Oriented cliques and colorings of graphs with low maximum degree

Janusz Dybizbański, Pascal Ochem, Alexandre Pinlou, Andrzej Szepietowski

Oriented cliques and colorings of graphs with low maximum degree☆

Janusz Dybizbański[a], Pascal Ochem[b], Alexandre Pinlou[b], Andrzej Szepietowski[a]

[a]*Institute of Informatics*
Faculty of Mathematics, Physics and Informatics
University of Gdańsk
80-308 Gdańsk, Poland
jdybiz@inf.ug.edu.pl, matszp@inf.ug.edu.pl

[b]*LIRMM, University of Montpellier, CNRS, Montpellier, France*
firstname.lastname@lirmm.fr

Abstract

An oriented clique, or oclique, is an oriented graph G such that its oriented chromatic number $\chi_o(G)$ equals its order $|V(G)|$. We disprove a conjecture of Duffy, MacGillivray, and Sopena [Oriented colourings of graphs with maximum degree three and four, *Discrete Mathematics* 342(4) (2019) 959–974] by showing that for maximum degree 4, the maximum order of an oclique is equal to 12. For maximum degree 5, we prove that the maximum order of an oclique is between 16 and 18. In the same paper, Duffy et al. also proved that the oriented chromatic number of *connected* oriented graphs with maximum degree 3 and 4 is at most 9 and 69, respectively. We improve these results by showing that the oriented chromatic number of *non-necessarily connected* oriented graphs with maximum degree 3 (resp. 4) is at most 9 (resp. 26). The bound of 26 actually follows from a general result which determines properties for a target graph to be universal for graphs of bounded maximum degree. This generalization also allows us to get the upper bound of 90 (resp. 306, 1322) for the oriented chromatic number of graphs with maximum degree 5 (resp. 6, 7).

Keywords: Oriented graph; Homomorphism; Oriented clique; Bounded degree graph.

2010 Mathematics Subject Classification: 05C15, 05C20, 05C69

1. Introduction

Oriented graphs are directed graphs with neither loops nor opposite arcs. Unless otherwise specified, the term *graph* refers to *oriented graph* in the sequel.

For a graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v, or simply uv whenever its orientation is not relevant (therefore, $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$).

Given two graphs G and H, a *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs, that is, $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An *oriented k-coloring* of G can be defined as a homomorphism from G to H, where H is a graph with k vertices. The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The *oriented chromatic number* of a graph G, denoted by $\chi_o(G)$, is defined as the smallest number of vertices of a graph H such that $G \to H$. If $\mathcal{F}$ is a family of oriented graphs, then $\chi_o(\mathcal{F})$ denotes the maximum of $\chi_o(G)$ over all $G \in \mathcal{F}$.

The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors and the problem of bounding the oriented chromatic number has been investigated for various

☆The second and the third author were partially supported by the French ANR grant HOSIGRA under contract ANR-17-CE40-0022.

graph classes: outerplanar graphs (with given minimum girth) [18, 20], 2-outerplanar graphs [9, 16], planar graphs (with given minimum girth) [1, 2, 3, 4, 14, 16, 17, 19], graphs with bounded maximum average degree [3, 4], graphs with bounded degree [7, 11, 23], graphs with bounded treewidth [15, 20, 21], Halin graphs [8], graph subdivisions [25]. A survey on the study of oriented colorings has been done by Sopena in 2001 and recently updated [22].

For bounded degree graphs, Kostochka et al. [11] proved as a general bound that graphs with maximum degree Δ have oriented chromatic number at most $2\Delta^2 2^\Delta$. They also showed that, for every Δ, there exists graphs with maximum degree Δ and oriented chromatic number at least $2^{\Delta/2}$. For low maximum degrees, specific results are only known for graphs with maximum degree 3 and 4. Sopena [20] proved that graphs with maximum degree 3 have an oriented chromatic number at most 16 and conjectured that any such connected graphs have an oriented chromatic number at most 7. The upper bound was later improved by Sopena and Vignal [23] to 11. Recently, Duffy et al. [7] proved that 9 colors are enough for *connected* graphs with maximum degree 3. They proved in the same paper that *connected* graphs with maximum degree 4 have oriented chromatic number at most 69. Lower bounds are given by Duffy et al. [7] who exhibit a graph with maximum degree 3 (resp. 4) and oriented chromatic number 7 (resp. 11). Note that the above-mentioned conjecture of Sopena is thus best possible. In each of the above cases, the lower bound is achieved by presenting an oclique. An *oclique*, or oriented clique, is an oriented graph G such that $\chi_o(G) = |V(G)|$.

