and green on $\varphi^{\bar{v}}$. It is a well-known fact that a 2-colored K_n must have a monochromatic spanning tree. Suppose without loss of generality that such spanning tree is red. Furthermore, notice that if there is only one green edge, then the original coloring, φ , on K_5 will be vertex-special and that is not possible.

Case $\varphi(uv) = \text{blue}$. The number of extensions in this case is simply $w(\varphi^{\bar{v}})$, which by Lemma 4 is at most $3 \cdot 2^3 + 1 = 25$.

Case $\varphi(uv) = \text{red}$. We are able to use only colors red and blue on the edges between u and $V(K_n) \setminus \{v\}$. This would give at most 2^4 extensions. But, since $\varphi^{\bar{v}}$ has at least two green edges, there are at most $2^2 = 4$ ways to color the edges between u and $V(K_n) \setminus \{v\}$.

Case $\varphi(uv) = \text{green}$. All edges between u and $V(K_n) \setminus \{v\}$ must be blue or green. The fact that $\varphi^{\bar{v}}$ has a red spanning tree implies that all edges between u and $V(K_n) \setminus \{v\}$ must receive the same color. So, there are only two ways to color those edges.

In total, there are at most $25 + 4 + 2 = 31$ extensions. □

The colorings of K_5 depicted in Figure 6 are important for the proof of the next result, Lemma 14.

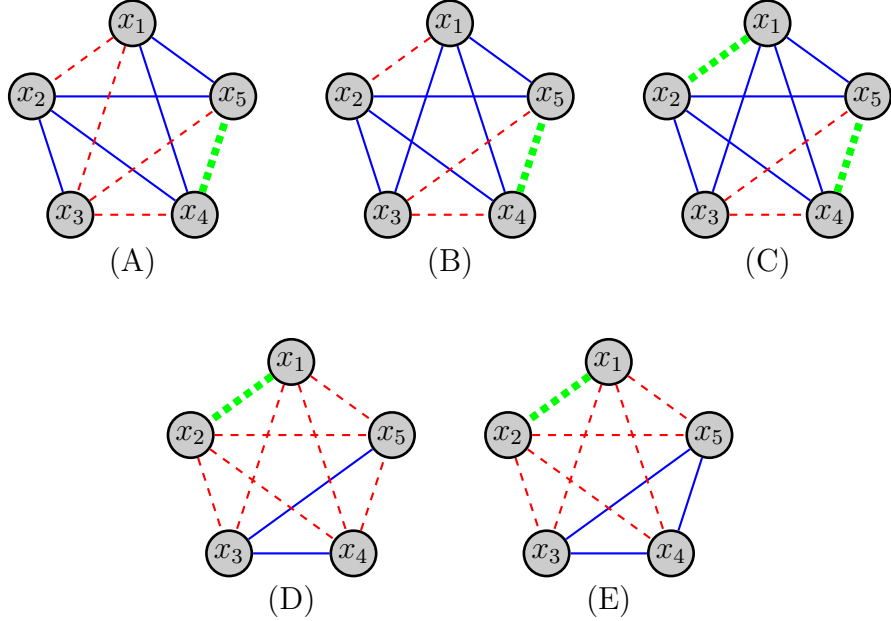


FIGURE 6. Colorings of K_5 that are non-special and have no monochromatic vertex.

Lemma 14. *Every non-special Gallai coloring of K_5 that uses exactly three colors and does not have a monochromatic vertex is isomorphic to one of the colorings depicted in Figure 6.*

Proof. By Fact 5, there exists a vertex v such that $\varphi^{\bar{v}}$ uses all three colors. Since $K_5 - v$ has only 4 vertices, it follows that $\varphi^{\bar{v}}$ is a special coloring. Let $V(K_5 - v) = \{a, b, c, d\}$. By Lemma 7, we have $w(\varphi^{\bar{v}}) = 19$. This means that there are 19 ways to color the edges between v and $\{a, b, c, d\}$ for every Gallai coloring of $V(K_5 - v)$ with exactly 3 colors.

We consider separately the cases where $\varphi^{\bar{v}}$ is vertex-special or edge-special. Assume first that $\varphi^{\bar{v}}$ is vertex-special and, without loss of generality, that a is monochromatic in blue (inside K_4), bd is green, and the other two edges are red (see Figure 4). We partition the set of the 19 extensions of $\varphi^{\bar{v}}$ into three classes (as in the proof of Lemma 7), according to the color of the edge va . Following the counting in the proof of Lemma 7, for va being

blue, red, or green, we have, respectively, 13, 4, and 2 extensions ($19 = 13 + 4 + 2$). If va were blue, then v would be a monochromatic vertex in φ , a contradiction (so the first 13 extensions are not feasible). If va is red, then vb , vc , and vd must be either red or blue. This implies that $\varphi(vb) = \varphi(vd)$ (as bd is green). Among the four options for the colors of vb and vd , there is one in which v becomes monochromatic in red. From the other three colorings, two are isomorphic to the coloring in Figure 6-(A) (swapping colors red with blue in one of the colorings) and the third one is the coloring 6-(B). Finally, assume that va is green. In this case, vb , vc , and vd must be either green or blue and must all have the same color. From these two options, one of them makes v become monochromatic (green). The other one is given in Figure 6-(C).

