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# Communication Complexity of the Secret Key Agreement in Algorithmic Information Theory

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## Abstract

It is known that the mutual information, in the sense of Kolmogorov complexity, of any pair of strings  $x$  and  $y$  is equal to the length of the longest shared secret key that two parties can establish via a probabilistic protocol with interaction on a public channel, assuming that the parties hold as their inputs  $x$  and  $y$  respectively. We determine the worst-case communication complexity of this problem for the setting where the parties can use private sources of random bits.

We show that for some pairs  $x, y$  the communication complexity of the secret key agreement does not decrease even if the parties have to agree on a secret key whose size is much smaller than the mutual information between  $x$  and  $y$ . On the other hand, we discuss examples of  $x, y$  such that the communication complexity of the protocol declines gradually with the size of the resulting secret key.

The proof of the main result uses spectral properties of appropriate graphs and the expander mixing lemma, as well as information theoretic inequalities.

**Keywords:** Kolmogorov complexity, mutual information, communication complexity, expander mixing lemma, finite geometry

## 1 Introduction

In this paper we deal with *Kolmogorov complexity* and *mutual information*, which are the central notions of algorithmic information theory. Kolmogorov complexity  $C(x)$  of a string  $x$  is the length of the shortest program that prints  $x$ . Similarly, Kolmogorov complexity  $C(x|y)$  of a string  $x$  given  $y$  is the length of the shortest program that prints  $x$  when  $y$  is given as the input. Let us consider two strings  $x$  and  $y$ . The mutual information  $I(x : y)$  can be defined by a formula:  $I(x : y) = C(x) + C(y) - C(x, y)$ . Intuitively, this quantity is the information shared by  $x$  and  $y$ . In general, it cannot be “materialized”

as one object of complexity  $I(x : y)$  that can be easily extracted from both  $x$  and  $y$ . However, this quantity has a sort of *operational interpretation*. The mutual information of  $x$  and  $y$  is essentially equal to the size of a longest shared secret key that two parties, one having  $x$  and the other one having  $y$ , and both parties also possessing the complexity profile of the two strings can establish via a probabilistic protocol:

**Theorem 1** (sketchy version; see [22] for a more precise statement). *(a) There is a secret key agreement protocol that, for every  $n$ -bit strings  $x$  and  $y$ , allows Alice and Bob to compute with high probability a shared secret key  $z$  of length equal to the mutual information of  $x$  and  $y$  (up to an  $O(\log n)$  additive term).*

*(b) No protocol can produce a longer shared secret key (up to an  $O(\log n)$  additive term).*

In this paper we study the communication complexity of the protocols that appear in this theorem. Before we proceed with our results, we should clarify the statement of Theorem 1.

**Clarification 1: secrecy.** In this theorem we say that the obtained key  $z$  is “secret” in the sense that it looks random. Technically, it must be (almost) incompressible, even from the point of view of the eavesdropper who does not know the inputs  $x$  and  $y$  but intercepts the communication between Alice and Bob. More formally, if  $t$  denotes the transcript of the communication, we require that  $C(z|t) \geq |z| - O(1)$ . We will need to make this requirement even slightly stronger, see below.

**Clarification 2: randomized protocols.** In this communication model we assume that Alice and Bob may use additional randomness. Each of them can toss a fair coin and produce a sequence of random bits with a uniform distribution. The private random bits produced by Alice and Bob are accessible only to Alice and Bob respectively. (Of course, Alice and Bob can send the produced random bits to each other, but then this information becomes visible to the eavesdropper.)

In an alternative setting, Alice and Bob use a common *public* source of randomness (also accessible to the eavesdropper). The model with public randomness is easier to analyze, see [22], and in this paper we focus on the setting with private randomness.

**Clarification 3: minor auxiliary inputs.** We assume also that besides the main inputs  $x$  and  $y$  Alice and Bob both are given the *complexity profile* of the input, i.e., the values  $C(x)$ ,  $C(y)$ , and  $I(x : y)$ . Such a concession is unavoidable for the positive part of the theorem. Indeed, Kolmogorov complexity and mutual information are non-computable; so there is no computable protocol that finds a  $z$  of size  $I(x : y)$  unless the value of the mutual information is given to Alice and Bob as a promise. This supplementary information is rather small, it can be represented by only  $O(\log n)$  bits. The theorem remains valid if we assume that this auxiliary data is known to the eavesdropper. So, formally speaking, the protocol should find a key  $z$  such that

$$C(z|t, \text{complexity profile of } (x, y)) \geq |z| - O(1).$$

Now the statement of Theorem 1 is clarified, and we can formulate the main question studied in this paper:

**Central Question.** What is the optimal communication complexity of the communication problem from Theorem 1? That is, how many bits should Alice and Bob send to each other to agree on a common secret key?

A protocol proposed in [22] allows to compute for *all* pairs of inputs a shared secret key of length equal to the mutual information of  $x$  and  $y$  with communication complexity

$$\min\{C(x|y), C(y|x)\} + O(\log n). \quad (1)$$

Alice and Bob may need to send to each other different number of bits for different pairs of input (even with the same mutual information). It was proven in [22] that in the worst case (i.e., for *some* pairs of inputs  $(x, y)$ ) the communication complexity (1) is optimal for communication protocols using only *public randomness*. The natural question whether this bound remains optimal for protocols with *private* sources of random bits remained open (see *Open Question 1* in [22]). The main result of this paper is the positive answer to this question. More specifically, we provide explicit examples of pairs  $(x, y)$  such that

$$\begin{cases} I(x : y) &= 0.5n + O(\log n) \\ C(x|y) &= 0.5n + O(\log n) \\ C(y|x) &= 0.5n + O(\log n) \end{cases} \quad (2)$$

and in every communication protocol satisfying Theorem 1 (with private random bits) Alice and Bob must exchange approximately  $0.5n$  bits of information. Moreover, the same communication complexity is required even if Alice and Bob want to agree on a much smaller secret key of size, say,  $\omega(\log n)$ .

**Theorem 2.** *Let  $\pi$  be a communication protocol such that given inputs  $x$  and  $y$  satisfying (2) Alice and Bob use  $\text{poly}(n)$  private random bits and compute with probability  $> 1/2$  a shared secret key  $z$  of length  $\delta(n) = \omega(\log n)$ . Then for every  $n$  there exists a pair of  $n$ -bit strings  $(x, y)$  satisfying (2) such that following this communication protocol with inputs  $x$  and  $y$ , Alice and Bob send to each other messages with a total length of at least  $0.5n - O(\log n)$  bits. In other words, the worst-case communication complexity of the protocol is at least  $0.5n - O(\log n)$ .*

**Remark 1.** We assume that the computational protocol  $\pi$  used by Alice and Bob is computable, i.e., the parties send messages and compute the final result by following rules that can be computed given the length of the inputs. We may assume that the protocol is public (known to the eavesdropper). The constants hidden in the  $O(\cdot)$  notation may depend on the protocol, as well as on the choice of the optimal description method in the definition of Kolmogorov complexity.

An alternative approach might be as follows. We might assume that the protocol  $\pi$  is not uniformly computable (but its description is available to Alice, Bob, and to the eavesdropper). Then substantially the same result can be proven for Kolmogorov complexity relativized conditional on  $\pi$ . That is,

we should define Kolmogorov complexity and mutual information in terms of programs that can access  $\pi$  is an oracle, and the inputs  $x$  and  $y$  should satisfy a version of (2) with the relativized Kolmogorov complexity. Our main result can be proven for this setting (literally the same arguments applies). However, to simplify the notation, we focus on the setting with only computable communication protocols (whose size does not depend on  $n$ ).

Theorem 2 can be viewed as a special case of the general question of “extractability” of the mutual information studied in [12]. We prove this theorem for two specific examples of pairs  $(x, y)$ . In the first example  $x$  and  $y$  are a *line* and a *point* incident with each other in a discrete affine plane. In the second example  $x$  and  $y$  are points of the discrete plane with a fixed distance between them. The proof consists in a combination of a spectral and information-theoretical techniques. In fact, our argument applies to all pairs with similar spectral properties. Our main technical tools are the Expander Mixing Lemma (see Lemma 5) and the lemma on non-negativity of the triple mutual information (see Lemma 7). We also use Muchnik’s theorem on conditional descriptions with multiple conditions (see Proposition 1).

The communication protocol proposed in [22] and Theorem 2 imply together that we have the following phase transition phenomenon. When the inputs given to Alice and Bob are a line and a point (incident with each other in a discrete affine plane), then the parties can agree on a secret key of size  $I(x : y)$  with a communication complexity slightly *above*  $\min\{C(x|y), C(y|x)\}$ . But when a communication complexity is slightly *below* this threshold, the optimal size of the secret key sinks immediately to  $O(\log n)$ .

*Historical digression: classical information theory.* The problem of *secret key agreement* was initially proposed in the framework of classical information theory by Ahlswede and Csiszár, [2] and Maurer, [3]. In these original papers the problem was studied for the case when the input data is a pair of random variables  $(X, Y)$  obtained by  $n$  independent draws from a joint distribution (Alice can access  $X$  and Bob can access  $Y$ ). In this setting, the mutual information between  $X$  and  $Y$  and the secrecy of the key are measured in terms of Shannon entropy. Ahlswede and Csiszár in [2] and Maurer in [3] proved that the longest shared secret key that Alice and Bob can establish via a communication protocol is equal to Shannon’s mutual information between  $X$  and  $Y$ . This problem was extensively studied by many subsequent works in various restricted settings, see the survey [23]. The optimal communication complexity of this problem for the general setting remains unknown, though substantial progress has been made (see, e.g., [15, 17]).