Theorem 1 ([10]). *An oriented graph is an oclique if and only if any two vertices are connected by a directed path of length* 1 *or* 2*.*

In their paper, Duffy et al. proved the following upper bound on maximum size of an oclique with maximum degree Δ.

Theorem 2 ([7]). *Every oclique with maximum degree* Δ *has at most* $\left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor$ *vertices.*

The theorem gives the upper bound 8 (resp. 13, 18) for $\Delta = 3$ (resp. $\Delta = 4$, $\Delta = 5$). They improved the above general result for $\Delta = 3$ by showing that the largest number of vertices in a subcubic oclique is 7. They also prove that there exists an oclique of size 11 with maximum degree $\Delta = 4$. Moreover they conjectured that the maximum order of an oclique with maximum degree $\Delta = 4$ is 11.

In this paper, we first improve the known upper bounds for graphs with low maximum degree. Secondly, we consider ocliques of maximum degree 4 and 5, and disprove the above-mentioned conjecture of Duffy et al. [7].

We prove in Section 4 that the oriented chromatic number of graphs with maximum degree 3 is at most 9 ($\chi_o(\mathcal{G}_3) \leqslant 9$), that is, we remove the condition of connectivity; see Theorem 8. In Section 5, we prove a general result which determines properties of a target graph to be universal for (non-necessarily connected) graphs of maximum degree $\Delta \geqslant 4$; see Theorem 10. As a consequence of this general result, we obtain that the oriented chromatic number of graphs with maximum degree 4 is at most 26 ($\chi_o(\mathcal{G}_4) \leqslant 26$), substantially decreasing the bound of 69 due to Duffy et al. [7]. We also get that $\chi_o(\mathcal{G}_5) \leqslant 90$, $\chi_o(\mathcal{G}_6) \leqslant 306$, and $\chi_o(\mathcal{G}_7) \leqslant 1322$.

In Section 6, we disprove the conjecture of Duffy et al. [7] by showing that the maximum order of an oclique maximum degree 4 equals 12 ($\chi_o(\mathcal{G}_4) \geqslant 12$). More precisely, we exhibit an oclique of order 12 and maximum degree 4, and show that there is no such oclique of order at least 13. Similarly in Section 7, we exhibit an oclique of order 16 and maximum degree 5 ($\chi_o(\mathcal{G}_5) \geqslant 16$).

The next two sections will be devoted to define the notation and to present the properties of the target graphs we use to prove our upper bounds.

2. Notation

In the remainder of this paper, we use the following notions. For a vertex v of a graph G, we denote by $N^+_G(v)$ the set of outgoing neighbors of v, by $N^-_G(v)$ the set of incoming neighbors of v and by $N_G(v) = N_G^+(v) \cup N_G^-(v)$ the set of neighbors of v (subscripts are omitted when the considered graph is clearly identified from the context). The *degree* of a vertex v (resp. *in-degree*, *out-degree*), denoted by $d(v)$ (resp. $d^-(v)$, $d^+(v)$), is the number of its neighbors $|N(v)|$ (resp. incoming neighbors $|N^-(v)|$, outgoing neighbors $|N^+(v)|$). Let $\mathcal{G}_\Delta$ denote the family of oriented graphs with maximum degree Δ. If two graphs G and H are isomorphic, we denote this by $G \cong H$.

3. Paley tournaments and Tromp digraphs

In this section, we describe the general construction of graphs that will be used to prove Theorems 8 and 10, and present some of their useful properties.