Now, consider the case where $\varphi^{\bar{v}}$ is edge-special, say ab is blue, cd is green, and the other four edges are red. Again, following the cases in Lemma 7, we may partition the set of extensions of $\varphi^{\bar{v}}$ according to the colors of the edges va and vb . Lemma 7 gives $7 + 3 \cdot 2 + 4 + 2 = 19$ extensions. All of them are shown in Figure 7. Three of them, (7-(1), 7-(9), and 7-(17)), have a monochromatic vertex (so φ cannot be one of them). Ten of them, (7-(3), 7-(4), 7-(6), 7-(11), 7-(13), 7-(14), 7-(15), 7-(16), 7-(18), and 7-(19)), have a vertex z (that we marked with a different color in Figure 7) such that $\varphi^{\bar{z}}$ is isomorphic to a vertex-special K_4 on 3 colors. Therefore, they were already treated in the case where $\varphi^{\bar{v}}$ is edge-special. Four of the remaining colorings, (7-(2), 7-(5), 7-(10), and 7-(12)), are isomorphic to Figure 6-(D). And the remaining two colorings, (7-(7) and 7-(8)), are isomorphic to the one in Figure 6-(E). \square

Lemma 15. *Every non-special Gallai coloring of K_5 that uses exactly three colors has at most 31 extensions.*

Proof. Let $\varphi \in \Phi_5(3)$ be a non-special Gallai coloring of K_5 that uses exactly three colors. If K_5 has a monochromatic vertex, we are done by Lemma 13. So, assume this is not the case. By Lemma 14, we only need to compute the number of extensions for each one of the colorings in Figure 6. We let $V(K_5) = \{x_1, \dots, x_5\}$ as in Figure 6.

We split the colorings into five types, A, B, D, C, and E (following the labels in Figure 6). Let u be a vertex not in K_5 . In the colorings of type B, D, C, and E there exists an edge, x_1x_2 , such that all edges from $\{x_1, x_2\}$ to $\{x_3, x_4, x_5\}$ have the same color. So we can first color the edges ux_1 and ux_2 , and then use Fact 10 to restrict the colors allowed for the edges from u to $\{x_3, x_4, x_5\}$. We organize the seven ways to color the edges ux_1 and ux_2 into four cases, following the same structure as in the proof of the edge-special case of Lemma 7. To keep it compact, we will write sentences of the form “case $\{rr, bg\}$ ” to mean $(\varphi(ux_1), \varphi(ux_2)) \in \{(\text{red}, \text{red}), (\text{blue}, \text{green})\}$.

