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Abstract

Golumbic, Hirst, and Lewenstein define a matching in a simple, finite, and undirected graph G to be uniquely restricted if no other matching covers exactly the same set of vertices. We consider uniquely restricted edge-colorings of G defined as partitions of its edge set into uniquely restricted matchings, and study the uniquely restricted chromatic index $\chi'_{ur}(G)$ of G, defined as the minimum number of uniquely restricted matchings required for such a partition.

For every graph G,

$$\chi'(G) \leq a'(G) \leq \chi'_{ur}(G) \leq \chi'_s(G),$$

where $\chi'(G)$ is the classical chromatic index, $a'(G)$ is the acyclic chromatic index, and $\chi'_s(G)$ is the strong chromatic index of G, respectively. While Vizing's famous theorem states that $\chi'(G)$ is either the maximum degree $\Delta(G)$ of G or $\Delta(G)+1$, two famous open conjectures due to Alon, Sudakov, and Zaks, and to Erdős and Nešetřil concern upper bounds on $a'(G)$ and $\chi'_s(G)$ in terms of $\Delta(G)$. Since $\chi'_{ur}(G)$ is sandwiched between these two parameters, studying upper bounds in terms of $\Delta(G)$ is a natural problem.

We show that $\chi'_{ur}(G) \leq \Delta(G)^2$ with equality if and only if some component of G is $K_{\Delta(G),\Delta(G)}$. If G is connected, bipartite, and distinct from $K_{\Delta(G),\Delta(G)}$, and $\Delta(G)$ is at least 4, then, adapting Lovász's elegant proof of Brooks' theorem, we show that $\chi'_{ur}(G) \leq \Delta(G)^2 - \Delta(G)$. Our proofs are constructive and yield efficient algorithms to determine the corresponding edge-colorings.

Keywords: Uniquely restricted matching; edge-coloring; chromatic index; acyclic chromatic index; strong chromatic index

1 Introduction

Motivated by a problem about matrices studied by Hershkowitz and Schneider [12], Golumbic, Hirst, and Lewenstein [11] introduced the notion of a uniquely restricted matching. In the present paper we consider the edge-coloring notion derived from this type of matching and provide best possible upper bounds on the corresponding chromatic index in terms of the maximum degree.

Before we explain our results in detail and discuss related research, we collect some terminology and notation. We consider finite, simple, and undirected graphs. For a graph G, let $V(G)$ denote its vertex set, and let $E(G)$ denote its edge set. For a vertex u of G, the neighborhood $N_G(u)$ of u in G is $\{v \in V(G) : uv \in E(G)\}$, and the closed neighborhood $N_G[u]$ of u in G is $\{u\} \cup N_G(u)$. A *matching* in G is a set of pairwise non-adjacent edges of G. For a matching M, let $V(M)$ be the set of vertices incident with an edge in M. A matching M in G is *induced* if the subgraph $G[V(M)]$ of G induced by $V(M)$ is 1-regular. Golumbic, Hirst, and Lewenstein [11] define a matching M in G to be *uniquely restricted* if there is no matching M' in G with $M' \neq M$ and $V(M') = V(M)$, that is, no other matching covers exactly the same set of vertices.

Each type of matching naturally leads to an edge-coloring notion. For a graph G, let $\chi'(G)$ be the *chromatic index* of G, which is the minimum number of matchings into which the edge set $E(G)$ of G can be partitioned. Similarly, let the *strong chromatic index* $\chi'_s(G)$ [8] and the *uniquely restricted chromatic index* $\chi'_{ur}(G)$ of G be the minimum number of induced matchings and uniquely restricted matchings into which the edge set of G can be partitioned, respectively. A partition of the edges of a graph G into uniquely restricted matchings is a *uniquely restricted edge-coloring* of G.

Another related notion is that of an *acyclic edge-coloring*, which is a partition of the edge set into matchings such that the union of every two of these matchings is a forest. The minimum number of matchings in an acyclic edge-coloring of a graph G is its *acyclic chromatic index* $a'(G)$ [1, 9].

