Excessive Transverse Coordinates for Orbital Stabilization of (Underactuated) Mechanical Systems

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Abstract— The transverse linearization is a useful tool for the design of feedback controllers that orbitally stabilizes (periodic) motions of mechanical systems. Yet, in an n-dimensional state-space, this requires knowledge of a set of $(n-1)$ independent transverse coordinates, which at times can be difficult to find and whose definitions might vary for different motions (trajectories). Motivated by this, we present in this paper a generic choice of *excessive* transverse coordinates defined in terms of a particular parameterization of the motion and a projection operator recovering the "position" along the orbit. We present a constructive procedure for obtaining the corresponding excessive transverse linearization and state a sufficient condition for the existence of a feedback controller rendering the desired trajectory (locally) asymptotically orbitally stable. The approach is demonstrated through numerical simulation by stabilizing oscillations around the unstable upright position of the underactuated cart-pendulum system, in which a novel motion planning approach based on virtual constraints is utilized for trajectory generation.

Index Terms— Underactuated mechanical systems, orbital stabilization, transverse coordinates, transverse linearization.

I. INTRODUCTION

We consider the task of stabilizing periodic trajectories of underactuated Euler-Lagrange systems, defined by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{F}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{B}\mathbf{u}. \qquad (1)$$

Here $\mathbf{q} \in \mathbb{R}^{n_q}$ are the generalized coordinates and $\dot{\mathbf{q}} = \frac{d}{dt}\mathbf{q}$ the generalized velocities; $\mathbf{u} \in \mathbb{R}^{n_u}$ is a vector of $n_u(< n_q)$ control inputs; $\mathbf{M} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ is the symmetric, positive definite inertia matrix; $\mathbf{C} : \mathbb{R}^{n_q} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ is the matrix of Coriolis and centrifugal terms that satisfies $\mathbf{C}(\mathbf{q}, X)Y = \mathbf{C}(\mathbf{q}, Y)X$ for any $X, Y \in \mathbb{R}^{n_q}$; $\mathbf{F} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ is a matrix function of damping and friction terms; $\mathbf{G} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q}$ is the gradient of the system's potential energy; while $\mathbf{B} \in \mathbb{R}^{n_q \times n_u}$ is a constant matrix of full rank.

The general problem of stabilizing a predetermined motion (trajectory) of such systems can be highly challenging due to both the nonlinear dynamics and the underactuation. For instance, this prohibits the use of simplifying strategies such feedback linearization, while alternative techniques such as partial feedback linearization [1] will result in some remaining internal dynamics, which can be (made) unstable (non-minimum phase), and consequently must be considered in any control design.

Linearization of the dynamics along a nontrivial orbit is also of limited use for the purpose of control design. Indeed, whereas the exponential stability of an equilibrium point, $\mathbf{x}_e \in \mathbb{R}^n$, of an autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ can be determined simply from the stability of the linearized (first approximation) system $\delta\dot{\mathbf{x}} = [\partial f(\mathbf{x})/\partial \mathbf{x}]_{\mathbf{x}=\mathbf{x}_e}\delta\mathbf{x}$, determining the stability of a trajectory $\mathbf{x}_*(t)$, $\|\dot{\mathbf{x}}_*(t)\| > 0$, is slightly more intricate. Consider, for example, a trajectory slightly perturbed from the nominal motion. Although it might stay close to the orbit in space, it will eventually "out-run" it in time, making the nominal motion unstable in the sense of Lyapunov. This leads to the well known fact that a periodic solution cannot be asymptotically stable, even if the periodic *orbit* is. It is therefore beneficial to instead look at the stability of the corresponding orbit (the set of all states along the solution) as a whole. This is the notion of *orbital (Poincaré) stability* [2], [3], in which asymptotic orbital stability simply means asymptotic convergence to the orbit itself, and not to a specific point (moving) along it.

It is known that a periodic orbit is exponentially stable in the orbital sense if and only if the linearization of the dynamics transverse to the orbit is exponentially stable [4]. Indeed, this is true for any orbit if the corresponding linearized system is regular [3], [5]. Thus, if one can find $(2n_q - 1)$ independent *transverse coordinates* which vanish on the nominal orbit, and then exponentially stabilize the corresponding *linearized transverse dynamics* (the first approximation), then one simultaneously asymptotically stabilizes the trajectory in the orbital sense. However, finding a set of $(2n_q - 1)$ independent transverse coordinates can sometimes be nontrivial. That is, finding a set of coordinates $\mathbf{x}_\perp \in \mathbb{R}^{2n_q-1}$ together with a scalar variable, $s \in \mathbb{R}$, parameterizing the trajectory, such that there exists a (local) diffeomorphism $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto (s, \mathbf{x}_\perp)$. However, as we will see, there exists a generic choice of $2n_q$ *excessive transverse coordinates*. Since these excessive coordinates, by definition, are dependent on a minimal set of $(2n_q - 1)$ coordinates, the stability of their origin implies the stability of the origin of the minimal coordinates, and consequently the orbital stability of the trajectory.[1]

The same set of excessive coordinates we consider in this paper, together with the linearization of their dynamics, has previously been considered in [6] for stabilizing periodic motions of a fully actuated robot manipulator. Moreover, they

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[1]For more details, a note containing a simple example illustrating the difference between orbital stabilization and reference tracking, as well as the notion of (excessive) transverse coordinates is available at `http://folk.ntnu.no/christfs/OrbStabVsRefTrack`.

were utilized in [7] for the stabilization of a hybrid walking cycle of a three-link biped robot with two degrees of under-actuation. There, a particular choice of the parameterizing variable s allowed for one coordinate to be trivially omitted in order to obtain a minimal set of transverse coordinates.

In this paper, we build upon the aforementioned work by presenting several original contributions, providing new insight into excessive transverse coordinates and their linearization. The main contributions of the present paper are:

- Extending and generalizing the method presented in [6] to underactuated mechanical systems;
- Stating analytical expressions for the transverse linearization of the excessive transverse coordinates without the need to numerically solve a matrix equation as in [6];
- Allowing for the projection operator, which recovers the parameterizing variable of the nominal orbit, to be implicitly defined, and which can depend on all of the system's states, not only its configuration as in [6];
- Illustrating that the transverse coordinates need only be locally transverse to the flow of the nominal orbit rather than restricted to being locally orthogonal as in [4], [8];
- Providing an explicit procedure for obtaining an asymptotically orbitally stabilizing feedback controller.

Furthermore, it is worth to note that the proposed method is not sensitive to singularities in the reduced dynamics (see e.g. [9]). Thus it can be utilized for the stabilization of a richer set of trajectories than the method in [10], [11], such as trajectories whose generating equations have singularities [12] (see also Sec. VI) and even certain non-periodic trajectories (assuming the linearization is regular). The method is also easily applicable to systems of any degree of underactuation, as well fully- and even redundantly actuated systems; and has the added benefit that any change in actuation will require minor changes in the presented procedure.