There is a deep connection between the frameworks of classical information theory (based in Shannon entropy) and algorithmic information theory (based on Kolmogorov complexity). It can be shown that the statements of Theorem 1 and Theorem 2 imply similar statements in Shannon’s theory. We refer the reader to [22] for a more detailed discussion of parallels and differences between Shannon’s and Kolmogorov’s version of the problem of secret key agreement. Here we only mention two important distinctions between Shannon’s and Kol-

mogorov’s framework. The first one regards ergodicity of the input data. Most results on secret key agreement in Shannon’s framework are proven with the assumption that the input data are obtained from a sequence of independent identically distributed random variables (or at least enjoy some properties of ergodicity and stationarity). In the setting of Kolmogorov complexity we usually deal with inputs obtained in “one shot” without any assumption of ergodicity of the sources (see, in particular, Example 1 and Example 2 below). Another distinction regards the definition of correctness of the protocol. The usual paradigm in classical information theory is to require that the communication protocol works properly for *most* randomly chosen inputs. In our approach, we prove a stronger property: for *each* valid pair of input data, the protocol works properly with high probability (this approach is more typical for the theory of communication complexity).

The rest of the paper is organized as follows. In Preliminaries (Section 2) we sketch the basic definitions and notations for Kolmogorov complexity and communication complexity. In Section 3 we translate information theoretic properties of pairs  $(x, y)$  in the language of graph theory and present three explicit examples of pairs  $(x, y)$  satisfying (2):

- the first example involves finite geometry,  $x$  and  $y$  are incident points and lines on a finite plane;
- the second example uses a discrete version of the Euclidean distance,  $x$  and  $y$  are points on the discrete plane with a known quasi-Euclidean distance between them;
- the third example  $x$  and  $y$  are binary strings a fixed Hamming distance between them.

The pairs  $(x, y)$  from these examples have pretty much the same complexity profile, but the third example has significantly different spectral properties. In Section 4 we use a spectral technique to analyze combinatorial properties of graphs and prove our main result (Theorem 2) for the pairs  $(x, y)$  from Example 1 and Example 2 mentioned above. In Section 5 we show that Theorem 2 is not true for the pairs  $(x, y)$  from our Example 3: for those  $x$  and  $y$  there is no “phase transition” mentioned above, and the size of the optimal secret key decreases gradually with the communication complexity of the protocol, see Theorem 3 and Theorem 4.

## 2 Preliminaries

**Kolmogorov complexity.** Given a Turing machine  $M$  with two input tapes and one output tape, we say that  $p$  is a program that prints a string  $x$  conditional on  $y$  (a description of  $x$  conditional on  $y$ ) if  $M$  prints  $x$  on the pair of inputs  $p, y$ . Here  $M$  can be understood as an interpreter of some programming language that simulates a program  $p$  on a given input  $y$ . We denote the length of a binary

string  $p$  by  $|p|$ . The *algorithmic complexity* of  $x$  conditional on  $y$  relative to  $M$  is defined as

$$C_M(x|y) = \min\{|p| : M(p, y) = x\}.$$

It is known that there exists an *optimal* Turing machine  $U$  such that for every other Turing machine  $M$  there is a number  $c_M$  such that for all  $x$  and  $y$

$$C_U(x|y) \leq C_M(x|y) + c_M.$$

Thus, if we ignore the additive constant  $c_M$ , the algorithmic complexity of  $x$  relative to  $U$  is minimal. In the sequel we fix an optimal machine  $U$ , omit the subscript  $U$  and denote

$$C(x|y) := C_U(x|y).$$

This value is called *Kolmogorov complexity* of  $x$  conditional on  $y$ . Kolmogorov complexity of a string  $x$  is defined as the Kolmogorov complexity of  $x$  conditional on the empty string  $\Lambda$ ,

$$C(x) := C(x|\Lambda).$$

We fix an arbitrary computable bijection between binary strings and all finite tuple of binary strings (i.e., each tuple is encoded in a single string). Kolmogorov complexity of a tuple  $\langle x_1, \dots, x_k \rangle$  is defined as Kolmogorov complexity of the code of this tuple. For brevity we denote this complexity by  $C(x_1, \dots, x_k)$ .

We use the conventional notation

$$I(x : y) := C(x) + C(y) - C(x, y)$$

and

$$I(x : y|z) := C(x|z) + C(y|z) - C(x, y|z).$$

In this paper we use systematically the Kolmogorov–Levin theorem, [1], which claims that all  $x, y$ ,

$$|C(x|y) + C(y) - C(x, y)| = O(\log(|x| + |y|)).$$

It follows, in particular, that

$$I(x : y) = C(x) - C(x|y) + O(\log |x| + |y|) = C(y) - C(y|x) + O(\log(|x| + |y|)).$$

We also use the notation

$$I(x : y : z) := C(x) + C(y) + C(z) - C(x, y) - C(x, z) - C(y, z) + C(x, y, z).$$

Using the Kolmogorov–Levin theorem it is not hard to show that

$$I(x : y : z) = I(x : y) - I(x : y|z) = I(x : z) - I(x : z|y) = I(y : z) - I(y : z|x).$$

These relations can be observed on a Venn-like diagram, as shown in Fig. 1. In the usual jargon,  $x$  is said to be (almost) *incompressible* given  $y$  if

$$C(x|y) \geq |x| - O(\log |x| + |y|),$$

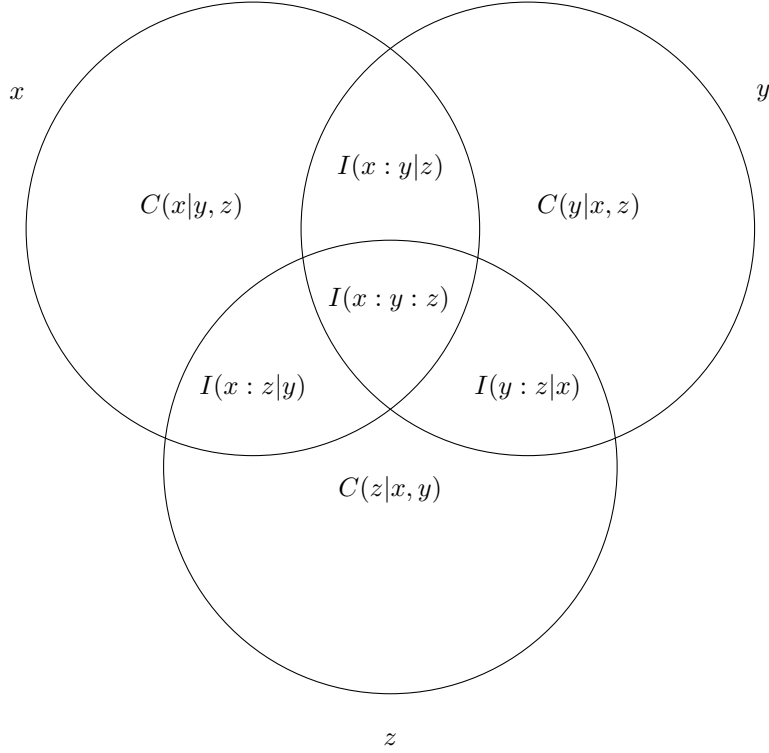


Figure 1: Complexity profile for a triple  $x, y, z$ . On this diagram it is easy to observe several standard equations:

- $C(x) = C(x|y, z) + I(x : y|z) + I(x : z|y) + I(x : y : z)$   
(the sum of all quantities inside the left circle representing  $x$ );
  - $I(x : y) = I(x : y|z) + I(x : y : z)$   
(the sum of the quantities in the intersection of the left and the right circles representing  $x$  and  $y$  respectively);
  - $C(x, y) = C(x|y, z) + C(y|x, z) + I(x : y|z) + I(x : z|y) + I(y : z|x) + I(x : y : z)$   
(the sum of all quantities inside the union of the left and the right circles);
  - $C(x|y) = C(x|y, z) + I(x : z|y)$   
(the sum of the quantities inside the left circle but outside the right one);
- and so on; all these equations are valid up to  $O(\log |x| + |y| + |z|)$ .



and  $x$  and  $y$  are said to be *independent*, if  $I(x : y) = O(\log(|x| + |y|))$ .

Since many natural equalities and inequalities for Kolmogorov complexity hold up to a logarithmic term, we abbreviate some formulas by using the notation  $A =^+ B$ ,  $A \leq^+ B$ , and  $A \geq^+ B$  for

$$|A - B| = O(\log n), \quad A \leq B + O(\log n), \quad \text{and} \quad B \leq A + O(\log n)$$

respectively, where  $n$  is clear from the context. Usually  $n$  is the sum of the lengths of all strings involved in the inequality. In particular, the Kolmogorov–Levin theorem can be rewritten as  $C(x, y) =^+ C(x|y) + C(y)$ .

For a survey of basic properties of Kolmogorov complexity we refer the reader to the introductory chapters of [24] and [18].

**Communication complexity.** In what follows we use the conventional notion of a communication protocol for two parties (traditionally called Alice and Bob), see for detailed definitions [4]. In a deterministic communication protocol the inputs of Alice and Bob are denoted  $x$  and  $y$  respectively. A deterministic communication protocol is said to be correct for inputs of length  $n$  if for all  $x, y \in \{0, 1\}^n$ , following this protocol Alice and Bob obtain a valid result  $z = z(x, y)$ . The sequence of messages sent by Alice and Bob to each other while following the protocol’s step is called a *transcript* of the communication.

In the setting of randomized protocols with *private* sources of randomness, Alice can access  $x$  and an additional string of bits  $r_A$ , and Bob can access  $y$  and an additional string of bits  $r_B$ . A randomized communication protocol is said to be correct for inputs of length  $n$  if for *all*  $x, y \in \{0, 1\}^n$  and for *most*  $r_A$  and  $r_B$ , following this protocol Alice and Bob obtain a valid result  $z = z(x, r_A, y, r_B)$ .

