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# Repetition avoidance in products of factors

Pamela Fleischmann\* Pascal Ochem† Kamellia Reshadi‡

## Abstract

We consider a variation on a classical avoidance problem from combinatorics on words that has been introduced by Mousavi and Shallit at DLT 2013. Let  $\mathbf{pexp}_i(w)$  be the supremum of the exponent over the products of  $i$  factors of the word  $w$ . The repetition threshold  $\text{RT}_i(k)$  is then the infimum of  $\mathbf{pexp}_i(w)$  over all words  $w \in \Sigma_k^\omega$ . Mousavi and Shallit obtained that  $\text{RT}_i(2) = 2i$  and  $\text{RT}_2(3) = \frac{13}{4}$ . We show that  $\text{RT}_i(3) = \frac{3i}{2} + \frac{1}{4}$  if  $i$  is even and  $\text{RT}_i(3) = \frac{3i}{2} + \frac{1}{6}$  if  $i$  is odd and  $i \geq 3$ .

**Keywords:** Words; Repetition avoidance.

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## 1 Introduction

A *repetition* in a word  $w$  is a pair of words  $p$  and  $e$  such that  $pe$  is a factor of  $w$ ,  $p$  is non-empty, and  $e$  is a prefix of  $pe$ . If  $pe$  is a repetition, then its *period* is  $|p|$  and its *exponent* is  $\frac{|pe|}{|p|}$ . A word is  $\alpha^+$ -free (resp.  $\alpha$ -free) if it contains no repetition with exponent  $\beta$  such that  $\beta > \alpha$  (resp.  $\beta \geq \alpha$ ).

Given  $k \geq 2$ , Dejean [2] defined the repetition threshold  $\text{RT}(k)$  for  $k$  letters as the smallest  $\alpha$  such that there exists an infinite  $\alpha^+$ -free word over a  $k$ -letter alphabet. Dejean initiated the study of  $\text{RT}(k)$  in 1972 for  $k = 2$  and  $k = 3$ . Her work was followed by a series of papers which determine the exact value of  $\text{RT}(k)$  for any  $k \geq 2$ .

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- $\text{RT}(2) = 2$  [2];
- $\text{RT}(3) = \frac{7}{4}$  [2];
- $\text{RT}(4) = \frac{7}{5}$  [7];
- $\text{RT}(k) = \frac{k}{k-1}$ , for  $k \geq 5$  [1, 4, 8].

Mousavi and Shallit [5] have considered two notions related to the repetition threshold.

The first notion considers repetitions in conjugates of factors of the infinite word. A word is circularly  $r^+$ -free if it does not contain a factor  $pxs$  such that  $sp$  is a repetition of exponent strictly greater than  $r$ . Let  $\Sigma_k = \{0, 1, \dots, k-1\}$ . The smallest real number  $r$  such that  $w$  is circularly  $r^+$ -free is denoted by  $\text{cexp}(w)$ . Let  $\text{RTC}(k)$  be the minimum of  $\text{cexp}(w)$  over every  $w \in \Sigma_k^\omega$ .

The second notion considers repetitions in concatenations of a fixed number of factors of the infinite word. Let  $\text{pexp}_i(w)$  be the smallest real number  $r$  such that every product of  $i$  factors of  $w$  is  $r^+$ -free. Let  $\text{RT}_i(k)$  be the minimum of  $\text{pexp}_i(w)$  over every  $w \in \Sigma_k^\omega$ . Notice that  $\text{RT}_i(k)$  generalizes the classical notion of repetition threshold which corresponds to the case  $i = 1$ , that is,  $\text{RT}_1(k) = \text{RT}(k)$  for every  $k \geq 2$ .

Our first result shows that the case  $i = 2$  corresponds to the first notion of repetition avoidance in conjugates.