A brief outline of this paper is as follows. In the following section, we start by defining a set of excessive transverse coordinates for a given trajectory and then derive the linearization of their dynamics in Sec. III. We then state the main result of this paper in Sec. IV on the form of Theorem 1, which gives sufficient conditions for attaining an orbitally stabilizing controller. In Sec. V we give a statement which allows for the construction of projection operators. While lastly, in Sec. VI, we illustrate the proposed procedure by stabilizing upright oscillations of the underactuated cart-pendulum system.

II. ON EXCESSIVE TRANSVERSE COORDINATES

Let $\mathbf{x} = [\mathbf{q}^{\mathsf{T}}, \dot{\mathbf{q}}^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{2n_q}$ denote the state vector of (1). Suppose a non-trivial (non-vanishing) T-periodic trajectory

$$\mathbf{x}_*(t, \mathbf{x}_0) = \mathbf{x}_*(t + T, \mathbf{x}_0), \ \mathbf{x}_*(0, \mathbf{x}_0) = \mathbf{x}_0, \ T > 0,$$

is known, as well as the corresponding nominal control input $\mathbf{u}_*(t, \mathbf{x}_0)$. Further suppose that the corresponding orbit (set of all points along the trajectory), denoted as η_*, admits a reparametrization in terms of a (strictly) monotonically increasing scalar variable $s \in \mathcal{S}$, whose domain $\mathcal{S} \subset \mathbb{R}$ is homeomorphic to the unit circle. We will refer to the parameterizing variable s as the the *motion generator* (MG) of the reparameterized trajectory, defined by

$$\mathbf{x}_s(s) = \begin{bmatrix} \mathbf{q}_*(s) \\ \dot{\mathbf{q}}_*(s) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}(s) \\ \mathbf{\Phi}'(s)\zeta(s) \end{bmatrix}, \quad s \in \mathcal{S} \subset \mathbb{R}, \quad (2)$$

with $\mathbf{\Phi}'(s) = \frac{d}{ds}\mathbf{\Phi}(s)$. Here $\mathbf{\Phi} : \mathcal{S} \to \mathbb{R}^{n_q}$ is at least thrice continuously differentiable, while $\zeta : \mathcal{S} \to \mathbb{R}$ is a $\mathcal{C}^2$-function that recovers the nominal velocity of the MG along the orbit, i.e. $\dot{s}_*(t) = \zeta(s(t)) > 0$, and whose existence is guaranteed by the monotonicity of s and the existence of the orbit η_*.

Note that we will use the subscript-notation "s" throughout this paper to denote that a function is evaluated along the trajectory parameterized by the MG, e.g. $h_s(s) := h(\mathbf{x}_s(s))$ for any $h : \mathbb{R}^{2n_q} \to H$ and an arbitrary space H. Moreover, $h'_s(s) = \frac{dh_s}{ds}(s)$ for any function $h_s : \mathcal{S} \to H$.

Suppose that within some non-zero tubular neighbourhood $\mathcal{X} \subset \mathbb{R}^{2n_q}$ of the orbit η_*, the MG can be found from a projection of the system states upon the orbit by the operator

$$P : \mathbb{R}^{2n_q} \to \mathcal{S}, \quad \mathbf{x} \mapsto P(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}, \quad (3)$$

i.e. $s = P(\mathbf{x})$, which is assumed to be at least twice continuously differentiable and well defined in $\mathcal{X}$. We will denote by $\mathbf{D}P(\cdot) = [\frac{\partial P}{\partial \mathbf{q}}(\cdot), \frac{\partial P}{\partial \dot{\mathbf{q}}}(\cdot)]$ the gradient of $P(\cdot)$ and $\mathbf{DD}P(\cdot) = \frac{\partial^2 P}{\partial \mathbf{x}^2}(\cdot)$ its $2n_q \times 2n_q$ symmetric Hessian matrix. (In Sec. V we will show that at least one such (local) projection is guaranteed to exist for smooth periodic orbits).

The idea behind this projection operator is simply that it allows us to project the current state, at least within some tubular neighbourhood, down upon the nominal orbit to recover the "position" along it. This then allows us to define some measure of the distance to this orbit, which, unlike regular reference tracking, will only depend on the current state of the system and not on some time-varying reference, giving rise to a completely state-dependent feedback.

More specifically, consider the following coordinates

$$\mathbf{x}_\perp := \mathbf{x} - \mathbf{x}_s(s) \quad (4)$$

which are well defined for all $\mathbf{x} \in \mathcal{X}$. Differentiating (4) with respect to time leads to

$$\dot{\mathbf{x}}_\perp = \left(\mathbf{I}_{2n_q} - \mathbf{x}'_s(s)\mathbf{D}P(\mathbf{x})\right)\dot{\mathbf{x}} =: \mathbf{\Omega}(\mathbf{x})\dot{\mathbf{x}}. \quad (5)$$

It thus follows that sufficiently close to the orbit, a small variation in the states, $\delta\mathbf{x}$, is related to a small variation of the coordinates (4) as follows

$$\delta\mathbf{x}_\perp = \mathbf{\Omega}_s(s)\delta\mathbf{x}. \quad (6)$$

Here $\mathbf{\Omega}_s(s)$ is of particular interest.

Lemma 1: Let $P(\cdot)$ be defined as in (3) and let $x_s : \mathcal{S} \to \eta_*$ be a $\mathcal{C}^2$-curve defined by (2). Then for all $s \in \mathcal{S}$, the matrix function

$$\mathbf{\Omega}_s(s) := \mathbf{I}_{2n_q} - \mathbf{x}'_s(s)\mathbf{D}P_s(s) \quad (7)$$

is a projection matrix, i.e. $\mathbf{\Omega}_s(s)^2 = \mathbf{\Omega}_s(s)$, and its rank is always $(2n_q - 1)$. Moreover, $\mathbf{D}P_s(s)$ and $\mathbf{x}'_s(s)$ are its left- and right annihilators, respectively. $\square$

This statement naturally follows by observing from (3) that we need to have $\zeta(s) = \mathbf{D}P_s(s)\mathbf{x}'_s(s)\zeta(s)$; and hence

$$\mathbf{D}P_s(s)\mathbf{x}'_s(s) \equiv 1, \quad \forall s \in \mathcal{S}, \qquad (8)$$

where $\mathbf{x}'_s(s) = [\mathbf{\Phi}'(s)^\mathsf{T}, \mathcal{F}(s)^\mathsf{T}]^\mathsf{T}$ with $\mathcal{F}(\cdot)$ defined as

$$\mathcal{F}(s) := \mathbf{\Phi}'(s)\zeta'(s) + \mathbf{\Phi}''(s)\zeta(s). \qquad (9)$$

Note that this does *not* necessarily imply that $\mathbf{D}P_s(s) = \mathbf{x}'_s(s)^\mathsf{T}/\|\mathbf{x}'_s(s)\|^2$ in general. Rather, the following statement, which follows directly from (8) and the definition of the dot product, is always true.