In a secret key agreement protocol, *correctness* of the result  $z$  means that (i)  $z$  is of the required size and (ii) it is almost incompressible even given the transcript of the communication. That is, if  $t = t(x, r_A, y, r_B)$  denotes the transcript of the communication, then  $C(z|t)$  must be close enough to  $|z|$ . Note that in this setting the transcript  $t$  and the final result  $z$  are not necessarily functions of the inputs  $x, y$ , they may depend also on the random bits used by Alice and Bob. For a more detailed discussion of this setting we refer the reader to [22].

Throughout this paper we assume that the number of private random bits used by Alice and Bob (to handle the inputs  $x, y$ ) is polynomial, i.e.,  $|r_A|$  and  $|r_B|$  are not greater than  $\text{poly}(n)$ , where  $n = |x| = |y|$ .

Communication complexity of a protocol is the maximal length of its transcript, i.e.,  $\max_{x, r_A, y, r_B} |t(x, r_A, y, r_B)|$ .

### 3 From information theoretic properties to combinatorics of graphs

To study information theoretic properties of a pairs  $(x, y)$  we will embed this pair of strings in a large set of pairs that are in some sense similar to each

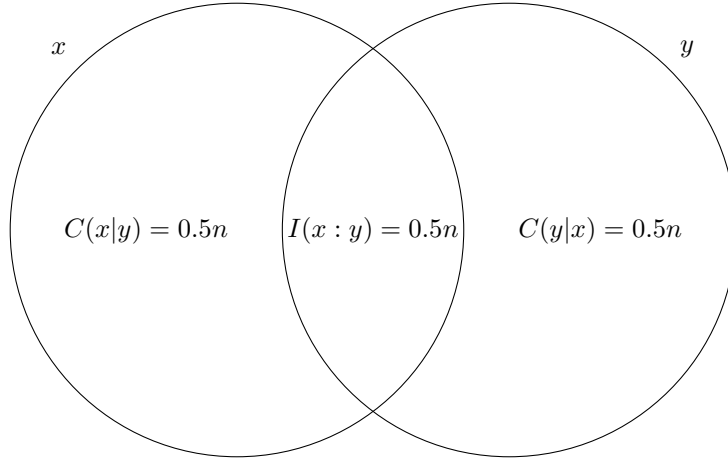


Figure 2: A diagram for the complexity profile of two strings  $x, y$ : from the Kolmogorov–Levin theorem we have  $C(x) = C(x|y) + I(x : y) = n$ ,  $C(y) = C(y|x) + I(x : y) = n$ , and  $C(x, y) = C(x|y) + C(y|x) + I(x : y) = 1.5n$ .

other. We will do it in the language of bipartite graphs. The information theoretic properties of the initial pair  $(x, y)$  will be determined by combinatorial properties of these graphs. In their turn, combinatorial properties of these graphs will be proven using the spectral technique. In this section we present three examples of  $(x, y)$  corresponding to three different constructions of graphs. In next sections we will study spectral and combinatorial properties of these graphs and, accordingly, information theoretic properties of these pairs  $(x, y)$ .

We start with a simple lemma that establishes a correspondence between information theoretic and combinatorial language for the properties of pairs  $(x, y)$ .

**Lemma 1.** *Let  $G = (L \cup R, E)$  be a bipartite graph such that  $|L| = |R| = 2^{n+O(1)}$  and the degree of each vertex is  $D = 2^{0.5n+O(\log n)}$ . We assume that this graph has an explicit construction in the sense that the complete description of this graph (its adjacency matrix) has Kolmogorov complexity  $O(\log n)$ . Then most  $(x, y) \in E$  (pairs of vertices connected by an edge) have the following complexity profile:*

$$\begin{cases} C(x) &= n + O(\log n), \\ C(y) &= n + O(\log n), \\ C(x, y) &= 1.5n + O(\log n), \end{cases} \quad (3)$$

(which is equivalent to the triple of equalities in (2)), see Fig. 2.

*Proof.* There are  $D \cdot |L| = D \cdot |R| = 2^{1.5n+O(\log n)}$  edges in the graph. Each of them can be specified by its index in the list of elements of  $E$ , and this index should consist of only  $\log |E|$  bits; the set  $E$  itself can be described by  $O(\log n)$

bits. Therefore,

$$C(x, y) \leq \log |E| + O(\log n) = 1.5n + O(\log n).$$

For most  $(x, y) \in E$  this bound is tight. Indeed, for every number  $c$  there are at most  $2^{1.5n-c}$  objects with complexity less than  $1.5n - c$ . Therefore, for all but  $|E|/2^{O(\log n)} = 2^{1.5n-O(\log n)}$  edges in  $E$  we have  $C(x, y) = 1.5n + O(\log n)$ .

Similarly, for each  $x \in L$  and for each  $y \in R$  we have

$$C(x) \leq \log |L| + O(\log n) = n + O(\log n), \quad C(y) \leq \log |R| + O(\log n) = n + O(\log n),$$

and for most  $x$  and  $y$  these bounds are tight.  $\square$

**Remark 2.** In a graph satisfying the conditions of Lemma 1 each vertex has  $D = 2^{0.5n}$  neighbors. Therefore, for *all*  $(x, y) \in E$  we have

$$C(x|y) \leq 0.5n + O(\log n), \quad C(y|x) \leq 0.5n + O(\log n),$$

(given  $x$ , we can specify  $y$  by its index in the list of all neighbors of  $x$  and vice-versa.) From Lemma 1 it follows that for *most*  $(x, y) \in E$  these inequalities are tight, i.e.,  $C(x|y) = 0.5n + O(\log n)$  and  $C(y|x) = 0.5n + O(\log n)$ .

**Remark 3.** The density of edges in the graph  $G = (L \cup R, E)$  (i.e., the ratio  $\frac{|E|}{|L| \cdot |R|}$ ) corresponds on the logarithmic scale to the mutual information between  $x$  and  $y$ . Indeed, the equations in (3) mean that for most  $(x, y) \in E$

$$\frac{|E|}{|L| \cdot |R|} = \frac{2^{C(x,y)+O(\log n)}}{2^{C(x)+O(\log n)} \cdot 2^{C(y)+O(\log n)}} = 2^{-I(x;y)+O(\log n)}.$$

In the sequel we pay attention to the density of edges in *subgraphs* of  $G$ . We will see that this ratio corresponds in some sense to the property of “extractability” of the mutual information. We will show that for some specific graphs  $G$  satisfying Lemma 1, in all large enough induced subgraphs, the density of edges is close to  $2^{-I(x;y)}$ .

**Example 1** (discrete plane). Let  $\mathbb{F}_q$  be a finite field of cardinality  $q = 2^n$ . Consider the set  $L$  of points on plane  $\mathbb{F}_q^2$  and the set  $R$  of non-vertical lines, which can be represented as affine functions  $y = ax - b$  for  $(a, b) \in \mathbb{F}_q^2$ . Let  $G = (L \cup R, E)$  be the bipartite graph where a point  $(x_0, y_0)$  is connected to a line  $y = ax - b$  if and only if it is on the line i.e.  $y_0 = ax_0 - b$ . Clearly  $|L| = |R| = 2^{2n}$ . The degree of each vertex is  $2^n$  since there are exactly  $q$  points on each line and there are exactly  $q$  lines on each point. In the sequel we denote this graph by  $G_n^{Pl}$ .

This graph (or its adjacency matrix) can be constructed effectively when the field  $\mathbb{F}_q$  is given. We assume a standard construction of  $\mathbb{F}_{2^n}$  to be fixed. Thus, we need only  $O(\log n)$  bits to describe the graph (as a finite object). Lemma 1 applies to this graph, so for most  $(x, y) \in E$  the equalities in (3) are satisfied.

**Example 2** (discrete Euclidean distance). Let  $\mathbb{F}_q$  be a finite field of order  $q$ , where  $q$  is an odd prime power. Let us define the distance function between two points in  $\mathbb{F}_q^2$  as

$$\text{dist}((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

For every  $r \in \mathbb{F}_q \setminus \{0\}$  we define the *finite Euclidean distance graph*  $G = (L \cup R, E)$  as follows:  $L = R = \mathbb{F}_q^2$ , and

$$E = \{((x_1, x_2), (y_1, y_2)) : \text{dist}((x_1, x_2), (y_1, y_2)) = r\}.$$

Obviously,  $|L| = |R| = q^2$ . It can be shown that the degree of this graph is  $O(q)$ , and  $|E| = O(q^3)$ , see [5].

For every integer  $n > 0$  we fix a prime number  $q_n$  such that  $\lceil 2 \log q_n \rceil = n$ . For the defined above graph  $G = (L \cup R, E)$  for this  $\mathbb{F}_{q_n}$  we have  $|L| = |R| = 2^{n+O(1)}$  and  $|E| = 2^{1.5n+O(1)}$ , and Lemma 1 applies to this graph. We should also fix the value of  $r$ . Any non-zero element of  $\mathbb{F}_{q_n}$  would serve the purpose, it only should be computable from  $n$ . For simplicity we may assume that  $r = 1$ . In the sequel we denote this graph by  $G_n^{Euc}$ .

In our next example we use the following standard lemma.

**Lemma 2.** Denote  $h(t) := -t \log t - (1-t) \log(1-t)$ . For any real number  $\gamma \in (0, 1)$  and every positive integer  $m$ ,  $\binom{m}{\gamma m} = 2^{h(\gamma)m + O(\log m)}$ .

**Example 3** (Hamming distance). We choose  $\theta \in (0, \frac{1}{2})$  such that  $h(\theta) = 0.5$ . Let  $L = R = \{0, 1\}^n$ . We define the bipartite graph  $G = (L \cup R, E)$  so that two strings (vertices) from  $L$  and  $R$  are connected if and only if the Hamming distance between them is  $\theta n$ . Clearly  $|L| = |R| = 2^n$ . By Lemma 2 the degree of each vertex is  $D = \binom{n}{\theta n} = 2^{0.5n + O(\log n)}$ . Lemma 1 applies to this graph. Therefore, for most  $(x, y) \in E$  we have (3). In the sequel we denote this graph by  $G_{\theta, n}^{Ham}$ .