**Theorem 1.**  $\text{RT}_2(k) = \text{RTC}(k)$  for every  $k \geq 2$ .

Mousavi and Shallit [5] have considered the binary alphabet and obtained that  $\text{RT}_i(2) = 2i$  for every  $i \geq 1$ . Our second result considers the ternary alphabet and gives the value of  $\text{RT}_i(3)$  for every  $i \geq 1$ . This extends the result of Dejean [2] that  $\text{RT}_1(3) = \frac{7}{4}$  and the result of Mousavi and Shallit [5] that  $\text{RT}_2(3) = \frac{13}{4}$ .

**Theorem 2.**

- $\text{RT}_i(3) = \frac{3i}{2} + \frac{1}{4}$  if  $i = 1$  or  $i$  is even.
- $\text{RT}_i(3) = \frac{3i}{2} + \frac{1}{6}$  if  $i$  is odd and  $i \geq 3$ .

## 2 Proofs

*Proof of Theorem 1.*

The language of words in  $\Sigma_k^*$  avoiding circular repetitions of exponent at least  $e$  (or strictly greater than  $e$ ) is a factorial language. As it is well-known [3], if a factorial language is infinite, then it contains a uniformly recurrent word  $w$ . By Proposition 14 in [5],  $\text{pexp}_2(w) = \text{cexp}(w)$ . This implies that  $\text{RT}_2(k) = \text{RTC}(k)$ .  $\square$

To obtain the two equalities of Theorem 2, we show the two lower bounds and then the two upper bounds.

*Proof of  $\text{RT}_i(3) \geq \frac{3i}{2} + \frac{1}{4}$  for every even  $i$ .*

Mousavi and Shallit [5] have proved that  $\text{RT}_2(3) = \frac{13}{4}$ , which settles the case  $i = 2$ . We have double checked their computation of the lower bound  $\text{RT}_2(3) \geq \frac{13}{4}$ . Suppose that  $i$  is a fixed even integer and that  $w_3$  is an infinite ternary word. The lower bound for  $i = 2$  implies that there exists two factors  $u$  and  $v$  such that  $uv = t^e$  with  $e \geq \frac{13}{4}$ . Thus, the prefix  $t^3$  of  $uv$  is also a product of two factors of  $w_3$ . So we can form the  $i$ -terms product  $(t^3)^{i/2-1}uv$  which is a repetition of the form  $t^x$  with exponent  $x = 3\left(\frac{i}{2} - 1\right) + e \geq 3\left(\frac{i}{2} - 1\right) + \frac{13}{4} = \frac{3i}{2} + \frac{1}{4}$ . This is the desired lower bound.  $\square$

*Proof of  $\text{RT}_i(3) \geq \frac{3i}{2} + \frac{1}{6}$  for every odd  $i \geq 3$ .*

Suppose that  $i \geq 3$  is a fixed odd integer, that is,  $i = 2j + 1$ . Suppose that  $w_3$  is a recurrent ternary word such that the product of  $i$  factors of  $w_3$  is never a repetition of exponent at least  $\frac{3i}{2} + \frac{1}{6} = 3j + \frac{5}{3}$ . First,  $w_3$  is square-free since otherwise there would exist an  $i$ -terms product of exponent  $2i$ . Also,  $w_3$  does not contain two factors  $u$  and  $v$  with the following properties:

- $uv = t^3$ ,
- $u = t^e$  with  $e \geq \frac{5}{3}$ .

Indeed, this would produce the  $i$ -terms product  $(uv)^j u$  which is a repetition of the form  $t^x$  with exponent  $x = 3j + e \geq 3j + \frac{5}{3}$ .