For a prime power $p \equiv 3 \pmod 4$, the *Paley tournament* QR_p is defined as the graph whose vertices are the elements of the field $\mathbb{F}_p$ and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of $\mathbb{F}_p$. Clearly QR_p is vertex- and arc-transitive.

An *orientation n-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{-1, 1\}^n$ of n elements. Let $S = (v_1, v_2, \ldots, v_n)$ be a sequence of n (not necessarily distinct) vertices of a graph G. The vertex u is said to be an *α-successor of S* if for any i, $1 \leqslant i \leqslant n$, we have $\overrightarrow{uv_i} \in A(G)$ whenever $\alpha_i = 1$ and $\overrightarrow{v_iu} \in A(G)$ otherwise. A sequence $S = (v_1, v_2, \ldots, v_n)$ of n (not necessarily distinct) vertices of QR_p is said to be *compatible* with an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if $\alpha_i = \alpha_j$ whenever $v_i = v_j$ since graphs do not contain opposite arcs. We say that a graph G has *Property $P_{n,k}$* if, for every sequence S of n vertices of G and any compatible orientation n-vector α, there exist k distinct α-successors of S. Such Properties $P_{n,k}$ have been extensively used in many papers dealing with oriented coloring.

Proposition 3. *The Paley tournament QR_p has Properties $P_{1,\frac{p-1}{2}}$ and $P_{2,\frac{p-3}{4}}$.*

Proof. By the vertex-transitivity of QR_p, the in-degree of every vertex is equal to its out-degree. This implies that QR_p has Property $P_{1,\frac{p-1}{2}}$.

Let us prove that QR_p has Property $P_{2,\frac{p-3}{4}}$. To do so, by arc-transitivity of QR_p, we just have to show that there exist at least $\frac{p-3}{4}$ α-successors of the sequence $S = (0, 1)$ for any of the four orientation vector $\alpha \in \{-1, 1\}^2$.

We first need to count the transitive triangles with arcs $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ in QR_p. There are p choices for the source vertex x of a transitive triangle. The number of transitive triangles such that $x = 0$ is equal to the number of arcs in $N^+(0)$, that is, $\binom{(p-1)/2}{2} = \frac{(p-1)(p-3)}{8}$. Thus, QR_p contains $\frac{p(p-1)(p-3)}{8}$ transitive triangles.

Considering $\alpha = (+1, +1)$, we can notice that $|N^+(0) \cap N^+(1)|$ is the number of transitive triangles such that $\overrightarrow{xy} = \overrightarrow{01}$. Since QR_p is arc-transitive and contains $\frac{p(p-1)}{2}$ arcs, $|N^+(0) \cap N^+(1)| = \frac{p(p-1)(p-3)/8}{p(p-1)/2} = \frac{p-3}{4}$. Similarly for $\alpha = (-1, -1)$ and $\alpha = (+1, -1)$, considering $\overrightarrow{yz} = \overrightarrow{01}$ gives $|N^-(0) \cap N^-(1)| = \frac{p-3}{4}$ and considering $\overrightarrow{xz} = \overrightarrow{01}$ gives $|N^+(0) \cap N^-(1)| = \frac{p-3}{4}$. Finally for $\alpha = (-1, +1)$, we have $N^-(0) \cap N^+(1) = V(QR_p) \setminus \{0, 1, N^+(0) \cap N^+(1), N^-(0) \cap N^-(1), N^+(0) \cap N^-(1)\}$, so that $|N^-(0) \cap N^+(1)| = p - 2 - \frac{3(p-3)}{4} = \frac{p+1}{4} > \frac{p-3}{4}$. This proves $P_{2,\frac{p-3}{4}}$. $\square$

Paley tournaments will be used as basic brick to build new graphs as explained below. Tromp (unpublished manuscript) proposed the following construction. Let G be a graph and let $G' \cong G$. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$

Figure 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between u and u'. Such pairs of vertices will be called *anti-twin vertices*, and we denote by $\mathrm{at}(u) = u'$ the anti-twin vertex of u.