We are interested in properties of  $(x, y)$  that are much subtler than those from Lemma 1. For example, we would like to know whether there exists a  $z$  materializing a part of the mutual information between  $x$  and  $y$  (i.e., such that  $C(z|x) \approx 0$ ,  $C(z|y) \approx 0$ , and  $C(z) \gg 0$ ). These subtler properties are not determined completely by the “complexity profile” of  $(x, y)$  shown in Fig. 2. In particular, we will see that some of these properties are different for pairs  $(x, y)$  from Example 1 and Example 2 on the one hand and from Example 3 on the other hand. In the next section we will show that some information theoretic properties of  $(x, y)$  are connected with the spectral properties of these graphs.

**Randomized communication protocols in the information theoretic framework.** In our main results we discuss communication protocols with two parties, Alice and Bob, who are given inputs  $x$  and  $y$ . We will assume that Alice and Bob are given the ends of some edge  $(x, y)$  from  $G_n^{Pl}$ , from  $G_n^{Euc}$ , or from  $G_{\theta, n}^{Ham}$ .

We admit randomized communication protocols with private sources of randomness. Technically this means that besides the inputs  $x$  and  $y$ , Alice and Bob are given strings of random bits,  $r_A$  and  $r_B$  respectively. We assume that both  $r_A$  and  $r_B$  are binary strings from  $\{0, 1\}^m$  for some  $m = \text{poly}(n)$ .

It is helpful to represent the entire inputs available to Alice and Bob as an edge in a graph. The data available to Alice are  $x' := \langle x, r_A \rangle$  and the data available to Bob are  $y' := \langle y, r_B \rangle$ . We can think of the pair  $(x', y')$  as an edge in the graph

$$\widehat{G}_n^{Pl} := G_n^{Pl} \otimes K_{M,M}$$

(if  $(x, y)$  is an edge in  $G_n^{Pl}$ ), or

$$\widehat{G}_n^{Euc} := G_n^{Euc} \otimes K_{M,M}$$

(if  $(x, y)$  is an edge in  $G_n^{Euc}$ ), or

$$\widehat{G}_{\theta,n}^{Ham} := G_{\theta,n}^{Ham} \otimes K_{M,M}$$

(if, respectively,  $(x, y)$  is an edge in  $G_{\theta,n}^{Ham}$ ). Here  $K_{M,M}$  is a complete bipartite graph with  $M = 2^m$  vertices in each part, and  $\otimes$  denotes the usual tensor product of bipartite graphs.

Keeping in mind Example 1, Example 2, and Example 3, we obtain that for *most* edges  $(x', y')$  in  $\widehat{G}_n^{Pl}$ , in  $\widehat{G}_n^{Euc}$ , and in  $\widehat{G}_{\theta,n}^{Ham}$  we have

$$\begin{aligned} C(x') &=^+ n + m, \\ C(y') &=^+ n + m, \\ C(x', y') &=^+ 1.5n + 2m. \end{aligned}$$

## 4 Bounds with the spectral method

### 4.1 Information inequalities from the graph spectrum

In this section we show that spectral properties of a graph can be used to prove information theoretic inequalities for pairs  $(x, y)$  corresponding to the edges in this graph. We start with a reminder of the standard considerations involving the spectral gap of a graph.

Let  $G = (L \cup R, E)$  be a regular bipartite graph of degree  $D$  on  $2N$  vertices ( $|L| = |R| = N$ , each edge  $e \in E$  connects one vertex from  $L$  with another one from  $R$ , and each vertex is incident to exactly  $D$  edges). The adjacency matrix of such a graph is a  $(2N) \times (2N)$  zero-one matrix  $H$  of the form

$$\begin{pmatrix} 0 & J \\ J^\top & 0 \end{pmatrix}$$

(the  $N \times N$  submatrix  $J$  is usually called *bi-adjacency* matrix of the graph;  $J_{ab} = 1$  if and only if there is an edge between the  $a$ -th vertex in  $L$  and the  $b$ -th vertex in  $R$ ). Let

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2N}.$$

be the eigenvalues of  $H$ . Since  $H$  is symmetric, all  $\lambda_i$  are real numbers. It is well known that for a bipartite graph the spectrum is symmetric, i.e.,  $\lambda_i = -\lambda_{2N+1-i}$  for each  $i$ . As the degree of each vertex in the graph is equal to  $D$ , we have  $\lambda_1 = -\lambda_{2N} = D$ . We focus on the second eigenvalue of the graph  $\lambda_2$ ; we are interested in graphs such that  $\lambda_2 \ll \lambda_1$  (that is, the *spectral gap* is large).

**Remark 4.** If the bi-adjacency matrix of the graph is symmetric, then the spectrum of the  $(2N) \times (2N)$  matrix  $H$  consists of the eigenvalues of the  $N \times N$  matrix  $J$  and their opposites. This observation makes the computation of the eigenvalues simpler.

It is immediately clear that the bi-adjacency matrices of the bipartite graphs from Example 2 and Example 3 are symmetric. For Example 1 this is also true, since a point with coordinates  $(x, y)$  and a line indexed  $(a, b)$  are incident if  $a \cdot x - y - b = 0$ .

In the sequel we will use the fact that for the graphs from Example 1 and Example 2 the value of  $\lambda_2$  is much less than  $\lambda_1 = D$ :

**Lemma 3** (see lemma 5.1 in [7]). *For the bipartite graph  $G_n^{Pl}$  from Example 1 (incident points and lines on plane  $\mathbb{F}_q^2$ ) the second eigenvalue is equal to  $\sqrt{q} = \sqrt{D}$ .*

**Remark 5.** We prove the main result of this paper (Theorem 2) for the construction of  $(x, y)$  from Example 1. The same result can be proven for a similar (and even somewhat more symmetric) construction: we can take lines and points in the *projective* plane over a finite field. The construction based on the projective plane has spectral properties similar to Lemma 3.

**Lemma 4** (see theorem 3 in [5]). *For the bipartite graph  $G_n^{Euc}$  from Example 2 (a discrete version of the Euclidean distance) the second eigenvalue is equal to  $O(\sqrt{q}) = O(\sqrt{D})$ .*

**Remark 6.** For the tensor product of two graphs  $G_1 \otimes G_2$ , the eigenvalues can be obtained as pairwise products of the eigenvalues of  $G_1$  and  $G_2$ . So, for the graph  $\widehat{G}_n^{Pl}$  (see p. 12) the eigenvalues are all pairwise products of the graph of incident lines and points  $G_n^{Pl}$  and the complete bipartite graph  $K_{M,M}$ . For  $G_n^{Pl}$  the first eigenvalue  $D$  and the second eigenvalue  $\sqrt{D}$ . The bi-adjacency matrix of  $K_{M,M}$  is the  $M \times M$  matrix with 1's in each cell. It is not hard to see that its maximal eigenvalue is  $M$  and all other eigenvalues are 0. Therefore, the first eigenvalue of  $\widehat{G}_n^{Pl}$  is equal to  $MD$  and the second one is equal to  $M\sqrt{D}$ . A similar observation is valid for  $G_n^{Euc}$ .

It is well known that the graphs with a large gap between the first and the second eigenvalues have nice combinatorial properties (vertex expansion, strong connectivity, mixing). One version of this property is expressed by the expander mixing lemma, which was observed by several researchers (see, e.g., [13, lemma 2.5] or [6, theorem 9.2.1]). We use a variant of the expander mixing lemma for bipartite graphs (see [16]):

**Lemma 5** (Expander Mixing Lemma for bipartite graphs). *Let  $G = (L \cup R, E)$  be a regular bipartite graph, where  $|L| = |R| = N$  and each vertex has degree  $D$ . Then for each  $A \subseteq L$  and  $B \subseteq R$  we have*

$$\left| E(A, B) - \frac{D \cdot |A| \cdot |B|}{N} \right| \leq \lambda_2 \sqrt{|A| \cdot |B|},$$

where  $\lambda_2$  is the second largest eigenvalue of the adjacency matrix of  $G$  and  $E(A, B)$  is the number of edges between  $A$  and  $B$ .

**Remark 7.** In what follows we apply Lemma 5 to the graphs with a large gap between  $D$  and  $\lambda_2$ . This technique is pretty common. See, e.g., [14, Theorem 3] where the Expander Mixing Lemma was applied to the graph from Example 1. Due to technical reasons, we will need to apply the Expander Mixing Lemma not only to the graph  $G_n^{Pl}$  from Example 1 and  $G_n^{Euc}$  from Example 2 but also to the tensor product of these graphs and a complete bipartite graph, see below.

In what follows we use a straightforward corollary of the expander mixing lemma:

**Corollary 1.** (a) Let  $G = (L \cup R, E)$  be a graph satisfying the same conditions as in Lemma 1 with  $\lambda_2 = O(\sqrt{D})$ . Then for  $A \subseteq L$  and  $B \subseteq R$  such that  $|A| \cdot |B| \geq N^2/D$  we have

$$|E(A, B)| = O\left(\frac{D \cdot |A| \cdot |B|}{N}\right). \quad (4)$$

(b) Let  $G = (L \cup R, E)$  be the same graph as in (a), and let  $K_{M,M}$  be a complete bipartite graph for some integer  $M$ . Define the tensor product of these graphs  $\hat{G} := G \otimes K_{M,M}$  (this is a bipartite graph  $(\hat{L} \cup \hat{R}, \hat{E})$  with  $|\hat{L}| = |\hat{R}| = N \cdot M$ , with degree  $D \cdot M$ ).

Then for all subsets  $A \subset \hat{L}$  and  $B \subset \hat{R}$  such that  $|A| \cdot |B| \geq (MN)^2/D$  inequality (4) holds true.