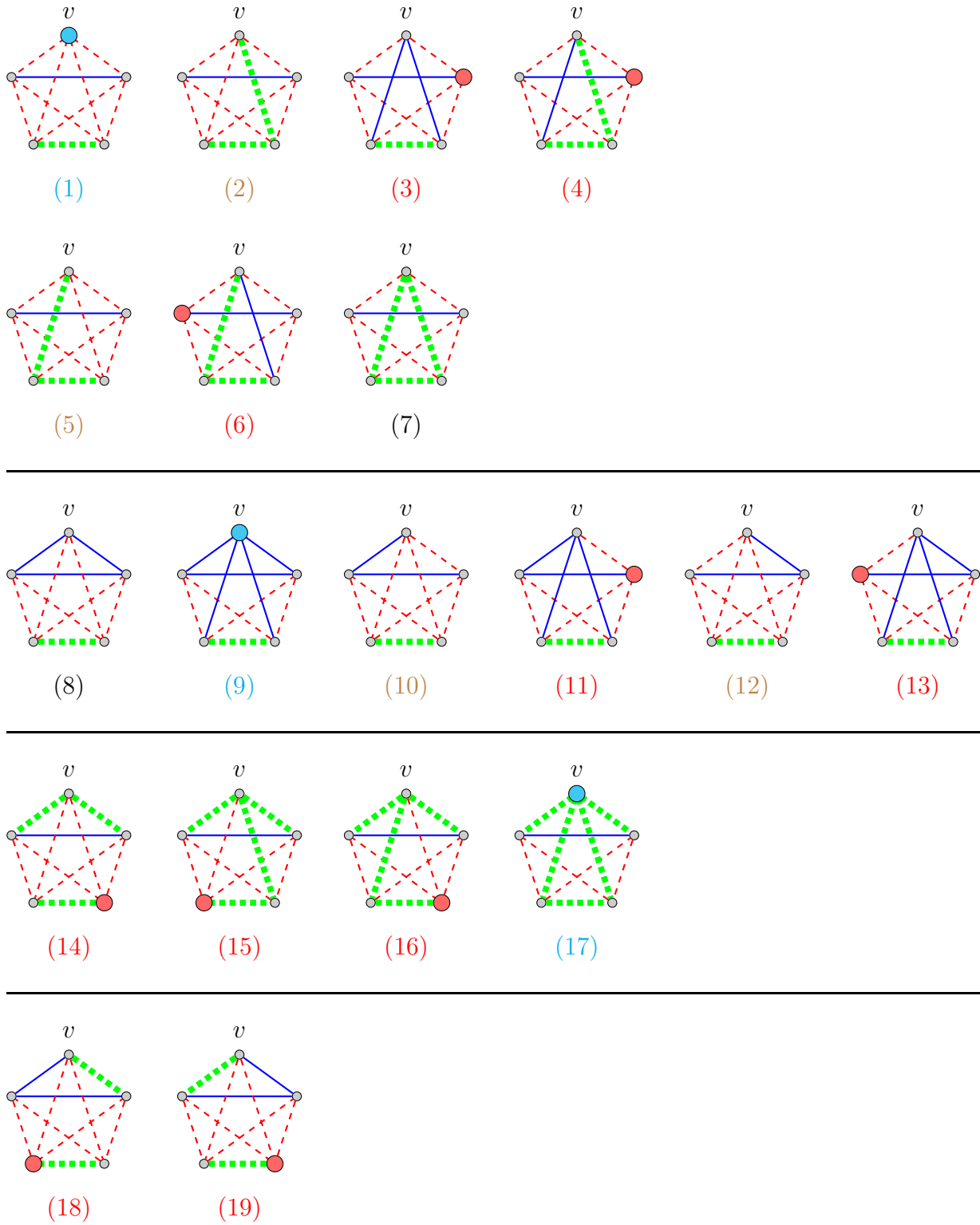


FIGURE 7. Colorings of K_5 with a vertex v such that $K_5 - v$ is a 3-colored edge-special K_4 .

Extensions of (B): For the cases $\{bb\}$, $\{rb, rr, br\}$, $\{gg\}$, and $\{gr, rg\}$, in this order, we add to a total of $(3 \cdot 2^2 + 1) + 3 \cdot 4 + 2 + 2 \cdot 1 = 29$ extensions. For case $\{bb\}$ we used Lemma 4, and for case $\{gg\}$ we used that there is a red spanning tree in $\{x_3, x_4, x_5\}$ (so all edges from u to this set are either green or blue, hence they must have the same color).

Extensions of (C): For the cases $\{bb\}$, $\{gb, gg, bg\}$, $\{rr\}$, and $\{rg, gr\}$, in this order, we add to a total of $(3 \cdot 2^2 + 1) + 3 \cdot 2 + 4 + 2 \cdot 1 = 25$ extensions. For case $\{bb\}$ we used Lemma 4, and for cases $\{gb, gg, bg\}$ we used that $\{x_3, x_4, x_5\}$ has a red spanning tree.

Extensions of (D): For the cases $\{rr\}$, $\{gr, gg, rg\}$, $\{bb\}$, and $\{bg, gb\}$, in this order, we add to a total of $(3 \cdot 2^2 + 1) + 3 \cdot 2 + 8 + 2 \cdot 1 = 29$ extensions. For case $\{rr\}$, we have used Lemma 4. For the cases $\{gr, gg, rg\}$ we have used that $\{x_3, x_4, x_5\}$ has a spanning blue tree.

Extensions of (E): For the cases $\{rr\}$, $\{gr, gg, rg\}$, $\{bb\}$, and $\{bg, gb\}$, in this order, we add to a total of $(2^4 - 1) + 3 \cdot 2 + 8 + 2 \cdot 1 = 31$ extensions. For case $\{rr\}$, we have used Fact 2, and for case $\{gr, gg, rg\}$ we used that $\{x_3, x_4, x_5\}$ has a blue spanning tree.

Finally, a coloring of type **A** has to be treated in an ad-hoc way.

Extensions of (A): We consider cases depending on the colors of ux_4 and ux_5 . For $i \in \{1, 2, 3\}$, we denote $\varphi(ux_i)$ by c_i and consider $c_i \in \{\text{red, green, blue}\}$.