Lemma 2: Let $\theta(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ denote the angle between $\mathbf{D}P_s(s)$ and $\mathbf{x}'_s(s)^\mathsf{T}$ in their common plane. Then there exists a $\mathcal{C}^1$-vector function $\mathbf{n}_\perp : \mathcal{S} \to \mathbb{R}^{2n_q}$ of unit length within the span of the kernel of $\mathbf{x}'_s(s)^\mathsf{T}$, such that

$$\mathbf{D}P_s(s) = \frac{\mathbf{x}'_s(s)^\mathsf{T}}{\|\mathbf{x}'_s(s)\|^2} + \frac{\tan(\theta(s))}{\|\mathbf{x}'_s(s)\|}\mathbf{n}_\perp(s)^\mathsf{T}. \qquad (10)$$

$\square$

A straightforward consequence of Lemma 1 together with (6) is that $\mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp = \delta\mathbf{x}_\perp$, and hence the relation

$$\mathbf{D}P_s(s)\delta\mathbf{x}_\perp = \mathbf{D}P_s(s)\mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp \equiv 0$$

must hold for all $s \in \mathcal{S}$. This, together with Lemma 2, allows us to conclude that, sufficiently close to the orbit, the coordinates $\mathbf{x}_\perp$ are orthogonal to $\mathbf{D}P_s(s)$, and thus transverse to the flow of the orbit. Consequently, they are transverse coordinates. However, the matrix function $\mathbf{\Omega}_s(s)$ is not invertible (its rank is always $(2n_q - 1)$), so they are *excessive* transverse coordinates. Nevertheless, by similar arguments to [6, Theorem 3], if the origin of these coordinates is asymptotically stable, then the periodic orbit is orbitally asymptotically stable. Indeed, this is also implied by the following statement.

Lemma 3: Let $\mathbf{y}_\perp : \mathcal{S} \times \mathbb{R}^{2n_q} \to \mathbb{R}^{2n_q - 1}$ be a valid set of minimal transverse coordinates together with $s = P(\mathbf{x})$ defined in (3). That is, $\mathbf{x} \mapsto (s, \mathbf{y}_\perp)$ is a local diffeomorphism and $\mathbf{y}_\perp$ vanishes on η_*. Then the origin of $\mathbf{y}_\perp$ is asymptotically stable if and only if the origin of the excessive coordinates $\mathbf{x}_\perp$ is asymptotically stable. $\square$
The proof of Lemma 3 is stated in Appendix A.

The value of these excessive transverse coordinates should therefore be evident: given a known solution to (1), they are a valid set of transverse coordinates for *any* parameterization of the form (2) and *any* projection operator (3). They also allows one to easily change between different sets of coordinates by simply changing either (or both) the parameterization or the projection.

With the aim of asymptotically stabilizing the origin of these coordinates, and consequently the orbit, we will show next how one can derive the linearization (first approximation) of their dynamics along the target motion.

III. DERIVING THE TRANSVERSE LINEARIZATION

Let $\mathbf{B}^\dagger \in \mathbb{R}^{n_u \times n_q}$ denote the left-inverse of $\mathbf{B}$, that is $\mathbf{B}^\dagger\mathbf{B} = \mathbf{I}_{n_u}$, and define the following matrix function

$$\mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s) := \mathbf{M}(\mathbf{q})\mathcal{F}(s)\zeta(s) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{F}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}),$$

with $\ddot{\mathbf{q}}_*(s) = \mathcal{F}(s)\zeta(s)$ given $\mathcal{F}(\cdot)$ as defined in (9).

It is not difficult to see that $\mathbf{B}^\dagger\mathbf{U}(\mathbf{q}, \mathbf{\Phi}'(s)\zeta(s), s)$ corresponds to the nominal control input $\mathbf{u}_*$ when on the nominal orbit.[2] Thus, consider the following controller:

$$\mathbf{u} = \mathbf{B}^\dagger\mathbf{U}(\mathbf{q}, \mathbf{\Phi}'(s)\zeta(s), s) + \mathbf{v} \qquad (11)$$

for some stabilizing control input $\mathbf{v} \in \mathbb{R}^{n_u}$ to be defined.

Lemma 4 (Excessive transverse linearization): Under the control law (11), the first approximation (linearization) of the dynamics of (4) along (2) can be written as

$$\delta\dot{\mathbf{x}}_\perp = \mathbf{A}_\perp(s)\delta\mathbf{x}_\perp + \mathbf{B}_\perp(s)\mathbf{v}, \quad \mathbf{D}P_s(s)\delta\mathbf{x}_\perp = 0, \quad (12)$$

where

$$\mathbf{A}_\perp(s) := \mathbf{\Omega}_s(s)\mathbf{A}(s) - \mathbf{x}'_s(s)\mathbf{x}'_s(s)^\mathsf{T}\mathbf{DD}P_s(s)\zeta(s)$$

$$\mathbf{A}(s) \;\; := \begin{bmatrix} \mathbf{0}_{n_q} & \mathbf{I}_{n_q} \\ \mathbf{a}_{21}(s) & \mathbf{a}_{22}(s) \end{bmatrix},$$

$$\mathbf{B}_\perp(s) := \mathbf{\Omega}_s(s)\begin{bmatrix} \mathbf{0}_{n_q \times n_u} \\ \mathbf{M}(\mathbf{\Phi}(s))^{-1}\mathbf{B} \end{bmatrix}$$

$$\mathbf{a}_{21}(s) := \mathbf{M}(\mathbf{\Phi}(s))^{-1}\left(\mathbf{BB}^\dagger - \mathbf{I}_{n_q}\right)\frac{\partial\mathbf{U}}{\partial\mathbf{q}}(\mathbf{\Phi}(s), \mathbf{\Phi}'(s)\zeta, s)$$

$$\mathbf{a}_{22}(s) := -\mathbf{M}(\mathbf{\Phi}(s))^{-1}\left(2\mathbf{C}(\mathbf{\Phi}(s), \mathbf{\Phi}'(s)\zeta) + \mathbf{F}(\mathbf{\Phi}(s))\right).$$

$\square$
The proof of Lemma 4 is found in Appendix B.

Remark 1: The matrix $\mathbf{A}_\perp(s)$ is not unique. Indeed, as $\delta\mathbf{x}_\perp = \mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp$ and $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$, the matrix function $\mathbf{A}_\perp(s)\mathbf{\Omega}_s(s) + \mathbf{X}(s)\mathbf{D}P_s(s)$ would also be a valid choice for any smooth (bounded) vector function $\mathbf{X} : \mathcal{S} \to \mathbb{R}^{2n_q}$. $\square$

Remark 2: When considering fully actuated systems ($n_u \equiv n_q$), one has $\mathbf{BB}^\dagger = \mathbf{I}_{n_q}$, and so $\mathbf{a}_{21} \equiv \mathbf{0}_{n_q}$. Furthermore, this would allow for the eliminations of $\mathbf{a}_{22}$ by the use of a partial feedback linearizing-like controller of the form $\mathbf{u} = \mathbf{B}^\dagger\mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s) + \mathbf{v}$. $\square$

The main value of the above Lemma is that it clearly shows that for a particular choice of a "feedforward"-like input[3], in this case (11), one can find, utilizing the particular structure of mechanical systems, explicit expressions for the excessive linearized transverse dynamics, valid for any trajectory of the form (2) and any projection (3).