*Proof.* (a) From Lemma 5 it follows that

$$|E(A, B)| \leq \frac{D \cdot |A| \cdot |B|}{N} + \lambda_2 \sqrt{|A| \cdot |B|} \quad (5)$$

Assuming that  $\lambda_2 = O(\sqrt{D})$  and  $|A| \cdot |B| \geq N^2/D$  we conclude that the first term on the right-hand side of (5) is dominating:

$$\lambda_2 \sqrt{|A| \cdot |B|} = O\left(\frac{D \cdot |A| \cdot |B|}{N}\right)$$

Given this and Lemma 5 we obtain (4).

(b) Let us recall that the eigenvalues of  $G \otimes K_{M,M}$  are pairwise products of the eigenvalues of  $G$  and  $K_{M,M}$ . Therefore, the maximal eigenvalue of  $G \otimes K_{M,M}$  is  $MD$  and the second one is  $O(M\sqrt{D})$ , see Remark 6. The rest of the proof is similar to the case (a).  $\square$

Now we translate the combinatorial property of *mixing* in the information-theoretic language. We show that a large spectral gap in a graph implies some inequality for Kolmogorov complexity that is valid for each pair of adjacent vertex in this graph. We do it in the next lemma, which is the main technical ingredient of the proof of our main result.

**Lemma 6.** *Let  $G = (L \cup R, E)$  be a bipartite graph satisfying the same conditions as in Lemma 1, with  $|L| = |R| = N$  and degree  $D = O(\sqrt{N})$ . Assume also that the second largest eigenvalue of this graph is  $\lambda_2 = O(\sqrt{D})$ . Let  $K_{M,M}$  be a complete bipartite graph for some  $M = 2^m$ . Define the tensor product of these graphs  $\hat{G} := G \otimes K_{M,M}$ .*

*For each edge  $(x, y)$  in  $\hat{G}$  and for all  $w$ , if*

$$C(x|w) + C(y|w) > 1.5n + 2m$$

*then*

$$I(x : y|w) \geq 0.5n + O(\log k),$$

*where  $k = n + m$ .*

**Remark 8.** Note that Lemma 6 applies to the graphs from Example 1 and Example 2 due to Lemma 3 and Lemma 4 respectively.

*Proof.* Denote  $a = C(x|w)$  and  $b = C(y|w)$ . By the assumption of the lemma we have  $a + b > 1.5n + 2m$ . Let  $A$  be the set of all  $x' \in L$  such that  $C(x'|w) \leq a$  and  $B$  be the set of all  $y' \in R$  such that  $C(y'|w) \leq b$ . Note that by definition  $A$  contains  $x$  and  $B$  contains  $y$ . In what follows we show that for all pairs  $(x', y') \in (A \times B) \cap E$  we have  $C(x, y) \leq a + b - 0.5n$ .

*Claim 1.* We have  $|A| = 2^{a+O(\log k)}$  and  $|B| = 2^{b+O(\log k)}$ .

*Proof of the claim 1:* We start with a proof of the upper bounds. Each element of  $A$  can be obtained from  $w$  with a programs (description) of length at most  $a$ . Therefore, the number of elements in  $A$  is not greater than the number of such descriptions, which is at most  $1 + 2 + \dots + 2^a < 2^{a+1}$ . Similarly, the number of elements in  $B$  is not greater than  $2^{b+1}$ .

Now we proceed with the lower bounds. Given  $w$  and an integer number  $a$  we can take all programs of size at most  $a$ , apply them to  $w$  and run in parallel. As some programs converge, we will discover one by one all elements in  $A$  (though we do not know when the last stopping program terminates, and when the last element of  $A$  is revealed). The element  $x$  must appear in this enumeration. Therefore, we can identify it given its position in this list, which requires only  $\log |A|$  bits. Thus, we have

$$C(x|w) \leq \log |A| + O(\log k)$$

(the logarithmic additive term is needed to specify the binary expansion of  $a$ ). On the other hand, we know that  $C(x|w) = a$ . It follows that  $|A| \geq 2^{a-O(\log k)}$ , and we are done. The lower bound  $|B| \geq 2^{b-O(\log k)}$  can be proven in a similar way.



*Claim 2.* The number of edges between  $A$  and  $B$  is rather small:

$$|(A \times B) \cap E| \leq O\left(\frac{D \cdot |A| \cdot |B|}{N}\right).$$

*Proof of the claim 2:* By Claim 1 we have  $|A| = 2^{a+O(\log k)}$  and  $|B| = 2^{b+O(\log k)}$ . Since  $a + b > 1.5n$  we obtain

$$|A| \cdot |B| = 2^{a+b+O(\log k)} > 2^{1.5n+2m} = (NM)^2/D.$$

Hence, we can apply Corollary 1 (b) and obtain the claim.

*Claim 3.* For all pairs  $(x', y') \in (A \times B) \cap E$  we have

$$C(x', y'|w) \leq \log |E(A, B)| + O(\log k).$$

*Proof of the claim 3:* Given a string  $w$  and the integer numbers  $a, b$ , we can run in parallel all programs of length at most  $a$  and  $b$  (applied to  $w$ ) and reveal one by one all elements of  $A$  and  $B$ . If we have in addition the integer number  $n$ , then we can construct the graph  $G$  and enumerate all edges between  $A$  and  $B$  in the graph  $G$ . The pair  $(a', b')$  must appear in this enumeration. Therefore, we can identify it by its ordinal number in this enumeration. Thus,

$$C(x', y'|w) \leq \log |E(A, B)| + O(\log k),$$

where the logarithmic term involves the binary expansions of  $n, a$ , and  $b$ .

Now we can finish the proof of the lemma. By claim 3, we have

$$C(x', y'|w) \leq \log |E(A, B)| + O(\log k)$$

for all pairs  $x', y' \in (A \times B) \cap E$ . By using claim 2, we obtain

$$C(x', y'|w) \leq \log D + \log |A| + \log |B| - \log N + O(1).$$

With claim 1 this rewrites to

$$C(x', y'|w) \leq a + b - 0.5n + O(1).$$

Now we apply this inequality to the initial  $x$  and  $y$ :

$$\begin{aligned} I(x : y|w) &=^+ C(x'|w) + C(y'|w) - C(x', y'|w) \\ &\geq^+ a + b - (a + b - 0.5n) + O(\log k) =^+ 0.5n. \end{aligned}$$

□

## 4.2 Information inequalities for a secret key agreement

In this section we prove some information theoretic inequalities that hold true for the objects involved in a communication protocol: the inputs given to Alice and Bob, the transcript of the communication, and the final result computed by Alice and Bob.

In the sequel we use the following lemma from [22] (see also a similar result proven for Shannon entropy in [21]):

**Lemma 7** ([22]). *Let us consider a communication protocol with two parties. Denote by  $x$  and  $y$  the inputs of the parties, and denote by  $t = t(x, y)$  the transcript of the communication between the parties. Then  $I(x : y : t) \geq^+ 0$ .*

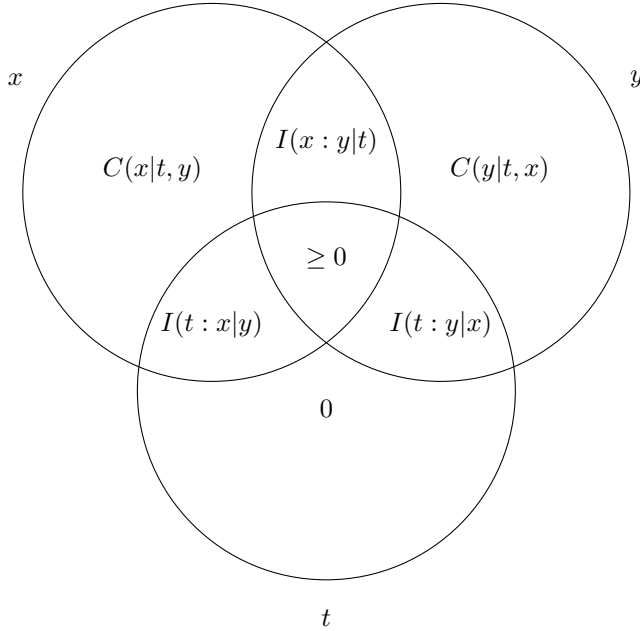


Figure 3: Complexity profile for inputs  $x, y$ , and the transcript  $t$  of a communication protocol with given inputs. Note that  $C(t|x, y)$  is negligibly small (we can compute  $t$  by simulating the communication protocol) and  $I(x : y|t) \leq^+ I(x : y)$  due to Lemma 7.

**Proposition 1** (Muchnik’s theorem on conditional descriptions, [11]). *(a) Let  $a$  and  $b$  be arbitrary strings of length at most  $n$ . Then there exists a string  $p$  of length  $C(a|b)$  such that*

- $C(p|a) = O(\log n)$  and
- $C(a|p, b) = O(\log n)$ .

*(b) Let  $a, b_1, b_2$  be arbitrary strings of length at most  $n$ . Then there exists a string  $q$  such that*

- $C(q|a) = O(\log n)$  and
- $C(a|b_j, q_j) = O(\log n)$  for  $j = 1, 2$ , where  $q_j$  is the prefix of  $q$  having length  $C(a|b_j)$ .

*As usual, the constants in  $O(\log n)$ -notation do not depend on  $n$ , see Fig. 3.*

**Remark 9.** In Proposition 1(a) the string  $q$  can be interpreted as an (almost) shortest description of  $a$  conditional on  $b$  that satisfies some additional nice properties. As an (almost) shortest description, it must be (almost) incompressible given  $b$ . Similarly, in Proposition 1(b) the strings  $q_1$  and  $q_2$  can be interpreted as almost shortest description of  $a$  given  $b_1$  and  $b_2$  respectively. In particular, they are almost incompressible given  $b_1$  and  $b_2$  respectively.

We use Lemma 7 and Proposition 1 to prove the next lemma.