Clearly, every induced matching is uniquely restricted. Furthermore, it is easy to see that a matching M in G is uniquely restricted if and only if there is no M*-alternating cycle* in G, which is a cycle in G that alternates between edges in M and edges not in M. This implies that every uniquely restricted edge-coloring is also an acyclic edge-coloring. These observations imply that, for every graph G,

$$\chi'(G) \leq a'(G) \leq \chi'_{ur}(G) \leq \chi'_s(G). \tag{1}$$

Vizing's classical result [18] states that $\chi'(G)$ of a graph G of maximum degree Δ is either Δ or $\Delta + 1$, and two well known open conjectures concern upper bounds on $\chi'_s(G)$ and $a'(G)$ in terms of Δ. Erdős and Nešetřil (see [8]) conjectured $\chi'_s(G) \leq \frac{5}{4}\Delta^2$, and much of the research on the strong chromatic index is motivated by this conjecture. Building on earlier work of Molloy and Reed [17], and Bruhn and Joos [4], Bonamy, Perrett, and Postle [3] showed $\chi'_s(G) \leq 1.835\Delta^2$ provided that Δ is sufficiently large. For further results on the strong chromatic index we refer to [2, 8, 13, 14]. Fiamčik [9] and Alon, Sudakov, and Zaks [1] conjectured $a'(G) \leq \Delta + 2$. See [5, 6, 10] for further references and the currently best known results concerning general graphs and graphs of large girth. In view of these open conjectures, the inequality chain (1) motivates to study upper bounds on $\chi'_{ur}(G)$ in terms of the maximum degree Δ of a graph G. Our contribution are best-possible such bounds and the characterization of all extremal graphs. Since our proofs are constructive, it is easy to extract efficient algorithms finding the corresponding edge-colorings.

2 Upper bounds on $\chi'_{ur}(G)$

Our first result applies to general graphs, and its proof relies on a natural greedy strategy. Faudree, Schelp, Gyárfás, and Tuza [7] conjectured $\chi'_s(G) \leq \Delta^2$ for a bipartite graph G of maximum degree Δ, and our Theorem 2.3 can be considered to be a weak version of this conjecture. Theorem 2.3 below shows that excluding the unique extremal graph from Theorem 2.1, the uniquely restricted chromatic index of bipartite graphs drops considerably.

For an integer k, let $[k]$ be the set of all positive integers at most k.

Theorem 2.1. *If G is a connected graph of maximum degree at most Δ, then $\chi'_{ur}(G) \leq \Delta^2$ with equality if and only if G is $K_{\Delta,\Delta}$.*

Proof. Since no two edges of $K_{\Delta,\Delta}$ form a uniquely restricted matching in this graph, we obtain $\chi'_{ur}(K_{\Delta,\Delta}) = |E(K_{\Delta,\Delta})| = \Delta^2$. Now, let G be a connected graph of maximum degree at most Δ. We first show that $\chi'_{ur}(G) \leq \Delta^2$. In a second step, we show that $\chi'_{ur}(G) < \Delta^2$ provided that G is not $K_{\Delta,\Delta}$.

We consider the vertices of G in some linear order, say $u_1, \ldots, u_n$. For i from 1 up to n, we assume that the edges of G incident with vertices in $\{u_1, \ldots, u_{i-1}\}$ have already been colored, and we color all edges between u_i and $\{u_{i+1}, \ldots, u_n\}$ using distinct colors, and avoiding any color that has already been used on a previously colored edge incident with some neighbor of u_i. Since u_i has at most Δ neighbors, each of which is incident with at most Δ edges, this procedure requires at most Δ^2 many distinct colors.

Suppose, for a contradiction, that some color class M is not a uniquely restricted matching in G. Since M is a matching by construction, there is an M-alternating cycle C. Let $C : u_{r_1}u_{s_1}u_{r_2}u_{s_2}\ldots u_{r_k}u_{s_k}u_{r_1}$ be such that r_1 is the minimum index of any vertex on C, and $u_{r_1}u_{s_k} \in M$. These choices trivially imply $r_1 < s_1$ and $r_1 < r_2$. If $r_2 > s_1$, then $u_{r_1}u_{s_k} \in M$ implies that, when coloring the edge $u_{s_1}u_{r_2}$, some edge incident with the neighbor u_{r_1} of u_{s_1} would already have been assigned the color of the edges in M, and the above procedure would have avoided this color on $u_{s_1}u_{r_2}$. Therefore, since $u_{r_1}u_{s_k} \in M$ and $u_{r_2}u_{s_1} \in M$, the coloring rules imply $r_2 < s_1$, that is, $r_1 < r_2 < s_1$. Now, suppose that $r_i < r_{i+1} < s_i$ for some $i \in [k-1]$. Since $u_{r_{i+1}}u_{s_i} \in M$ and $u_{r_{i+2}}u_{s_{i+1}} \in M$, the coloring rules imply in turn