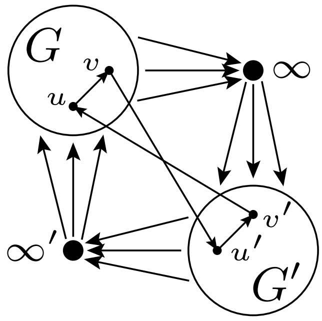


Figure 1: The Tromp graph $Tr(G)$.

In the following, we apply Tromp's construction to Paley tournaments QR_p which produces graphs with interesting structural properties. First of all, Marshall [12] proved that any $Tr(QR_p)$ is *vertex-transitive* and *arc-transitive*. He also prove that any $Tr(QR_p)$ is *triangle-transitive*, meaning that, given two triangles $u_1u_2u_3$ and $v_1v_2v_3$ of $Tr(QR_p)$ with the same orientation, there exists an automorphism that maps u_i to v_i. Secondly, it is possible to derive Properties $P_{n,k}$ for $Tr(QR_p)$ knowing those of QR_p (see Proposition 4). Let us first define the notion of compatible sequence of vertices of $Tr(QR_p)$ with an orientation vector: A sequence $S = (v_1, v_2, \dots, v_n)$ of n (not necessarily distinct) vertices of $Tr(QR_p)$ is said to be *compatible* with an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if for any $i \neq j$, we have $\alpha_i \neq \alpha_j$ whenever $v_i = \mathrm{at}(v_j)$, and $\alpha_i = \alpha_j$ whenever $v_i = v_j$. If the n vertices of S induce an n-clique subgraph of $Tr(QR_p)$ (*i.e.* $v_1, v_2, \dots, v_n$ are pairwise distinct and induce a complete graph), then S is compatible with any orientation n-vector since a vertex u and its anti-twin $\mathrm{at}(u)$ cannot belong together to the same clique. The authors already studied properties of $Tr(QR_{19})$ (see [16, Proposition 5]) and their results can be easily generalized to $Tr(QR_p)$:

Proposition 4. *[16] If QR_p has Property $P_{n-1,k}$, then $Tr(QR_p)$ has Property $P_{n,k}$.*

Let us now introduce another type of properties. We will say that a graph G has *Property $C_{n,k}$* if, given n vertices $v_1, v_2, \dots, v_n$ of G that form a clique subgraph, we have $|\bigcup_{1\leqslant i\leqslant n} N^+(v_i)| \geqslant k$ and $|\bigcup_{1\leqslant i\leqslant n} N^-(v_i)| \geqslant k$.

Remark 5. *Given two integers n and k, a graph having Property $C_{n,k}$ has Property $C_{n',k'}$ for any n' and k' such that $n' \geqslant n$ and $k' \leqslant k$.*

Proposition 6. *The graph $Tr(QR_p)$ has Properties $C_{2,\frac{3p+1}{2}}$ and $C_{3,\frac{7p+3}{4}}$.*

Proof. Recall that $Tr(QR_p)$ is built from two copies of QR_p, that will be named QR_p and QR'_p in the following (see Figure 1). In this proof, the in- and out-neighborhood of a vertex v of $Tr(QR_p)$ will be denoted by $N^-(v)$ and $N^+(v)$, while the in- and out-neighborhood of a vertex v in a subgraph H of $Tr(QR_p)$ will be denoted by $N^-_H(v)$ and $N^+_H(v)$.

Note that, given a set of n vertices $v_1, v_2, \dots, v_n$ of $Tr(QR_p)$ that form a clique subgraph, if $z \in \bigcup_{1\leqslant i\leqslant n} N^+(v_i)$ then $\mathrm{at}(z) = z' \in \bigcup_{1\leqslant i\leqslant n} N^-(v_i)$. Thus $|\bigcup_{1\leqslant i\leqslant n} N^+(v_i)| = |\bigcup_{1\leqslant i\leqslant n} N^-(v_i)|$.

- Let us first consider Property $C_{2,\frac{3p+1}{2}}$. We have to prove that given two adjacent vertices x and y of $Tr(QR_p)$, we have $|N^+(x) \cup N^+(y)| \geqslant \frac{3p+1}{2}$.