**Lemma 8.** *Assume a deterministic communication protocol for two parties on inputs  $x$  and  $y$  gives transcript  $t$  and denote  $n = C(x, y, t)$ .*

- (a)  $C(t|a, b) = O(\log n)$ .
- (b)  $C(t|x) =^+ I(t : y|x)$ .
- (c)  $C(t|y) =^+ I(t : x|y)$ .
- (d)  $C(t|x) + C(t|y) =^+ I(t : x|y) + I(t : y|x) + O(\log n) \leq^+ C(t)$ .
- (e) *There exist  $t_x$  and  $t_y$  such that*

- $C(t_x) = C(t|x)$  and  $C(t_y) = C(t|y)$ ,
- $C(t_x|t) = O(\log n)$  and  $C(t_y|t) = O(\log n)$ ,
- $C(t|t_x, x) = O(\log n)$  and  $C(t|t_y, y) = O(\log n)$ ,
- $C(t_x, t_y) =^+ C(t_x) + C(t_y)$ .

*Speaking informally,  $t_x$  and  $t_y$  are “fingerprints” of  $t$  that can play the roles of (almost) shortest descriptions of  $t$  conditional on  $x$  and  $y$  respectively. The last condition means that the mutual information between  $t_x$  and  $t_y$  is negligibly small.*

*The complexity profile for  $x$ ,  $y$ , and  $\langle t_x, t_y \rangle$  is shown in Fig. 4.*

*Proof.* (a) follows trivially from the fact that  $t$  can be computed given  $(x, y)$  (we may simulate the communication protocol on the given inputs). Note that the constant in the term  $O(\cdot)$  includes implicitly a description of the communication protocol.

(b) For all  $x, y, t$  we have

$$C(t|x) =^+ C(t|x, y) + I(t : y|x).$$

The term  $C(t|x, y)$  vanishes due to (a), and we are done.

(c) Is similar to (b).

(d) A routine check shows that for all  $x, y, t$  we have

$$C(t) =^+ I(t : x|y) + I(t : y|x) + (I(x : y) - I(x : y|t)) + C(t|x, y).$$

Due to Lemma 7 we have  $I(x : y) - I(x : y|t) =^+ I(x : y : t) \geq^+ 0$ , so (d) follows.

(e) First, we apply Proposition 1(a) with  $a = t$  and  $b = x$ ; we obtain a string  $p$  of length  $C(t|x)$  such that

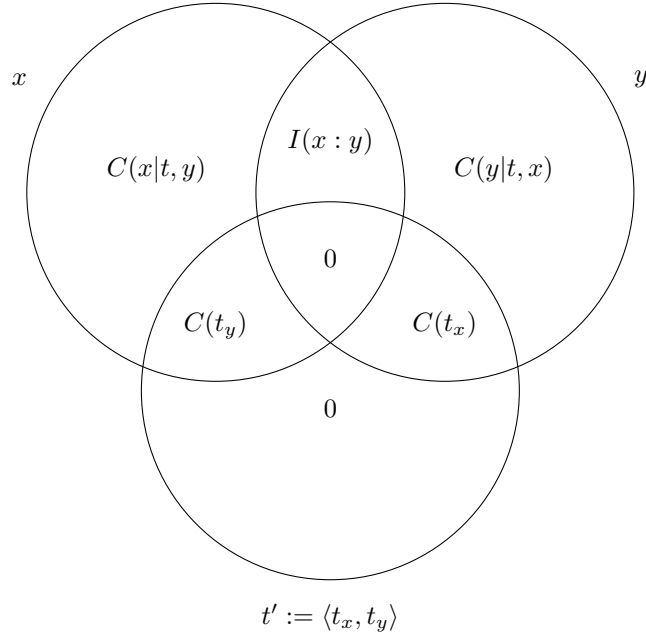


Figure 4: Complexity profile for  $x, y$ , and  $t' := \langle t_x, t_y \rangle$  from Lemma 8. Note that  $C(t_x) = I(x : t|y)$ ,  $C(t_y) = I(y : t|x)$ , and  $I(x : y|t') = I(x : y)$ .

- $C(p|t) = O(\log n)$  and
- $C(t|p, x) = O(\log n)$ .

From (b) we have  $C(p) =^+ I(t : y|x)$ . So we can let  $t_x := p$ . Observe that

$$C(t|t_x) =^+ C(t) - C(t_x) =^+ C(t) - I(t : y|x) \geq C(t|y)$$

(this inequality follow from (d)). Now we apply Proposition 1(b) with  $a = t$ ,  $b_1 = y$ , and  $b_2 = t_x$ . We obtain a string  $q$  such that

- $C(q|t) = O(\log n)$ ,
- $C(t|y, q_1) = O(\log n)$ , where  $q_1$  is the prefix of  $q$  having length  $C(t|y)$ , and
- $C(t|t_x, q_2) = O(\log n)$ , where  $q_2$  is the prefix of  $q$  having length  $C(t|t_x)$ .

Note that the length of  $q_2$  is not less (up to  $O(\log n)$ ) than the length of  $q_1$ . Since  $q_2$  is incompressible conditional on  $t_x$ , the shorter prefix  $q_1$  must be also incompressible conditional on  $t_x$ . Thus,  $t_x$  and  $q_1$  are independent. We let  $t_y := q_1$ , and (e) is proven.  $\square$

### 4.3 Proof of Theorem 2

Now we are ready to combine the spectral technique from Section 4.1 and the information theoretic technique from Section 4.2 and prove our main result.

*Proof of Theorem 2.* Let us take a pair of  $(x, y)$  from Example 1 or Example 2. We know that it satisfies (3) and, therefore, (2). Assume that in a communication protocol  $\pi$  Alice and Bob (given as inputs  $x$  and  $y$  respectively) agree on a secret key  $z$  of size  $\delta(n)$ . We will prove a lower bound on the communication in this protocol. To simplify the notation, in what follows we ignore the description of  $\pi$  in all complexity terms (assuming that it is a constant, which is negligible compared with  $n$ ).

In this proof we will deal with four objects: the inputs  $x' = \langle x, r_A \rangle$  and  $y' = \langle y, r_B \rangle$ , the transcript  $t$ , and the output of the protocol (secret key)  $z$ . Our aim is to prove that  $C(t)$  cannot be much less than  $0.5n$ . This is enough to conclude that the length of the transcript measured in bits (which is exactly the communication complexity of the protocol) also cannot be much less than  $0.5n$ . Due to some technical reasons that will be clarified below we need to reduce in some sense the sizes of  $t$  and  $z$ .

*Reduction of the key.* First of all, we reduce the size of  $z$ . This step might seem counterintuitive: we make the assumption of the theorem *weaker*, suggestion that Alice and Bob agree on a rather small secret key. We know from [22] that  $C(z)$  can be pretty large (more specifically, it can be of complexity  $0.5n + O(\log n)$ ). However, we prefer to deal with protocols where Alice and Bob agree on a moderately small (but still not *negligibly* small) key. To this end we may need to degrade the given communication protocol and reduce the size of the secret key to the value  $\mu \log n$  (the constant  $\mu$  to be chosen later). It is simple to make the protocol weaker: if the original protocol provides a common secret key  $z$  of bigger size, then in the degraded protocol Alice and Bob can take only the  $\delta(n)$  first bits of this key. Thus, without loss of generality, we may assume that the protocol gives a secret key  $z$  with complexity  $\delta(n) = \mu \log n$ .

*Reduction of the transcript.* Now we perform a reduction of  $t$ . We know from Lemma 7 that  $I(x' : y' : t)$  is non-negative. We want to reduce  $t$  to a  $t'$  such that  $I(x' : y' : t')$  is exactly 0 (here *exactly* means, as usual, an equality that holds up to  $O(\log n)$ ). To this end, we apply Lemma 8 to the triple  $(x', y', t)$  and obtain  $t_x$  and  $t_y$ , which play the roles of optimal descriptions of  $t$  given the conditions  $x'$  and  $y'$  respectively. We let  $t' := \langle t_x, t_y \rangle$ . Though technically this  $t'$  is not a transcript of any communication protocol, we will see that in some sense it behaves similar to the initial transcript.

We know from Lemma 8(d,e) that  $C(t') \leq^+ C(t)$ . Thus, to prove the theorem, it is enough to show that  $C(t') \geq^+ 0.5n - 2\delta(n)$ .

**Lemma 9.** *For  $t' = \langle t_x, t_y \rangle$  we have the following equalities:*

- (a)  $C(x'|t', z) =^+ n + m - C(t_y) - \delta(n)$ ,
- (b)  $C(y'|t', z) =^+ n + m - C(t_x) - \delta(n)$ ,

and

$$(c) I(x' : y'|t', z) =^+ I(x' : y') - C(z) + O(\log n) = 0.5n - \delta(n), \quad (6)$$

see Fig. 5.

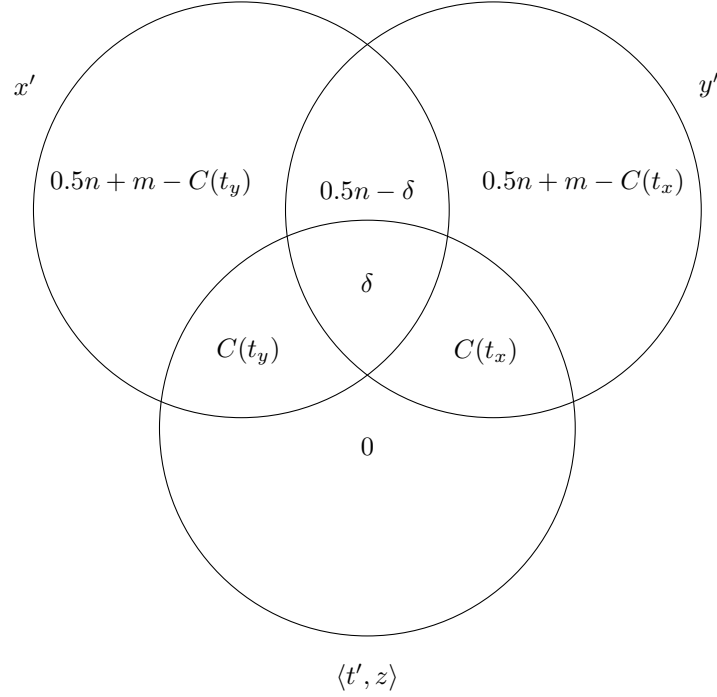


Figure 5: Complexity profile for the triple that consists of  $x'$ ,  $y'$ , and  $\langle t', z \rangle$  from Lemma 9.