- $r_{i+1} < s_{i+1}$, since otherwise we would have colored $u_{r_{i+1}}u_{s_i}$ differently,
- $r_{i+2} < s_{i+1}$, since otherwise we would have colored $u_{r_{i+2}}u_{s_{i+1}}$ differently, and
- $r_{i+1} < r_{i+2}$, since otherwise we would have colored $u_{r_{i+1}}u_{s_i}$ differently.

It follows that $r_{i+1} < r_{i+2} < s_{i+1}$, where we identify r_{k+1} with r_1. Now, by an inductive argument, we obtain $r_1 < r_2 < \cdots < r_k < r_1$, which is a contradiction.

Altogether, we obtain $\chi'_{ur}(G) \leq \Delta^2$.

Now, let G be distinct from $K_{\Delta,\Delta}$, and we want to prove that $\chi'_{ur}(G) < \Delta^2$. Among all uniquely restricted edge-colorings of G using colors in $[\Delta^2]$, we choose a coloring for which the number of edges with color 1 is as small as possible. Clearly, we may assume that some edge uv has color 1, as otherwise we already have that $\chi'_{ur}(G) < \Delta^2$.

If there is a color α in $[\Delta^2] \setminus \{1\}$ such that no edge incident with a neighbor of u has color α, then changing the color of uv to α yields a uniquely restricted edge-coloring of G with less edges of color 1, which is a contradiction. In view of the maximum degree, this implies that every vertex in $N_G[u]$ has degree Δ, the set $N_G(u)$ is independent, and, for every color α in $[\Delta^2]$, there is exactly one edge incident with a neighbor of u that has color α.

Since G is not $K_{\Delta,\Delta}$, some neighbor x of u has a neighbor y that does not lie in $N_G(v)$. Without loss of generality, let ux have color 2, and let xy have color 3. Let M be the set of edges with color 3.

If G does not contain an M-alternating path of odd length (number of edges) at least 3 between x and a vertex in $N_G(v) \setminus \{u\}$ that contains the edge xy, then changing the color of uv to 3 yields a uniquely restricted edge-coloring of G with less edges of color 1, which is a contradiction. Hence, G contains such a path, which implies that two edges incident with neighbors of y have color 3.

If there is a color α in $[\Delta^2] \setminus \{1\}$ such that no edge incident with a neighbor of y has color α, then changing the color of xy to α and the color of uv to 3 yields a uniquely restricted edge-coloring of G with less edges of color 1, which is a contradiction. Similarly as above, this implies that, for every color α in $[\Delta^2] \setminus \{1, 3\}$, there is exactly one edge incident with a neighbor of y that has color α. Now, changing the color of uv to 2, the color of ux to 3, and the color of xy to 2 yields a uniquely restricted edge-coloring of G with less edges of color 1, which is a contradiction. This completes the proof. □

As observed above, the proof of Theorem 2.1 is algorithmic; the simple greedy strategy considered in its first half efficiently constructs uniquely restricted edge-colorings using at most Δ^2 colors. Furthermore, also its second half can be turned into an efficient algorithm that finds uniquely restricted edge-colorings using at most $\Delta^2 - 1$ colors for connected graphs of maximum degree Δ that are distinct from $K_{\Delta,\Delta}$; the different cases considered in the proof correspond to simple manipulations of a given uniquely restricted edge-coloring that iteratively reduce the number of edges of color 1 down to 0. Golumbic, Hirst, and Lewenstein [11] showed that deciding whether a given matching is uniquely restricted can be done in polynomial time, and their algorithm can be used to decide which of the simple manipulations can be executed.