While it is known (see e.g. [6]) that the system (4) can be successfully stabilized in the fully actuated case ($n_u \equiv n_q$) by a linear feedback of the form $\mathbf{v} = \mathbf{K}\delta\mathbf{x}_\perp$ with $\mathbf{K} \in \mathbb{R}^{n_q \times 2n_q}$ constant, this will not be possible in general for underactuated systems. We therefore address this issue next.

IV. STABILIZATION OF THE TRANSVERSE DYNAMICS

Since we only consider the case of periodic orbits, for which it is well known that the (asymptotic) stability of the first approximation (linearization) implies (asymptotic) stability of the nonlinear system (see e.g. [3]), the following statement holds.

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[2]Note here that one instead can utilize $\mathbf{B}^\dagger\mathbf{U}(\mathbf{\Phi}(s), \mathbf{\Phi}'(s)\zeta(s), s)$ or $\mathbf{B}^\dagger\mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s)$ as they also corresponds to the nominal control input when on the orbit, but this will result in slightly different linearizations.

[3]The quotations marks are used as it is always state-dependent.

Lemma 5: Suppose that there exists a continuously differentiable matrix function $\mathbf{K} : \mathcal{S} \to \mathbb{R}^{n_u \times 2n_q}$ such that by taking $\mathbf{v} = \mathbf{K}(s)\delta\mathbf{x}_\perp$, the origin of the system (12) becomes asymptotically stable. Then the controller (11) with $\mathbf{v} = \mathbf{K}(P(\mathbf{x}))\mathbf{x}_\perp$ renders the desired periodic orbit η_* orbitally asymptotically stable. $\square$

The question then arises as to how one can find such a matrix function $\mathbf{K}(\cdot)$. If, for instance, the pair $(\mathbf{A}_\perp(\cdot), \mathbf{B}_\perp(\cdot))$ were stabilizable, then it is known (see e.g. [13]) that an exponentially stabilizing controller would be given by

$$\mathbf{v} = -\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}(s)\mathbf{R}(s)\mathbf{x}_\perp, \quad (13)$$

where the matrix function $\mathbf{R}(\cdot)$ is the symmetric, positive semi-definite solution of the differential Riccati equation,

$$\dot{\mathbf{R}}(s) + \mathbf{A}_\perp^\mathsf{T}(s)\mathbf{R}(s) + \mathbf{R}(s)\mathbf{A}_\perp(s) + \mathbf{Q}$$
$$+ \kappa\mathbf{R}(s) - \mathbf{R}(s)\mathbf{B}_\perp(s)\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}(s)\mathbf{R}(s) = 0,$$

for some $\mathbf{\Gamma} = \mathbf{\Gamma}^\mathsf{T} \succ 0$, $\mathbf{Q} = \mathbf{Q}^\mathsf{T} \succ 0$ and $\kappa \geq 0$.

Unfortunately, it can be shown that the pair $(\mathbf{A}_\perp(\cdot), \mathbf{B}_\perp(\cdot))$ is never stabilizable even though the origin of the system (12) can be asymptotically (exponentially) stabilized. This is due to the fact that the system $\dot{\mathbf{w}} = \mathbf{A}_\perp(s)\mathbf{w} + \mathbf{B}_\perp(s)\mathbf{v}$ always has a non-vanishing solution in the direction of $\mathbf{x}_s'(s)$, regardless of the control input $\mathbf{v}$. While showing this fact is beyond the scope of this paper, it can be readily seen by applying the coordinate change $\mathbf{w} = [\mathbf{x}_s'(s), \mathbf{p}_\perp(s)]\xi$, where $\mathbf{p}_\perp(s)$ is an orthonormal basis of the kernel (nullspace) of $\mathbf{D}P_s(s)$, and derive the dynamics of the coordinates ξ. Indeed, the n excessive coordinates $\mathbf{x}_\perp$ must locally (i.e. sufficiently close to η_*) be within the $(n-1)$-dimensional space spanned by $\mathbf{p}_\perp(s)$. But the linear system $\dot{\mathbf{w}} = \mathbf{A}_\perp(s)\mathbf{w} + \mathbf{B}_\perp(s)\mathbf{v}$ must have n linearly independent solutions. It turns out that this remaining solution, i.e. $\mathbf{x}_s'(s(t))\xi_1(t)$, is non-vanishing and independent of $\mathbf{v}$.

Regardless, this clearly shows the necessity of the condition $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp = 0$ in (12). Luckily, just minor modifications are needed in order to account for this, allowing for the generation of a stabilizing controller.

This leads us to the main result of this paper.

Theorem 1: Suppose there exists a symmetric, positive semi-definite solution $\mathbf{R}_\perp(\cdot)$ to the following modified periodic differential Riccati equation

$$\mathbf{\Omega}_s^\mathsf{T}(s)\Big[\mathbf{R}_\perp'(s)\zeta(s) + \mathbf{A}_\perp^\mathsf{T}(s)\mathbf{R}_\perp(s) + \mathbf{R}_\perp(s)\mathbf{A}_\perp(s) + \mathbf{Q}$$
$$+ \kappa\mathbf{R}_\perp(s) - \mathbf{R}_\perp(s)\mathbf{B}_\perp(s)\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}(s)\mathbf{R}_\perp(s)\Big]\mathbf{\Omega}_s(s) = 0 \quad (14)$$

with $\mathbf{R}_\perp'(s) = \frac{d}{ds}\mathbf{R}_\perp(s)$ and for some $\mathbf{\Gamma} = \mathbf{\Gamma}^\mathsf{T} \succ 0$, $\mathbf{Q} = \mathbf{Q}^\mathsf{T} \succ 0$ and $\kappa \geq 0$. Then the control law

$$\mathbf{u} = \mathbf{B}^\dagger\mathbf{U}(\mathbf{q}, \mathbf{\Phi}'(s)\zeta(s), s) - \mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}(s)\mathbf{R}_\perp(s)\mathbf{x}_\perp \quad (15)$$

renders the periodic orbit (2) of the mechanical system (1) locally orbitally asymptotically stable. $\square$
The proof of Theorem 1 is given in Appendix C.

It follows that a solution to the projected periodic differential Riccati equation (14) can only exist if the pair $(\mathbf{A}_\perp(s), \mathbf{B}_\perp(s))$ is stabilizable on the set of solutions satisfying the condition $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$. However, the question of the existence of solutions of this equation is, to our best knowledge, unknown. Although should a solution exists, it is likely not to be unique. Nevertheless, as we will see in the example of Sec. VI, we have been able to find very good approximate solutions using numerical methods.