*Proof of the lemma.* The proof is a routine check where we use repeatedly the Kolmogorov–Levin theorem. For (a) we have

$$\begin{aligned}
C(x'|t', z) &=^+ C(x', t', z) - C(t', z) \\
&\quad / \text{ from the Kolmogorov–Levin theorem } / \\
&=^+ C(x') + C(t'|x) + C(z|t', x') - (C(t') + C(z)) \\
&\quad / \text{ since } z \text{ is independent of } t' / \\
&=^+ C(x') + C(t'|x) - C(t') - C(z) \\
&\quad / z \text{ is computable given } t \text{ and } x', \text{ so } C(z|x', t') = O(\log n) / \\
&=^+ C(x') + I(y' : t|x') - I(x' : t|y') - I(y' : t|x') - C(z) \\
&\quad / \text{ from Lemma 8 } / \\
&=^+ C(x') - I(x' : t|y') - \delta(n) \\
&=^+ n + m - C(t_y) - \delta(n)
\end{aligned}$$

The proof of (b) is similar.

Since  $t$  and  $z$  can be computed from  $(x', y')$  by a simulation of the protocol and  $t'$  has negligibly small complexity conditional on  $t$ , we obtain from the Kolmogorov–Levin theorem

$$\begin{aligned}
C(x', y' | t', z) &=^+ C(x', y', t', z) - C(t', z) \\
&\quad / \text{ from the Kolmogorov–Levin theorem } / \\
&=^+ C(x', y') - C(t', z) \\
&\quad / \text{ since } t' \text{ and } z \text{ have logarithmic complexity conditional on } (x', y') / \\
&=^+ C(x', y') - C(t') - C(z) \\
&\quad / \text{ since } z \text{ is incompressible given } t' / \\
&=^+ 1.5n + 2m - I(x' : t | y') - I(y' : t | x') - \delta(n).
\end{aligned}$$

Combining this with (a) and (b) we obtain (c). □

Now we are ready to prove the theorem. Assume that

$$C(t_x) + C(t_y) < 0.5n - 2\delta(n) - \lambda \log n. \quad (7)$$

If the constant  $\lambda$  is large enough, then we obtain from Lemma 9(a,b)

$$C(x' | t', z) + C(y' | t', z) > 1.5n + 2m.$$

Now we can apply Lemma 6 (the spectral bound applies to Example 1 and Example 2, see Remark 8), which gives

$$I(x' : y' | t', z) \geq 0.5n + O(\log n). \quad (8)$$

Comparing (6) and (8) we conclude that  $\delta(n) = O(\log n)$  (the constant hidden in  $O(\cdot)$  depends only on the choice of optimal description method in the definition of Kolmogorov complexity). This contradicts the assumption  $\delta(n) = \mu \log n$ , if  $\mu$  is chosen large enough. Therefore, the assumption in (7) was false (without this assumption we cannot apply Lemma 6 and conclude with (8)).

The negation of (7) gives

$$C(t) \geq C(t_x) + C(t_y) - O(\log n) \geq 0.5n - 2\delta(n) - O(\log n),$$

and we are done. □

## 5 Paris with a fixed Hamming distance

Theorem 2 estimates communication complexity of the protocol in the worst case. For some classes of inputs  $(x, y)$  there might exist more efficient communication protocol. In this section we study one such special class — the pairs  $(x, y)$  from Example 3. The spectral argument from the previous section does not apply to this example. The spectral gap for the graph from Example 3 is too small: for this graph we have  $\lambda_2 = \Theta(\lambda_1)$ , while in Example 1 and Example 2

we had  $\lambda_2 = O(\sqrt{\lambda_1})$ . In fact, the spectrum of the graph from Example 3 can be computed explicitly: the eigenvalues of this graph are the numbers

$$K_{\theta n}(i) = \sum_{h=0}^{\theta n} (-1)^h \binom{i}{h} \binom{n-i}{\theta n-h} \text{ for } i \in \{0, 1, \dots, n\}$$

with different multiplicities, see [19] and the survey [20]. In particular, the maximal eigenvalue of this graph is  $K_{\theta n}(0) = \binom{n}{\theta n}$  and its second eigenvalue is  $K_{\theta n}(1) = \binom{n-1}{\theta n} - \binom{n-1}{\theta n-1}$ . It is not difficult to verify that  $K_{\theta n}(1) = \Omega\left(\binom{n}{\theta n}\right)$  (for a fixed  $\theta$  and  $n$  going to infinity), so the difference between the first and the second eigenvalues is only a constant factor. Thus, we cannot apply Lemma 6 to this graph.

It is no accident that our *proof* of Theorem 2 fails on Example 3. Actually, the *statement* of the theorem is not true for  $(x, y)$  from this example. In what follows we show that given these  $x$  and  $y$  Alice and Bob can agree on a secret key of any size  $m$  (intermediate between  $\log n$  and  $n/2$ ) with communication complexity  $\Theta(m)$ . The positive part of this statement (the existence of a communication protocol with communication complexity  $O(m)$ ) is proven in Theorem 3. The negative part of the statement (the lower bound  $\Omega(m)$  for all communication protocols) is proven in Theorem 4.

**Theorem 3.** *For every  $\delta \in (0, 1)$  there exists a two-parties randomized communication protocol  $\pi$  such that given inputs  $x$  and  $y$  from Example 3 (a pair of  $n$ -bit strings with the Hamming distance  $\theta n$  and complexity profile (2)) Alice and Bob with probability  $> 0.99$  agree on a secret key  $z$  of size  $\delta n/2 - o(n)$  with communication complexity  $O(\delta n)$ . (The constant hidden in the  $O(\cdot)$  does not depend on  $n$  or  $\delta$ .)*

**Theorem 4.** *For every  $\delta \in (0, 1)$  for every randomized communication protocol  $\pi'$  such that for inputs  $x$  and  $y$  from Example 3 Alice and Bob with probability  $> 0.99$  agree on a secret key  $z$  of size  $\geq \delta n$ , the communication complexity is at least  $\Omega(\delta n)$ . (The constant hidden in the  $\Omega(\cdot)$  does not depend on  $n$  or  $\delta$ .)*

*Proof of Theorem 3.* We start the proof with a lemma.

**Lemma 10.** *Let  $(x, y)$  be a pair from from Example 3 (two  $n$ -bit strings with the Hamming distance  $\theta n$  and complexity profile (2)). Let  $m = \delta n$  for some  $\delta \in (0, 1)$ . Denote by  $\hat{x}$  and  $\hat{y}$  the  $m$ -bit prefixes of  $x$  and  $y$  respectively. Then*

- $C(\hat{x}) = m + o(n)$ ,
- $C(\hat{y}) = m + o(n)$ ,
- $I(\hat{x} : \hat{y}) = 0.5m + o(n)$ ,
- the Hamming distance between  $\hat{x}$  and  $\hat{y}$  is  $\theta m + o(n)$ .

*Proof of lemma.* Denote by  $\hat{x}'$  and  $\hat{y}'$  the suffixes of length  $n - m$  of  $x$  and  $y$  respectively (so  $x$  is a concatenation of  $\hat{x}$  and  $\hat{x}'$ , and  $y$  is a concatenation of  $\hat{y}$



and  $\hat{y}'$ ). The idea of the proof is simple: the law of large number guarantees that for the vast majority of pairs  $(x, y)$  such that  $\text{HammingDist}(x, y) = \theta n$ , the fraction of positions where  $\hat{x}$  differs from  $\hat{y}$  and the fraction of positions where  $\hat{x}'$  differs from  $\hat{y}'$  are both close to  $\theta$ ; the pairs violating this rule are uncommon; therefore, Kolmogorov complexity of these “exceptional” pairs is small, and they cannot satisfy (2). To convert this idea into a formal proof, we need the following technical claim:

*Claim.* If the pair  $(x, y)$  satisfies (3) and the Hamming distance between  $x$  and  $y$  is  $\theta$ , then

$$\begin{aligned}\text{HammingDist}(\hat{x}, \hat{y}) &= \theta m + o(n), \\ \text{HammingDist}(\hat{x}', \hat{y}') &= \theta(n - m) + o(n).\end{aligned}$$

*Proof of the claim.* Denote

$$\begin{aligned}\theta_1 &:= \frac{1}{m} \cdot \text{HammingDist}(\hat{x}, \hat{y}), \\ \theta_2 &:= \frac{1}{n-m} \cdot \text{HammingDist}(\hat{x}', \hat{y}')\end{aligned}$$

(note that  $\theta_1 m + \theta_2(n - m) = \theta n$ ; as  $\theta$  is fixed, there is a linear correspondence between  $\theta_1$  and  $\theta_2$ ). For a fixed  $x$  of length  $n$ , the number strings  $y$  of the same length that matches the parameters  $m, \theta_1, \theta_2$  (i.e., that differ from  $x$  in exactly  $\theta_1 m$  bits in the first  $m$  positions and in  $\theta_2(n - m)$  bits in the last  $n - m$  positions) is

$$\begin{aligned}\binom{m}{\theta_1 m} \cdot \binom{n-m}{\theta_2(n-m)} &= 2^{h(\theta_1)m + O(\log n)} \cdot 2^{h(\theta_2)(n-m) + O(\log n)} \\ &= 2^{(\frac{m}{n}h(\theta_1) + \frac{n-m}{n}h(\theta_2))n + O(\log n)} \\ &= 2^{(\delta h(\theta_1) + (1-\delta)h(\theta_2))n + O(\log n)} \\ &\leq 2^{h(\theta)n + O(\log n)} = 2^{0.5n + O(\log n)},\end{aligned}$$

where  $h(\tau) = -\tau \log \tau - (1-\tau) \log(1-\tau)$  is the binary entropy function. The last inequality follows from the fact that the function  $h(\tau)$  is concave, and therefore

$$\delta h(\theta_1) + (1 - \delta)h(\theta_2) \leq h(\delta\theta_1 + (1 - \delta)\theta_2) = h(\theta). \quad (9)$$