  Since x and y are adjacent, w.l.o.g. $x = 0$ and $y = \infty$ by arc-transitivity of $Tr(QR_p)$. Then $N^+(0) \cup N^+(\infty)$ contains:

  - $N^+(\infty) = \{0', 1', \dots, (p-1)'\}$ (p vertices);
  - Note that $N^+(0) = N^+_{QR_p}(0) \uplus N^+_{QR'_p}(0) \uplus \{\infty\}$. Since $N^+_{QR'_p}(0) \subset N^+(\infty)$ is already counted in the previous point, we just consider $N^+_{QR_p}(0)$ (at least $\frac{p-1}{2}$ vertices by Proposition 3) and ∞ (1 vertex);

So that $N^+(\infty) \cup N^+(0)$ contains at least $p + \frac{p-1}{2} + 1 = \frac{3p+1}{2}$ vertices and thus $Tr(QR_p)$ has Property $C_{2,\frac{3p+1}{2}}$.

- Let us now consider Property $C_{3,\frac{7p+3}{4}}$. We have to prove that given three vertices x, y, and z of $Tr(QR_p)$ that form a triangle, we have $|N^+(x) \cup N^+(y) \cup N^+(z)| \geqslant \frac{7p+3}{4}$.

  We have to consider two cases depending on whether x, y, z form a transitive triangle or x, y, z form a directed triangle. By triangle-transitivity of $Tr(QR_p)$, it suffices to consider the cases $x, y, z = 0, 1, \infty$ (transitive triangle) and $x, y, z = 0, 1', \infty$ (directed triangle).

  Case $x, y, z = 0, 1, \infty$: Let $A = N^+(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. We clearly have $|N^+(0) \cup N^+(1) \cup N^+(\infty)| = |N^+(0) \cup N^+(\infty)| + |A|$. Since we already know that $|N^+(0) \cup N^+(\infty)| = \frac{3p+1}{2}$ (see the previous point), let us focus on the set A. We have $N^+(1) = N^+_{QR_p}(1) \uplus N^+_{QR'_p}(1) \uplus \{\infty\}$. Since $N^+_{QR'_p}(1) \subset N^+(\infty)$ and $\{\infty\} \subset N^+(0)$, we have $A = N^+_{QR_p}(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. Since vertex ∞ has no out-neighbor in QR_p, we have $A = N^+_{QR_p}(1) \setminus N^+(0)$, that corresponds to the set of out-neighbors of 1 in QR_p which are not out-neighbors of 0. Since QR_p is a tournament, the vertices which are not out-neighbors of a given vertex are the in-neighbors of this vertex. Therefore, the set A corresponds to the set of out-neighbors of 1 in QR_p which are in-neighbors of 0 and thus $A = N^+_{QR_p}(1) \cap N^-(0)$. This set has already been considered in the proof of Proposition 3 where we showed that $|A| = \frac{p+1}{4}$.

  Therefore, the set $N^+(0) \cup N^+(1) \cup N^+(\infty)$ contains $|N^+(0) \cup N^+(\infty)| + |A| \geqslant \frac{3p+1}{2} + \frac{p+1}{4} = \frac{7p+3}{4}$ vertices.

  Case $x, y, z = 0, 1', \infty$: Note that $N^+(1') = N^-(1)$. Thus $N^+(0) \cup N^+(1') \cup N^+(\infty) = N^+(0) \cup N^-(1) \cup N^+(\infty)$. Using the same kind of arguments as previous case, we get that $|N^+(0) \cup N^-(1) \cup N^+(\infty)| \geqslant \frac{7p+3}{4}$.

$\square$

4. Upper bound of the oriented chromatic number of graphs with maximum degree 3

In this section, we consider graphs with maximum degree 3 and we prove that they all admit a homomorphism to the same target graph on nine vertices.