V. ON OBTAINING A PROJECTION OPERATOR

We have so far assumed that the projection operator (3), i.e. $s = P(\mathbf{x})$, is known and given as an explicit equation which is at least well defined within some neighbourhood of the orbit. Yet, this might not necessarily always be the case. That is to say, if one has found a feasible trajectory of the system (1) parameterized on the form (2), the corresponding motion generator might only be known as a function of time, i.e. $s = s(t)$; indeed, $s = t$ is of course the most commonly used parameterization of trajectories. Therefore, we will briefly show next how one can generate projection operators given only knowledge of the nominal trajectory and the time evolution of its parameterizing variable (MG).

The following statement follows directly from the implicit function theorem.

Proposition 1: Assume that on a given subarc of the trajectory (2), denoted $\mathcal{S}_k \subseteq \mathcal{S}$, there exists a function $h_k : \mathbb{R}^{2n_q} \times \mathcal{S}_k \to \mathbb{R}$ satisfying

$$h_k(\mathbf{x}_s(s), s) \equiv 0 \ \text{ and } \ \frac{\partial h_k}{\partial s}(\mathbf{x}_s(s), s) \neq 0, \ \ \forall s \in \mathcal{S}_k. \ (16)$$

Then, in some non-zero tubular neighbourhood $\mathcal{X}_k \subset \mathbb{R}^{2n_q}$ of the orbit η_*, there exists a function $P_k : \mathcal{X}_k \to \mathcal{S}_k$ such that for all $\mathbf{x} \in \mathcal{X}_k$, we have $h_k(\mathbf{x}, P_k(\mathbf{x})) = 0$ as well as

$$\mathbf{D}P_k(\mathbf{x}) = -\left(\frac{\partial h_k}{\partial s}(\mathbf{x}, P_k(\mathbf{x}))\right)^{-1}\frac{\partial h_k}{\partial \mathbf{x}}(\mathbf{x}, P_k(\mathbf{x})).$$

$\square$

It follows that if a function $h_k(\cdot)$ satisfying the conditions of Proposition 1 for $\mathbf{x}(t) \in \mathcal{X}_k$ is found, then one can take

$$s = P_k(\mathbf{x}(t))$$

as the projection of the states at time t onto the $\mathcal{S}_k$ subarc of the orbit (2). Such a function $h(\cdot)$ can often be found satisfying (16) on the whole trajectory. Indeed, $s = \arg\min_{s \in \mathcal{S}} \|\mathbf{x}(t) - \mathbf{x}_s(s)\|^2$ is a generic choice for any trajectory of the form (2). This choice, which has been considered several times before in relation to stability analysis of autonomous dynamical systems (see e.g. [3], [14], [15], [4]), results in the condition $\mathbf{x}_s'(s)^\mathsf{T}\mathbf{x}_\perp \equiv 0$, and hence

$$\mathbf{D}P(\mathbf{x}) = \frac{\mathbf{x}_s'(s)^\mathsf{T}}{\|\mathbf{x}_s'(s)\|^2 - \mathbf{x}_s''(s)\mathbf{x}_\perp}.$$

Consequently, the projection (3) can be found numerically by, for instance, a few iterations of Newton's method.
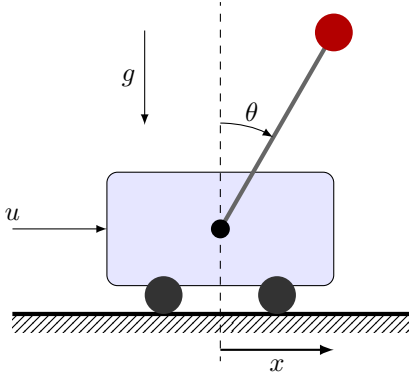
Fig. 1: The cart-pendulum system, consisting of an unactuated pendulum attached to an actuated cart which is driven along the horizontal by an external force.

Fig. 2: Solutions of (19) with the parameterization (20).

VI. EXAMPLE: UPRIGHT OSCILLATIONS OF THE CART-PENDULUM SYSTEM

Let us now illustrate the procedure outlined in Sec. III-IV by stabilizing oscillations around the unstable upright equilibrium of the cart-pendulum system, as illustrated in Fig. 1. To keep the notation simple, we consider unit masses, and consider the pendulum bob to be a point mass, while its rod is considered to be massless and of unit length. With the generalized coordinates $\mathbf{q} = [x, \theta]^\mathsf{T}$ as defined in Fig. 1, the equations of motion of the system are

$$2\ddot{x} + \cos(\theta)\ddot{\theta} - \sin(\theta)\dot{\theta}^2 = u, \tag{17a}$$

$$\ddot{\theta} + \cos(\theta)\ddot{x} - g\sin(\theta) = 0, \tag{17b}$$

where $g = 9.81\,\mathrm{m\,s^{-2}}$ is the gravitational acceleration.

This task has previously been considered in [10] utilizing the *virtual constraints approach*, where a feasible trajectory of the nonlinear system (17) was generated in the following way. Under the assumption that along a nominal trajectory of the system a set of relations of the form $(x_*, \theta_*)^\mathsf{T} = \mathbf{\Phi}(s) = [\phi_1(s), \phi_2(s)]^\mathsf{T}$ are kept invariant, one can write (17b) as

$$\underbrace{(\phi_2' + \phi_1' \cos(\phi_2))}_{\alpha(s)}\ddot{s} + \underbrace{(\phi_2'' + \phi_1'' \cos(\phi_2))}_{\beta(s)}\dot{s}^2 \underbrace{- g\sin(\phi_2)}_{\gamma(s)} = 0. \tag{18}$$

This constrains the time-evolution of the motion generator, s, for the particular choice of $\mathbf{\Phi}(s)$ and its initial velocity $\dot{s}_0 = \dot{s}(0)$. Moreover, the nominal velocity $\zeta(s) = \dot{s}_*$ can be found as (18) is integrable [10]; indeed, the equality

$$\frac{1}{2}\exp\left\{\int_{s_0}^{s} \frac{2\delta(\sigma)}{\alpha(\sigma)}d\sigma\right\}\alpha^2(s)\zeta^2(s) - \frac{1}{2}\alpha^2(s_0)\zeta^2(s_0)$$
$$+ \int_{s_0}^{s}\exp\left\{\int_{s_0}^{\tau}\frac{2\delta(\sigma)}{\alpha(\sigma)}d\sigma\right\}\alpha(\tau)\gamma(\tau)d\tau \equiv 0 \tag{19}$$

must hold, where $\delta(s) := \beta(s) - \frac{d}{ds}\alpha(s)$. Consequently, the nominal control input $u_* = u_*(s)$ can be found from (17a).