Case $(\varphi(ux_4), \varphi(ux_5)) = (\text{red, red})$: This imposes no restriction on the colors c_1 and c_2 , but forces $c_3 \in \{\text{blue, red}\}$. If none of c_1 , c_2 , and c_3 is green, we have a valid Gallai coloring (as $\{x_1, \dots, x_4\}$ also only has red and blue edges). This gives $2^3 = 8$ colorings. It remains to check the cases where either c_1 or c_2 is green. If c_1 is green, then we must have $c_2 = c_3 = \text{red}$. If c_2 is green, then $c_1 = \text{red}$ and $c_3 = \text{blue}$. Furthermore, it is not possible to have $c_1 = c_2 = \text{green}$ (as there would be no color available for c_3). This gives a total of $8 + 2 = 10$ colorings.

Case $(\varphi(ux_4), \varphi(ux_5)) \in \{\text{green, green}\}$: Analogously to the previous case, there is no restriction for c_3 , but we must have $\{c_1, c_2\} \subseteq \{\text{red, blue}\}$. If none of c_1 , c_2 and c_3 is green, we have $2^3 = 8$ valid colorings. Otherwise, c_3 is green and this forces $c_1 = \text{red}$ and $c_2 = \text{blue}$. So we have 9 colorings in this case.

Cases in $(\varphi(ux_4), \varphi(ux_5)) \in \{(\text{green, blue}), (\text{blue, green})\}$: The green edge (among ux_4 and ux_5) forces that each of c_1 and c_2 must be green or blue. While the fact that $\{\varphi(ux_4), \varphi(ux_5)\} = \{\text{green, blue}\}$ forces $c_3 = \text{red}$. Now $(c_3 = \text{red}) \implies (c_2 = \text{blue})$ and $(c_2 = \text{blue}) \implies (c_1 = \text{blue})$. So, for each of the two choices

for ux_4 and ux_5 there is only one way to complete the coloring. This gives us 2 colorings.

Cases in $(\varphi(ux_4), \varphi(ux_5)) \in \{(\text{red}, \text{green}), (\text{green}, \text{red})\}$: The colors of ux_4 and ux_5 already imply that $c_1 = c_2 = \text{blue}$ and $c_3 \in \{\text{red}, \text{green}\}$. But now, $(c_1 \text{blue}) \implies (c_3 = \text{red})$. As in the previous case, we have 2 colorings.

Case $\{gg\}$: We must have $\{c_1, c_2\} \subseteq \{\text{green}, \text{blue}\}$ and $c_3 \in \{\text{green}, \text{red}\}$. Furthermore, the color of c_2 determines the colors of c_1 and c_3 . So, we have 2 colorings in this case.

Total: These add up to a total of $10 + 9 + 2 + 2 + 2 = 25$ extensions.

As in all cases we had at most 31 extensions, the lemma is proved. \square

Lemma 16 provides the number of extensions of colorings isomorphic to the one in 5.

Lemma 16. *Let φ be a Gallai colouring of K_6 with exactly three colors that contains a matching of size three colored with exactly two colors and all the remaining edges are colored with the third color. Then, $w(\varphi) = 53$.*

Proof. Let φ be as in the statement. Let b_1b_2 , b_3b_4 , and r_1r_2 be the edges of the matching and suppose without loss of generality that $\varphi(b_1b_2) = \varphi(b_3b_4) = \text{blue}$ and $\varphi(r_1r_2) = \text{red}$. We will count the number of ways to extend φ to a Gallai coloring of K_7 when we add a vertex u to the initial K_6 .

Note that all edges between $\{b_1, b_2\}$ and $V(K_6) \setminus \{b_1, b_2\}$ are red. In our proof we consider separately some of the (seven) ways to color the edges ub_1 and ub_2 avoiding rainbow triangles.

Case $(\varphi(ub_1), \varphi(ub_2)) = (\text{red}, \text{red})$: In this case we do not have any additional restrictions on the choices of the colors of the remaining edges. Also, note that the coloring of the remaining graph is edge-special with three colors. Thus, we have $2^4 + 3$ ways to extend the coloring.

Cases $(\varphi(ub_1), \varphi(ub_2)) = (\text{green}, \text{green})$: We cannot use color blue on the remaining edges. So, we can only use red or green. Because b_3b_4 is blue, the edges ub_3 and ub_4 must have the same color. Other than that, there is no other restriction, so we have 2^3 ways to color those remaining edges.

Cases $(\varphi(ub_1), \varphi(ub_2)) \in \{(\text{blue}, \text{blue}), (\text{red}, \text{blue}), (\text{blue}, \text{red})\}$: Analogously to the previous case, all remaining edges must be red or blue. Additionally, because g_1g_2 is green, ug_1 and ug_2 must have the same color. Thus, there are 2^3 extensions in each of the three cases.

Cases $(\varphi(ub_1), \varphi(ub_2)) \in \{(\text{blue}, \text{green}), (\text{green}, \text{blue})\}$: These cases allow us to use only the red color on the remaining edges.

In total, there are $w(\varphi) = 19 + 8 + 3 \cdot 8 + 2 = 53$ extensions. □

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