Our next goal is to improve Theorem 2.1 for bipartite graphs. The following proof was inspired by Lovász's [16] elegant proof of Brooks' Theorem.

Lemma 2.2. *If G is a connected bipartite graph of maximum degree at most $\Delta \geq 4$ that is distinct from $K_{\Delta,\Delta}$, and M is a matching in G, then M can be partitioned into at most $\Delta - 1$ uniquely restricted matchings in G.*

Proof. Let A and B be the partite sets of G, and let $R = V(G) \setminus V(M)$. Note that M is perfect if and only if R is empty. Whenever we consider a coloring of the edges in M, and α is one of the colors, let M_α be the set of edges in M colored with α.

First, we assume that R is empty, and that G is not Δ-regular. By symmetry, we may assume that some vertex a in A has degree less than Δ. Let $ab \in M$. Let T be a spanning tree of G that contains the edges in M. Contracting within T the edges from M, rooting the resulting tree at the vertex corresponding to the edge ab, and considering a breadth-first search order, we obtain the existence of a linear order $a_1b_1, \ldots, a_nb_n$ of the edges in M such that $ab = a_nb_n$, and, for every $i \in [n-1]$, there is an edge between $\{a_i, b_i\}$ and $\{a_{i+1}, b_{i+1}, \ldots, a_n, b_n\}$. Since a_n has degree less than Δ, this implies that, for every $i \in [n]$, some vertex u_i in $\{a_i, b_i\}$ has at most $\Delta - 2$ neighbors in $\{a_1, b_1, \ldots, a_{i-1}, b_{i-1}\}$. Now, we color the edges in M greedily in the above linear order. Specifically, for every i from 1 up to n, we color the edge a_ib_i with some color α in $[\Delta-1]$ such that, for every $j \in [i-1]$, for which $u_i \in \{a_i, b_i\}$ has a neighbor in $\{a_j, b_j\}$, the edge a_jb_j is not colored with α. By the degree condition on u_i, such a coloring exists. Suppose, for a contradiction, that M_α is not uniquely restricted for some color α in $[\Delta-1]$. Let the edge a_ib_i in M_α be such that it belongs to some M_α-alternating cycle C, and, subject to this condition, the index i is maximum. If the neighbor of u_i on C outside of $\{a_i, b_i\}$ is in $\{a_j, b_j\}$, then the choice of the edge a_ib_i implies $j < i$, and the coloring rule implies that the edge a_jb_j is not colored with α, which is a contradiction. Altogether, the statement follows.

Next, we assume that R is non-empty. Let K be a component of $G - R$. Let M_K be the set of edges in M that lie in K. Since G is connected, the graph K is not Δ-regular. Therefore, proceeding exactly as above, we obtain a coloring of the edges in M_K using the colors in $[\Delta-1]$ such that each color class is a uniquely restricted matching in K. If $K_1, \ldots, K_k$ are the components of $G - R$, and M_i is a uniquely restricted matching in K_i for every $i \in [k]$, then $M_1 \cup \cdots \cup M_k$ is a uniquely restricted matching in G. Therefore, combining the colorings within the different components, we obtain that also in this case the statement follows.

At this point, we may assume that G is Δ-regular, and that M is perfect.

Next, we assume that there are two distinct edges e and e' in M such that $V(\{e, e'\})$ is a vertex cut of G. This implies that we can partition the set $M \setminus \{e, e'\}$ into two non-empty sets M_1 and M_2 such that there is no edge between $V(M_1)$ and $V(M_2)$. For $i \in [2]$, let G_i be the subgraph of G induced by $V(\{e, e'\} \cup M_i)$. Since G is connected, the graph G_i is not Δ-regular. In view of the above, this implies that there is a coloring c_i of the edges of the perfect matching $\{e, e'\} \cup M_i$ of G_i using the colors in $[\Delta-1]$ such that each color class of c_i is a uniquely restricted matching in G_i. If $c_i(e) \neq c_i(e')$ for both i in $[2]$, then we may assume that c_1 and c_2 assign the same colors to e and e', and it is easy to verify that the common extension c of c_1 and c_2 to M has the property that every color class of c is a uniquely restricted matching in G. Hence, we may assume that necessarily $c_1(e) = c_1(e')$. Note that this implies in particular that at least one of the two possible edges between $V(\{e\})$ and $V(\{e'\})$ is missing.