In [10], the holonomic relations $\mathbf{\Phi}(\theta) = [-1.5\sin(\theta), \theta]^\mathsf{T}$ are utilized, i.e. the motion generator is simply $s = \theta$, which results in a center at the equilibrium $\theta = \dot{\theta} = 0$. While $s = \theta$ is clearly a convenient choice in this case, it is not consistent
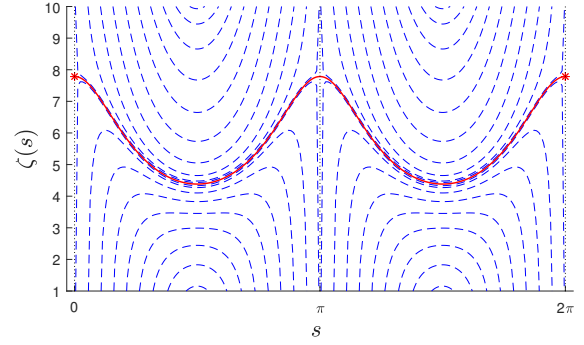
with our parameterization (2) as we require $\dot{s}_* > 0$. Thus, with $s \in [0, 2\pi]$ and $\dot{s}_* = \zeta(s) > 0$, let us instead consider

$$\mathbf{\Phi}(s) = \left[-1.5\sin\left(a_2\cos(s)\right), a_2\cos(s)\right]^\mathsf{T}, \tag{20}$$

which, as we will see, is not holonomic as the projection $P(\cdot)$ will depend on (some of) the generalized velocities. Hence, we can pick a_2 to get the appropriate amplitude of the oscillations of the pendulum, compared to [10] in which the amplitude was determined by the initial conditions $(\theta_0, \dot{\theta}_0)$. Furthermore, while the parameterization in [10] results in a family of periodic solutions around the equilibrium, there exists a unique function $\zeta(s)$ for each choice of a_2 in the parameterization (20). For example, the unique solution for the case of $a_2 = 0.5$ is the red line highlighted in Fig. 2.

It is here worth to note that, even though $\alpha(s_s) \equiv 0$ for $s_s \in \{0, \pi, 2\pi\}$, i.e. s_s are singular points[4] of the equation (18), the solution $\zeta(s)$ of (19) is well defined over the interval $[0, 2\pi]$ if we take $\zeta(s_0)$ satisfying $\beta(s_s)\zeta^2(s_s) + \gamma(s_s) = 0$. Therefore, unlike most existing methods (see, e.g. [10]) which require $\alpha(s) \neq 0$ for all $s \in \mathcal{S}$, the existence of such singularities is irrelevant for our method.

Now, with the parameterization (20), it is clear that we get $\dot{\theta}_*(s) = \phi_2'(s)\zeta(s) = -a_2\sin(s)\zeta(s)$. Hence we can find s as the root of the implicit equation

$$h(\mathbf{x}, s) = s - \arctan2\left(\frac{\dot{\theta}}{-\theta\zeta(s)}\right) = 0,$$

enabling us to use the method outlined in Section V. (Here $\arctan2(\cdot)$ denotes the four-quadrant arctangent function).

In order to demonstrate the proposed control scheme, we found that taking $a_2 = 0.1129$ resulted in a periodic orbit very close to the one considered in [10]. Figure 3 shows the results of from numerically simulating the system with initial conditions

$$x_0 = 0.1, \ \theta_0 = 0.4, \ \dot{x}_0 = -0.1, \ \dot{\theta}_0 = -0.2$$

and with white noise added to the measurements. The system is seen to converge to the orbit after approximately $13\,\mathrm{s}$.
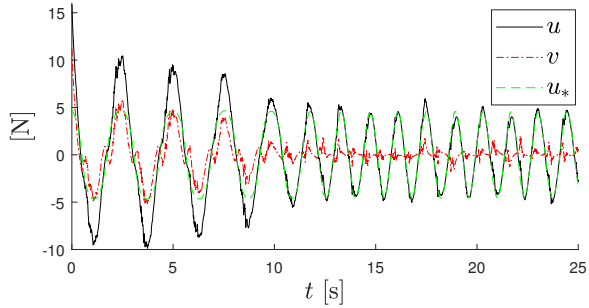
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[4]Note here that the type of (simple) singularities presented here are just a product of the choice of parameterization and not due to the non-uniqueness of (phase-space) solutions as in [12].
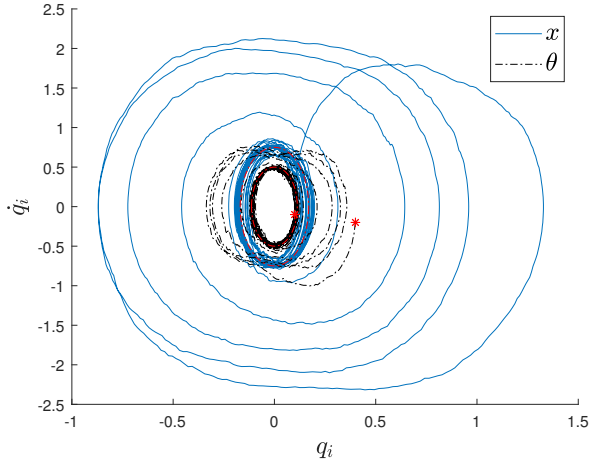
(a) Phase portraits of the cart (**blue**) and pendulum (**black**).
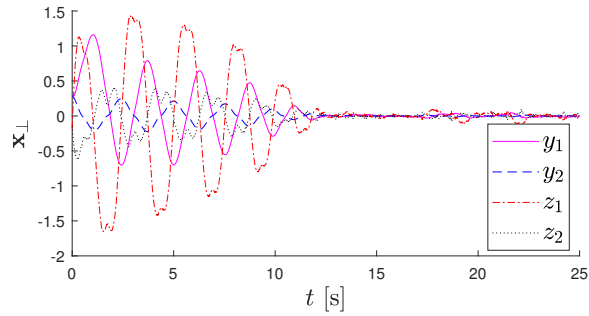
(b) Transverse coordinates.

(c) Control input.

Fig. 3: Results from simulating the cart-pendulum system with perturbed initial conditions and noisy measurements.

Here a feedback LQR-controller of the form (13) was generated by solving (14) with $\mathbf{Q} = \mathbf{I}_4$, $\Gamma = \kappa = 0.1$. This was achieved using the semi-definite programming method of [16] with a trigonometric polynomial of order 40 and utilizing the YALMIP toolbox [17] and the SDPT3 solver [18]. The resulting solution satisfied (14) within a maximum error norm of less than 2×10^{-4} for all $s \in [0, 2\pi]$.

VII. CONCLUDING REMARKS AND FUTURE WORK

In this paper, we have introduced a set of excessive transverse coordinates for the purpose of asymptotically orbitally stabilizing periodic trajectories of underactuated Euler-Lagrange systems. We have shown that one can derive analytical expressions for the corresponding excessive transverse linearization for a particular feedforward-like controller. We then stated a sufficient condition for stabilizing this linear system, which consequently allows for the construction of feedback controllers rendering the desired periodic motion orbitally asymptotically stable. The proposed method was implemented and successfully tested in numerical simulation for stabilizing oscillations of the cart-pendulum system around its unstable upright position. This example also illustrated that proposed methodology can be used to stabilize trajectories for which the reduced dynamics have singular points. For future work, experimental validation of the proposed scheme is currently being pursued.

APPENDIX A. PROOF OF LEMMA 3

It here suffices to show that the asymptotic stability of the variation $\delta\mathbf{x}_\perp$ is equivalent to that of $\delta\mathbf{y}_\perp$.