So if  $a$ ,  $b$ , and  $c$  are distinct letters, then  $w_3$  does not contain both  $u = abcab$  and  $v = cabc$  and  $w_3$  does not contain both  $u = abcbabc$  and  $v = babcb$ . A computer check shows that no infinite ternary square-free word satisfies this property. This proves the desired lower bound.  $\square$

*Proof of  $\text{RT}_i(3) \leq \frac{3i}{2} + \frac{1}{4}$  for every even  $i$ .*

Let  $i$  be any even integer at least 2. To prove this upper bound, it is sufficient to construct a ternary word  $w$  satisfying  $\text{pexp}_i(w) \leq \frac{3i}{2} + \frac{1}{4}$ . The ternary morphic word used in [5] to obtain  $\text{RT}_2(3) \leq \frac{13}{4}$  seems to satisfy the property. However, it is easier for us to consider another construction. Let us show that the image of every  $7/5^+$ -free word over  $\Sigma_4$  by the following 45-uniform morphism satisfies  $\text{pexp}_i \leq \frac{3i}{2} + \frac{1}{4}$ .

$0 \mapsto 010201210212021012102010212012101202101210212$   
 $1 \mapsto 010201210212012101202101210201021202101210212$   
 $2 \mapsto 010201210120212012102120210121021201210120212$   
 $3 \mapsto 010201210120210121021201210120212012102010212$

Recall that a word is  $(\beta^+, n)$ -free if it does not contain a repetition with period at least  $n$  and exponent strictly greater than  $\beta$ . First, we check that such ternary images are  $\left(\frac{202}{135}^+, 36\right)$ -free using the method in [6]. By Lemma 2.1 in [6], it is sufficient to check this freeness property for the image of every  $7/5^+$ -free word over  $\Sigma_4$  of length smaller than  $\frac{2 \times \frac{202}{135}}{\frac{202}{135} - 5} < 32$ . Since  $\frac{202}{135} < \frac{3}{2}$ , the period of every repetition formed from  $i$  pieces and with exponent at least  $\frac{3i}{2}$  must be at most 35. Then we check exhaustively by computer that the ternary images do not contain two factors  $u$  and  $v$  such that

- $uv = t^e$ ,
- $e > 3$ ,
- $9 \leq |t| \leq 35$ .

Thus, the period of every repetition formed from  $i$  pieces and with exponent strictly greater than  $\frac{3i}{2}$  must be at most 8. So we only need to check that  $\text{pexp}_i \leq \frac{3i}{2} + \frac{1}{4}$  for  $i$ -terms products that are repetitions of period at most 8.

Now the period is bounded, but  $i$  can still be arbitrarily large, a priori. For every factor  $t$  of length at most 8, we define  $\text{pexp}_{i,t}$  as the length of a largest factor of  $t^\omega$  that is a  $i$ -terms product, divided by  $|t|$ . We actually consider conjugacy classes, since if  $t'$  is a conjugate of  $t$ , then  $\text{pexp}_{i,t'} = \text{pexp}_{i,t}$ . Let  $t$  be such a factor. If, for some even  $j$ , we have  $\text{pexp}_{j+2,t} = \text{pexp}_{j,t} + 3$ , then it means that by appending a 2-terms product to a  $j$ -terms product that

corresponds to a maximum factor of  $t^\omega$ , that can only add a cube of period  $|t|$ . This implies that for every  $k$ ,  $\text{pexp}_{j+2k,t} = \text{pexp}_{j,t} + 3k$ .

We have checked by computer that for every conjugacy class of words  $t$  of length at most 8, there exists a (small) even  $j$  such that  $\text{pexp}_{j+2,t} = \text{pexp}_{j,t} + 3$ . Thus we have  $\text{pexp}_i \leq \frac{3i}{2} + \frac{1}{4}$  in all cases.  $\square$

*Proof of  $\text{RT}_i(3) \leq \frac{3i}{2} + \frac{1}{6}$  for every odd  $i \geq 3$ .*

Let us show that the image of every  $7/5^+$ -free word over  $\Sigma_4$  by the following 514-uniform morphism satisfies  $\text{pexp}_i \leq \frac{3i}{2} + \frac{1}{6}$  for every odd  $i \geq 3$ .