If  $\theta_1$  and  $\theta_2$  are not close enough to the average value  $\theta$ , then the gap between the left-hand side and the right-hand side in (9) is getting large. More specifically, it is not hard to verify that the difference

$$h(\theta) - \delta h(\theta_1) - (1 - \delta)h(\theta_2) \quad (10)$$

grows essentially proportionally to the square of  $|\theta_1 - \theta|$  (as the second term of the Taylor series of the function around the extremum point). So, if  $|\theta_1 - \theta|$  and  $|\theta_2 - \theta|$  are getting much bigger than  $\sqrt{(\log n)/n}$ , then the gap (10) becomes much bigger than  $(\log n)/n$ , and then we obtain

$$\binom{m}{\theta_1 m} \cdot \binom{n-m}{\theta_2(n-m)} < 2^{0.5n - \omega(\log n)}.$$

On the other hand,

$$C(x, y) =^+ C(x) + C(y|x) \leq^+ n + \log \left( \binom{m}{\theta_1 m} \cdot \binom{n-m}{\theta_2(n-m)} \right).$$

Thus, the assumption  $C(x, y) =^+ 1.5n$  can be true only if  $\theta_1 m = \theta m + o(n)$  and  $\theta_2(n-m) = \theta(n-m) + o(n)$ .  $\square$

Note that

$$\begin{aligned} C(\hat{x}) &\leq |\hat{x}| + O(1) = m + O(1), \\ C(\hat{y}) &\leq |\hat{y}| + O(1) = m + O(1). \end{aligned}$$

Further, from the Claim it follows that

$$C(\hat{y}|\hat{x}) \leq^+ \log \left( \binom{m}{\theta m + o(n)} \right) = h(\theta) \cdot m + o(n) = 0.5m + o(n). \quad (11)$$

Therefore,

$$C(\hat{x}, \hat{y}) =^+ C(\hat{x}) + C(\hat{y}|\hat{x}) \leq^+ 1.5m + o(n).$$

Thus, we have proven that

$$\begin{aligned} C(\hat{x}) &\leq m + o(n), \\ C(\hat{y}) &\leq m + o(n), \\ C(\hat{x}, \hat{y}) &\leq 1.5m + o(n). \end{aligned}$$

It remains to show that these three bounds are tight. To this end, we observe that a similar argument gives the upper bound

$$C(\hat{x}', \hat{y}') \leq 1.5m + o(n).$$

Since

$$C(\hat{x}, \hat{x}', \hat{y}, \hat{y}') =^+ C(x, y) = 1.5n + O(1),$$

we obtain

$$C(\hat{x}, \hat{y}) \geq 1.5m - o(n).$$

Due to (11), this implies  $C(\hat{x}) \geq m - o(n)$ , and similarly  $C(\hat{y}) \geq m - o(n)$ .  $\square$

Thus, Alice and Bob can take the prefixes of their inputs  $x, y$  of size  $m = \delta n$ . Lemma 10 guarantees that these prefixes  $\hat{x}$  and  $\hat{y}$  have the complexity profile (Kolmogorov complexities and mutual information) similar to the complexity profile of the original pair  $(x, y)$  scaled with the factor of  $\delta$  (up to an  $o(n)$ -term). Thus, Alice and Bob can apply to  $\hat{x}$  and  $\hat{y}$  the communication protocol from Theorem 1 and end up with a secret key  $z$  of size  $\delta n/2 - o(n)$ . It is shown in [22] that communication complexity of this protocol is  $C(\hat{x}|\hat{y}) + O(\log m)$  (note that it is enough for Alice and Bob to know the complexity profile of  $(\hat{x}, \hat{y})$  within a precision  $o(n)$ , see Remark 5 in [22]). In our setting this communication complexity is equal to  $\delta n/2 + o(n)$ .  $\square$

*Proof of Theorem 4.* In the proof of the theorem we use two lemmas. The first lemma gives us a pair of simple information inequalities:

**Lemma 11** (see, e.g., Ineq 6 in [9] or lemma 7 in [10]). *For all binary strings  $a, b, c, d$*

$$\begin{aligned} (i) \quad C(c) &\leq^+ C(c|a) + C(c|b) + I(a : b), \\ (ii) \quad C(c|d) &\leq^+ C(c|a) + C(c|b) + I(a : b|d). \end{aligned}$$

The other lemma is more involved:

**Lemma 12** ([8], see also exercise 316 in [18]). *There exists an integer number  $k$  with the following property. Let  $x = x_0$  and  $y = y_0$  be a pair of strings from Example 3 (two  $n$ -bit strings with the Hamming distance  $\theta n$  and complexity profile (2)). Then there exist two sequences of  $n$ -bit binary strings  $x_1, \dots, x_k$  and  $y_1, \dots, y_k$  such that*

- $I(x_i : y_i | x_{i+1}) = O(\log n)$  for  $i = 0, \dots, k-1$ ,
- $I(x_i : y_i | y_{i+1}) = O(\log n)$  for  $i = 0, \dots, k-1$ ,
- $I(x_k : y_k) = O(\log n)$ .

(Note that  $k$  is a constant that does not depend on  $n$ . It is determined uniquely by the value of  $\theta$ , which is the normalized Hamming distance between  $x$  and  $y$ .)

**Remark 10.** In the proof Lemma 12 suggested in [8], each pair  $(x_i, y_i)$  consists of two binary strings of length  $n$  with Hamming distance  $\theta_i n$  and maximal possible (for this value of  $\theta_i$ ) Kolmogorov complexity. In our case, the initial  $\theta_0 = \theta$  is chosen so that  $I(x_0 : y_0) =^+ n/2$ . Each next  $\theta_i$  is bigger than the previous one. For the last pair  $\theta_k = 1/2$ . This means that in the last pair  $(x_k, y_k)$  the strings differ in a half of the positions, so the mutual information is only  $O(\log n)$ .

Applying Lemma 11(ii), we obtain for every string  $w$  and for all  $x_i, y_i, x_{i+1}, y_{i+1}$  the inequalities

$$\begin{aligned} C(w|x_{i+1}) &\leq^+ C(w|x_i) + C(w|y_i) + I(x_i : y_i | x_{i+1}), \\ C(w|y_{i+1}) &\leq^+ C(w|x_i) + C(w|y_i) + I(x_i : y_i | y_{i+1}). \end{aligned}$$

Combining these inequalities for  $i = 0, \dots, k-1$  and taking into account  $I(x_i : y_i | x_{i+1}) = O(\log n)$  and  $I(x_i : y_i | y_{i+1}) = O(\log n)$ , we obtain

$$C(w|x_k) + C(w|y_k) \leq^+ 2^k \cdot (C(w|x_0) + C(w|y_0)).$$

Now we use Lemma 11(i) and obtain

$$C(w) \leq^+ C(w|x_k) + C(w|y_k) + I(x_k : y_k).$$

With the condition  $I(x_k : y_k) = O(\log n)$  we get

$$C(w) \leq^+ 2^k \cdot (C(w|x_0) + C(w|y_0)) + I(x_k : y_k) =^+ 2^k \cdot (C(w|x) + C(w|y)). \quad (12)$$

Denote by  $r_A$  and  $r_B$  the strings of random bits used in the protocol by Alice and Bob respectively. With high probability the randomly chosen  $r_A$  and  $r_B$

have negligibly small mutual information with  $w, x, y$ . Therefore, (12) rewrites to

$$C(w) =^+ 2^k \cdot (C(w|x, r_A) + C(w|y, r_B)). \quad (13)$$

We apply (13) to  $w := \langle t, z \rangle$ , where  $z$  is the secret key obtained by Alice and Bob, and  $t$  is the transcript of the communication protocol. Then

$$C(w) =^+ C(z) + C(t)$$

(the key has no mutual information with the transcript), and

$$C(w|x, r_A) \leq^+ C(t), \quad C(w|y, r_B) \leq^+ C(t)$$

(given the transcript and the data available to Alice or to Bob, we can compute  $z$ ). Plugging this in (13) we obtain

$$C(z) + C(t) \leq^+ 2^{k+1} \cdot C(t),$$

which implies  $C(t) = \Omega(C(z))$ . Therefore, the size of the transcript  $t$  is not less than  $\Omega(\delta n)$ , and we are done.  $\square$

## 6 Conclusion

In Theorem 2 we proved a lower bound for communication complexity of protocols with *private* randomness. The argument can be extended to the setting where Alice and Bob use both *private* and *public* random bits (the private sources of randomness are available only to Alice and Bob respectively; the public source of randomness is available to both parties and to the eavesdropper). Thus, the problem of the *worst case* complexity is resolved for the most general natural model of communication.

The same time, we have no characterization of the optimal communication complexity of the secret key agreement for pairs of inputs  $(x, y)$  that do not enjoy the spectral property required in Corollary 1. In particular, there is a large gap between constant hidden in the  $O(\delta n)$  notation in Theorem 3 and in the  $\Omega(\delta n)$  notation in and Theorem 4, so the question on the optimal trade-off between the secret key size and communication complexity for  $(x, y)$  from Example 3 remains open (*cf.* Conjecture 1 in [17] for an analogous problem in Shannon's setting).

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