Let $c_1(e) = \alpha$. Let $e = ab$, $e' = a'b'$, and $U = \{a, b, a', b'\}$. For every vertex $u \in U$, let $C_1(u)$ be the set of colors β for which M_1 contains an edge vw with $c_1(vw) = \beta$ such that u is adjacent to v or w. If there is some $u \in U$ and some color $\beta \in ([\Delta-1] \setminus \{\alpha\}) \setminus C_1(u)$, then changing the color of the unique edge in $\{e, e'\}$ incident with u from α to β yields a coloring c_1' of the edges in $\{e, e'\} \cup M_1$ using the colors in $[\Delta-1]$ such that each color class of c_1' is a uniquely restricted matching in G_1. Furthermore, $c_1'(e) \neq c_1'(e')$, which is a contradiction. This implies that $[\Delta-1] \setminus \{\alpha\} \subseteq C_1(u)$ for every $u \in U$. In particular, each vertex u in U has at least $\Delta - 2$ neighbors in $V(M_1)$, and, hence, at most one neighbor in $V(M_2)$. Let $C_2(u)$ for $u \in U$ be defined analogously as above. Clearly, the set $C_2(a) \cup C_2(a')$ contains at most two distinct colors. Since $\Delta - 1 \geq 3$, we may assume that c_2 is such that the set $C_2(a) \cup C_2(a')$ does not contain the color α. Now, let c_2' be a coloring of the edges in $\{e, e'\} \cup M_2$ that coincides with c_2 on M_2 and colors e and e' with color α. It is easy to see that each color class of c_2' is a uniquely restricted matching in G_2. Let c be the common extension of c_1 and c_2' to M. Suppose, for a contradiction, that the color class M_β of c is not uniquely restricted for some color β in $[\Delta-1]$. Clearly, we have $\beta = \alpha$. Let C be an M_α-alternating cycle in G. It is easy to see that C contains both edges e and e', but no edge between $\{a, b\}$ and $\{a', b'\}$. Furthermore, it follows that C contains an edge between $\{a, a'\}$ and $V(M_2)$. Since c coincides with c_2 on M_2, and $C_2(a) \cup C_2(a')$ does not contain α, we obtain a contradiction.

Altogether, we may assume that there are no two distinct edges e and e' in M such that $V(\{e,e'\})$ is a vertex cut of G.

Now, we show the existence of three edges ab, $a'b'$, and $a''b''$ in M such that some of the two possible edges between $\{a',b'\}$ and $\{a'',b''\}$ is missing, and either a is adjacent to b' as well as b'' or b is adjacent to a' as well as a''. Therefore, let a_1b_1 be an edge in M. Let $a_2b_2,\ldots,a_\Delta b_\Delta$ be the edges in M such that $N_G(a_1)=\{b_1,\ldots,b_\Delta\}$. We may assume that $\{a_2,b_2,\ldots,a_\Delta,b_\Delta\}$ induces a complete bipartite graph $K_{\Delta-1,\Delta-1}$; otherwise, we find the three edges with the desired properties. Since G is not $K_{\Delta,\Delta}$, the vertex b_1 is non-adjacent to some vertex a_i in $\{a_2,\ldots,a_\Delta\}$. Now, if $a_j\in\{a_2,\ldots,a_\Delta\}\setminus\{a_i\}$, then one of the two possible edges between $\{a_1,b_1\}$ and $\{a_i,b_i\}$ is missing, and b_j is adjacent to a_1 as well as a_i. Altogether, we obtain three edges ab, $a'b'$, and $a''b''$ in M with the desired properties.