0  $\mapsto$  01020120210120102120210201210120102012021020121021201020121012  
 02102012102120210120102012102120102012021020121012010212021020  
 12102120102012021012010212021020121021202101201020121021201020  
 12101202102012102120210120102120210201210120102012021020121012  
 01021202102012102120102012101202102012102120102012021012010212  
 02102012101201020120210201210212021012010201210120210201210212  
 01020120210201210120102120210201210212010201210120210201210212  
 02101201021202102012101201020120210201210120102120210201210212  
 021012010201210212

1  $\mapsto$  01020120210120102120210201210120102012021020121021201020121012  
 02102012102120102012021020121012010212021020121021201020120210  
 12010212021020121021202101201020121021201020121012021020121021  
 20210120102120210201210120102012021020121021201020121012021020  
 12101201021202102012102120210120102012102120102012021012010212  
 02102012101201020120210201210120102120210201210212010201210120  
 21020121021202101201021202102012101201020120210201210212021012  
 01020121012021020121021201020120210201210120102120210201210212  
 021012010201210212

2  $\mapsto$  01020120210120102120210201210120102012021020121021201020121012  
 02102012101201021202102012102120102012021012010212021020121021  
 20210120102012102120102012101202102012102120210120102120210201  
 21012010201202102012101201021202102012102120102012101202102012  
 10212010201202101201021202102012102120210120102012102120102012  
 02102012101201021202102012102120102012021012010212021020121012  
 01020120210201210212021012010201210212010201210120210201210212  
 02101201021202102012101201020120210201210120102120210201210212  
 021012010201210212

3  $\mapsto$  01020120210120102120210201210120102012021020121021201020121012  
02102012101201021202102012102120102012021012010212021020121021  
20210120102012101202102012102120102012021020121012010212021020  
12102120102012101202102012102120210120102012102120102012021012  
01021202102012101201020120210201210212010201210120210201210212  
01020120210201210120102120210201210212021012010201210212010201  
20210120102120210201210212010201210120210201210212021012010212  
02102012101201020120210201210120102120210201210212021012010201  
210120210201210212

First, we check that such ternary images are  $\left(\frac{3^+}{2}, 45\right)$ -free using the method in [6]. By Lemma 2.1 in [6], it is sufficient to check this freeness property for the image of every  $7/5^+$ -free word over  $\Sigma_4$  of length smaller than  $\frac{2 \times \frac{3}{2}}{\frac{3}{2} - \frac{1}{5}} = 30$ . Thus, the period of every repetition formed from  $i$  pieces and with exponent strictly greater than  $\frac{3i}{2}$  must be at most 44. Using the same argument as in the previous proof, we have checked by computer that for every conjugacy class of words  $t$  of length at most 44, there exists a (small) odd  $j$  such that  $\text{pexp}_{j+2,t} = \text{pexp}_{j,t} + 3$ . Thus we have  $\text{pexp}_i \leq \frac{3i}{2} + \frac{1}{6}$  in all cases.  $\square$

### 3 Concluding remarks

The next step would be to consider the 4-letter alphabet. Obviously,  $\text{RT}_{i+1}(k) \geq \text{RT}_i(k) + 1$  for every  $i \geq 1$  and  $k \geq 2$ . Mousavi and Shallit [5] verified that  $\text{RT}_2(4) \geq \frac{5}{2}$ , so that  $\text{RT}_i(4) \geq i + \frac{1}{2}$  for every  $i \geq 2$ . We conjecture that this is best possible, i.e., that  $\text{RT}_i(4) = i + \frac{1}{2}$  for every  $i \geq 2$ . However, a proof of an upper bound of the form  $\text{RT}_i(4) \leq i + c$  cannot be similar to the proof of the upper bounds of Theorem 2. The multiplicative factor of  $i$ , which drops from  $\frac{3}{2}$  when  $k = 3$  to 1 when  $k = 4$ , forbids that the constructed word is the morphic image of any (unspecified) Dejean word over a given alphabet.

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