By the hypothesis that the mapping $(s, \mathbf{y}_\perp) \mapsto \mathbf{x}$ is a diffeomorphism in a neighbourhood of η_*, it follows that

$$\frac{d\mathbf{y}_\perp}{d\mathbf{x}}(s, \mathbf{x}_s(s)) = \frac{\partial\mathbf{y}_\perp}{\partial\mathbf{x}}(s, \mathbf{x}_s(s)) + \frac{\partial\mathbf{y}_\perp}{\partial s}(s, \mathbf{x}_s(s))\mathbf{D}P_s(s)$$

has full (row) rank for all $s \in \mathcal{S}$. But as $\mathbf{y}_\perp(s, \mathbf{x}_s(s)) \equiv 0$, we have $\frac{d\mathbf{y}_\perp}{d\mathbf{x}}(s, \mathbf{x}_s(s))\mathbf{x}'_s(s) \equiv \mathbf{0}$, such that by (8) and by defining $\mathbf{\Pi}_s(s) := \frac{\partial\mathbf{y}_\perp}{\partial\mathbf{x}}(s, \mathbf{x}_s(s))$, we obtain the relation

$$\mathbf{\Pi}_s(s)\mathbf{x}'_s(s) = -\frac{\partial\mathbf{y}_\perp}{\partial s}(s, \mathbf{x}_s(s)).$$

This further implies that

$$\delta\mathbf{y}_\perp = \frac{d\mathbf{y}_\perp}{d\mathbf{x}}(s, \mathbf{x}_s(s))\delta\mathbf{x} = \mathbf{\Pi}_s(s)\mathbf{\Omega}_s(s)\delta\mathbf{x},$$

such that from (6), i.e. $\delta\mathbf{x}_\perp = \mathbf{\Omega}_s(s)\delta\mathbf{x}$, we obtain

$$\delta\mathbf{y}_\perp = \mathbf{\Pi}_s(s)\delta\mathbf{x}_\perp. \tag{21}$$

Therefore, from Lemma 1 and the fact that $\mathbf{\Pi}_s(s)\mathbf{\Omega}_s(s)$ is of full rank $(2n_q - 1)$, it follows from (21) that $\|\delta\mathbf{y}_\perp\| \to 0$ as $t \to \infty$ only if $\|\delta\mathbf{x}_\perp\| \to 0$.

Let us now prove that the converse is true as well, namely that $\|\delta\mathbf{x}_\perp\| \to 0$ as $t \to \infty$ only if $\|\delta\mathbf{y}_\perp\| \to 0$.

Take $\mathbf{p}_\perp : \mathcal{S} \to \mathbb{R}^{n\times(n-1)}$ to be some differentiable basis of the the kernel of $\mathbf{D}P_s(s)$. As then $\mathbf{p}_\perp(s) = \mathbf{\Omega}_s(s)\mathbf{p}_\perp(s)$, as well as the full rank property of $\mathbf{\Pi}_s(s)\mathbf{\Omega}_s(s)$, it follows that $\mathbf{\Pi}_s(s)\mathbf{p}_\perp(s)$ is invertible for all $s \in \mathcal{S}$. Hence

$$\begin{bmatrix} \mathbf{D}P_s(s) \\ \mathbf{\Pi}_s(s)\mathbf{\Omega}_s(s) \end{bmatrix} \begin{bmatrix} \mathbf{x}'_s(s) & \mathbf{p}_\perp(s)\left(\mathbf{\Pi}_s(s)\mathbf{p}_\perp(s)\right)^{-1} \end{bmatrix} = \mathbf{I}_n,$$

which implies that

$$\delta\mathbf{x} = \mathbf{p}_\perp(s)\left(\mathbf{\Pi}_s(s)\mathbf{p}_\perp(s)\right)^{-1}\delta\mathbf{y}_\perp + \mathbf{x}'_s(s)\delta s.$$

Left-multiplying the above equation by $\mathbf{\Omega}_s(s)$, we obtain

$$\delta\mathbf{x}_\perp = \mathbf{p}_\perp(s)\left(\mathbf{\Pi}_s(s)\mathbf{p}_\perp(s)\right)^{-1}\delta\mathbf{y}_\perp.$$

It follows that $\|\delta\mathbf{x}_\perp\| \to 0$ only if $\|\delta\mathbf{y}_\perp\| \to 0$. ∎

APPENDIX B. PROOF OF LEMMA 4

Let $\mathbf{z}_\perp := \dot{\mathbf{q}} - \mathbf{\Phi}'(s)\zeta(s)$ and note that the dynamics of the system (1) can be rewritten in the form

$$\dot{\mathbf{x}} = \mathbf{x}'_s(s)\zeta(s) + \begin{bmatrix} \mathbf{z}_\perp \\ -\mathbf{M}(\mathbf{q})^{-1}\mathbf{U}(\mathbf{q},\dot{\mathbf{q}},s) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_q \times n_u} \\ \mathbf{M}(\mathbf{q})^{-1}\mathbf{B} \end{bmatrix}\mathbf{u}.$$

Moreover, using the property $\mathbf{C}(\mathbf{q},X)Y = \mathbf{C}(\mathbf{q},Y)X$ for any $X, Y \in \mathbb{R}^{n_q}$, we can write $\mathbf{U}(\mathbf{q},\dot{\mathbf{q}},s)$ as

$$\begin{aligned}\mathbf{U}(\mathbf{q},\dot{\mathbf{q}},s) =& \mathbf{U}(\mathbf{q},\mathbf{\Phi}'(s)\zeta(s),s) + \mathbf{C}(\mathbf{q},\mathbf{z}_\perp)\mathbf{z}_\perp \\ &+ 2\mathbf{C}(\mathbf{q},\mathbf{\Phi}'(s)\zeta(s))\mathbf{z}_\perp + \mathbf{F}(\mathbf{q})\mathbf{z}_\perp.\end{aligned}$$

Thus taking the controller as in (11), we obtain

$$\dot{\mathbf{x}} = \mathbf{x}'_s(s)\zeta(s) + \mathbf{f}_U(\mathbf{q},s) + \mathbf{f}_{\dot{q}}(\mathbf{q},\mathbf{z}_\perp,s)\mathbf{z}_\perp + \begin{bmatrix} \mathbf{0}_{n_q \times n_u} \\ \mathbf{M}(\mathbf{q})^{-1}\mathbf{B} \end{bmatrix}\mathbf{v},$$

where

$$\mathbf{f}_U := \begin{bmatrix} \mathbf{0}_{n_q} \\ \mathbf{M}(\mathbf{q})^{-1}\left[(\mathbf{B}\mathbf{B}^\dagger - \mathbf{I}_n)\mathbf{U}(\mathbf{q},\mathbf{\Phi}'\zeta,s)\right] \end{bmatrix},$$

$$\mathbf{f}_{\dot{q}} := \begin{bmatrix} \mathbf{I}_{n_q} \\ -\mathbf{M}(\mathbf{q})^{-1}\left[\mathbf{C}(\mathbf{q},\mathbf{z}_\perp) + 2\mathbf{C}(\mathbf{q},\mathbf{\Phi}'\zeta) + \mathbf{F}(\mathbf{q})\right] \end{bmatrix}.$$

Note here that $(\mathbf{B}\mathbf{B}^\dagger - \mathbf{I}_n)\mathbf{U}(\mathbf{\Phi},\mathbf{\Phi}'\zeta,s) \equiv 0$.