By symmetry, we may assume that a is adjacent to b' and b'', and a' is non-adjacent to b''. In view of the above, the graph $G'=G-V(\{a'b',a''b''\})$ is connected, and $M'=M\setminus\{a'b',a''b''\}$ is a perfect matching of G'. Let T' be a spanning tree of G' that contains the edges in M'. Contracting within T' the edges from M', rooting the resulting tree in the vertex corresponding to the edge ab, and considering a breadth-first search order, we obtain the existence of a linear order $a_3b_3,\ldots,a_nb_n$ of the edges in M' such that $ab=a_nb_n$, and, for every $i\in[n-1]\setminus[2]$, there is an edge between $\{a_i,b_i\}$ and $\{a_{i+1},b_{i+1},\ldots,a_n,b_n\}$. Now, we color the edges in M greedily in the linear order $a_1b_1,a_2b_2,a_3b_3,\ldots,a_nb_n$, where $a_1b_1=a''b''$ and $a_2b_2=a'b'$. Note that, for every $i\in[n-1]\setminus[2]$, some vertex u_i in $\{a_i,b_i\}$ has at most $\Delta-2$ neighbors in $\{a_1,b_1,\ldots,a_{i-1},b_{i-1}\}$. We color a_1b_1 and a_2b_2 with the same color. For every i from 3 up to $n-1$, we color the edge a_ib_i with a color α in $[\Delta-1]$ such that, for every $j\in[i-1]$, for which u_i has a neighbor in $\{a_j,b_j\}$, the edge a_jb_j is not colored with α. By the degree condition on u_i, such a coloring exists. Finally, since a_n has neighbors in the two edges a_1b_1 and a_2b_2 that are colored with the same color, there is some color α in $[\Delta-1]$ for which no edge a_ib_i with $i\in[n-1]$ such that a_n is adjacent to b_i, is colored with α, and we color the edge a_nb_n with that color α. Suppose, for a contradiction, that M_β is not uniquely restricted for some color β in $[\Delta-1]$. Let the edge a_ib_i in M_β be such that it belongs to some M_β-alternating cycle C, and, subject to this condition, the index i is maximum. Since a' is non-adjacent to b'', we have $i\geq 3$. Let $u_n=a_n$. If the neighbor of u_i on C outside of $\{a_i,b_i\}$ is in $\{a_j,b_j\}$, then the choice of the edge a_ib_i implies $j<i$, and the coloring rule implies that the edge a_jb_j is not colored with β, which is a contradiction. This completes the proof. $\square$

Lemma 2.2 fails for $\Delta=3$; the matching $\{a_1b_1,a_2b_2,a_3b_3,a_4b_4,a_5b_5\}$ of the graph G in Figure 1 cannot be partitioned into two uniquely restricted matchings. Note that the matching $\{a_1b_3,a_2b_1,a_3b_5,a_4b_2,a_5b_4\}$ though is the union of the two uniquely restricted matchings $\{a_1b_3,a_3b_5\}$ and $\{a_2b_1,a_4b_2,a_5b_4\}$.

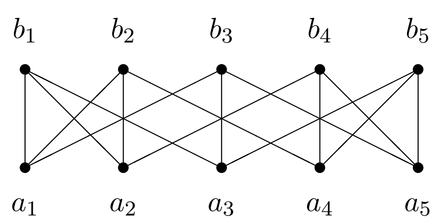

Figure 1: A bipartite graph G.

Lemma 2.2 also fails for non-bipartite graphs; in fact, if G arises from the disjoint union of two copies of K_Δ by adding a perfect matching M, then every partition of M into uniquely restricted matchings requires Δ sets.

With Lemma 2.2 at hand, the proof of our final result is easy.

Theorem 2.3. *If G is a connected bipartite graph of maximum degree at most $\Delta \geq 4$ that is distinct from $K_{\Delta,\Delta}$, then $\chi'_{ur}(G) \leq \Delta^2 - \Delta$.*

Proof. Since G is bipartite, its edge set can be partitioned into Δ matchings [15]. By Lemma 2.2, each of these matchings can be partitioned into $\Delta - 1$ uniquely restricted matchings. This completes the proof. $\square$

Note that the graph G in Figure 1 also satisfies $\chi'_{ur}(G) \leq \Delta^2 - \Delta = 9 - 3 = 6$. In fact, the uniquely restricted matchings $\{a_1b_1, a_4b_2, a_5b_4\}$, $\{a_1b_2, a_2b_4, a_5b_5\}$, $\{a_2b_1, a_3b_3, a_4b_5\}$, $\{a_1b_3, a_4b_4\}$, $\{a_2b_2, a_3b_5\}$, and $\{a_3b_1, a_5b_3\}$ partition $E(G)$.

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