Consider now (5). In order to linearize this system along the orbit, we note that for a differentiable function $\mathbf{h} : \mathbb{R}^{2n_q} \to \mathbb{R}^{2n_q}$, which, for all $s \in \mathcal{S}$, satisfies $\mathbf{h}(\mathbf{x}_s(s)) \equiv \mathbf{0}_{2n_q \times 1}$, then the relations

$$\frac{\partial \mathbf{h}}{\partial s}(\mathbf{x}_s(s)) \equiv 0 \quad \text{and} \quad \frac{\partial \mathbf{h}}{\partial \mathbf{x}_\perp}(\mathbf{x}_s(s))\mathbf{\Omega}_s(s) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}_s(s))$$

always hold [6]. Thus, if we write $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{v}$, then the matrix $\mathbf{B}_\perp(s) = \mathbf{\Omega}_s(s)\mathbf{g}(\mathbf{x}_s(s))$ follows from the fact that $\dot{\mathbf{x}}_\perp$ is affine in the control input $\mathbf{v}$, whereas the matrix $\mathbf{A}_\perp(s)$ must be the solution to the matrix equation

$$\mathbf{A}_\perp(s)\mathbf{\Omega}_s(s) = \mathbf{\Omega}_s(s)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_s(s)) - \Xi(s)\zeta(s)$$

with $\Xi(s) := \mathbf{x}'_s(s)\mathbf{x}'_s(s)^\mathsf{T}\mathbf{D}\mathbf{D}P_s(s) + \mathbf{x}''_s(s)\mathbf{D}P_s(s)$. However, as $\mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp = \delta\mathbf{x}_\perp$ and $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$, we must have $\mathbf{A}_\perp(s)\delta\mathbf{x}_\perp = \mathbf{A}_\perp(s)\mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp$; hence, one can take $\mathbf{A}_\perp(s) = \mathbf{\Omega}_s(s)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_s(s)) - \Xi(s)\zeta(s)$. Lastly, using that $\frac{\partial h(s)}{\partial \mathbf{x}}\delta\mathbf{x}_\perp = \frac{\partial h(s)}{\partial s}\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$ for any $h(s)$, we see that $\mathbf{A}_\perp(s)$ in (12) is a solution to this matrix equation. ∎

APPENDIX C. PROOF OF THEOREM 1

We will show that if a solution to (14) exists, then the system (12) is rendered asymptotically stable by controller

$$\mathbf{v} = -\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}(s)\mathbf{R}_\perp(s)\delta\mathbf{x}_\perp,$$

which lets us utilize Lemma 5. Towards this end, consider the following Lyapunov function candidate

$$V = \delta\mathbf{x}_\perp^\mathsf{T}\mathbf{R}_\perp(s)\delta\mathbf{x}_\perp.$$

Due to the condition $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$, and consequently $\delta\mathbf{x}_\perp = \mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp$, it can be equivalently rewritten as

$$V = \delta\mathbf{x}_\perp^\mathsf{T}\mathbf{\Omega}_s^\mathsf{T}(s)\mathbf{R}_\perp(s)\mathbf{\Omega}_s(s)\delta\mathbf{x}_\perp.$$

Now note that

$$\dot{\mathbf{\Omega}}_s(s) = -[\mathbf{x}''_s(s)\mathbf{D}P_s(s) + \mathbf{x}'_s(s)\mathbf{x}'_s(s)^\mathsf{T}\mathbf{D}\mathbf{D}P_s(s)]\zeta(s).$$

Hence

$$\begin{aligned}\dot{V} =& -2\delta\mathbf{x}_\perp^\mathsf{T}\left[\mathbf{\Omega}_s^\mathsf{T}\mathbf{R}_\perp\mathbf{x}''_s\mathbf{D}P_s\zeta\right]\delta\mathbf{x}_\perp + \delta\mathbf{x}_\perp^\mathsf{T}\left[\mathbf{\Omega}_s^\mathsf{T}\dot{\mathbf{R}}_\perp\mathbf{\Omega}_s + \right. \\ &\left.\mathbf{A}_\perp^\mathsf{T}\mathbf{R}_\perp\mathbf{\Omega}_s + \mathbf{\Omega}_s^\mathsf{T}\mathbf{R}_\perp\mathbf{A}_\perp - 2\mathbf{\Omega}_s^\mathsf{T}\mathbf{R}_\perp\mathbf{B}_\perp\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}\mathbf{R}_\perp\mathbf{\Omega}_s\right]\delta\mathbf{x}_\perp\end{aligned}$$

which, by the fact that $\mathbf{\Omega}_s(s)^2 = \mathbf{\Omega}_s(s)$, is equivalent to

$$\begin{aligned}\dot{V} = \delta\mathbf{x}_\perp^\mathsf{T}\mathbf{\Omega}_s^\mathsf{T}\Big[&\dot{\mathbf{R}}_\perp + \mathbf{A}_\perp^\mathsf{T}\mathbf{R}_\perp + \mathbf{R}_\perp\mathbf{A}_\perp \\ &- 2\mathbf{R}_\perp\mathbf{B}_\perp\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}\mathbf{R}_\perp\Big]\mathbf{\Omega}_s\delta\mathbf{x}_\perp,\end{aligned}$$

where $\dot{\mathbf{R}}_\perp(s) = \mathbf{R}'_\perp(s)\zeta(s)$. Thus by (14), we obtain

$$\begin{aligned}\dot{V} =& \delta\mathbf{x}_\perp^\mathsf{T}\mathbf{\Omega}_s^\mathsf{T}\Big[-\mathbf{Q} - \kappa\mathbf{R}_\perp - \mathbf{R}_\perp\mathbf{B}_\perp(s)\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}\mathbf{R}_\perp\Big]\mathbf{\Omega}_s\delta\mathbf{x}_\perp \\ =& \delta\mathbf{x}_\perp^\mathsf{T}\Big[-\mathbf{Q} - \kappa\mathbf{R}_\perp - \mathbf{R}_\perp\mathbf{B}_\perp(s)\mathbf{\Gamma}^{-1}\mathbf{B}_\perp^\mathsf{T}\mathbf{R}_\perp\Big]\delta\mathbf{x}_\perp.\end{aligned}$$

Consequently, for all $\|\delta\mathbf{x}_\perp\| \neq 0$ satisfying $\mathbf{D}P_s(s)\delta\mathbf{x}_\perp \equiv 0$, we have $\dot{V} < 0$, which implies asymptotic stability of the origin of (12). ∎

ACKNOWLEDGMENT

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