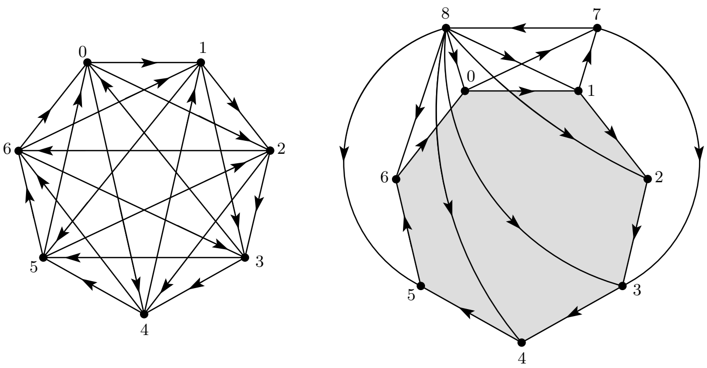
Duffy et al. [7] proved that every *connected* graph with maximum degree 3 has an oriented chromatic number at most 9. To achieve this bound, they use the Paley tournament QR_7 which has vertex set $V(QR_7) = \{0, 1, \dots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r \pmod 7$ for $r \in \{1, 2, 4\}$ (see Figure 2a). Here is a quick sketch of their proof. They first consider 2-degenerated graphs with maximum degree 3 (not necessarily connected) and prove the following:

Theorem 7. *[7] Every 2-degenerate graph with maximum degree 3 which does not contain a 3-source adjacent to a 3-sink is QR_7-colorable.*

Then, given a *connected* graph G with maximum degree 3, they firstly consider the case where G contains a 3-source. By removing all 3-sources from G, we obtain a graph G' that is QR_7-colorable by Theorem 7. It is then easy to put back all the 3-sources and color them with a new color 7. Then, subsequently, they consider the case where G does not contain 3-sources. Removing any arc $\overrightarrow{uv}$ from G leads to a graph G' which admits a QR_7-coloring φ by Theorem 7. To extend φ to G, it suffices to recolor u and v with two new colors so that $\varphi(u) = 7$ and $\varphi(v) = 8$. This gives that G has an oriented chromatic number at most 9.

The condition of connectivity of G is a necessary condition in their proof. Indeed, given a graph G with maximum degree 3 which is not connected, we need to remove one arc $\overrightarrow{u_iv_i}$ from each 3-regular component C_i of G to get a 2-degenerate graph G' which is QR_7-colorable by Theorem 7. However, to extend the coloring to G using two new colors, say color 7 for the u_i's and color 8 for the v_i's, the colorings of each component must agree on the neighbors of each u_i's and on the neighbors of each v_i's, which is not necessarily the case. This potentially leads to different target graphs on nine vertices for each component. Therefore, even if each component has an oriented chromatic number at most 9, the whole graph may have an oriented chromatic number strictly greater than 9.

In the following, we prove this is not the case by showing that it is possible to color each component with the same target graph T_9 on nine vertices whose construction is described below. This implies that the condition of connectivity is no more needed.



(a) QR_7.

(b) T_9.

Figure 2: The oriented graphs QR_7 and T_9.

The oriented graph T_9 is obtained from QR_7 (see Figure 2a) by adding two vertices labelled 7 and 8, and the arcs $\overrightarrow{07}$, $\overrightarrow{17}$, $\overrightarrow{73}$, $\overrightarrow{78}$, and $\overrightarrow{8i}$ for every $0 \leqslant i \leqslant 6$ (see Figure 2b where the grey part stands for QR_7).

We prove the following:

Theorem 8. *Every graph with maximum degree* 3 *admits a* T_9*-coloring and thus* $\chi_o(\mathcal{G}_3) \leqslant 9$.

Proof. It is sufficient to show that every connected graph G with maximum degree 3 admits a T_9-coloring. We consider the following cases.

- We suppose that G is 2-degenerate or G contains a 3-source. Let G' be the oriented graph obtained from G by removing every 3-source. Since G' is 2-degenerate and contains no 3-source, G' admits a QR_7-coloring φ by Theorem 7. We extend φ to a T_9-coloring of G by setting $\varphi(u) = 8$ for every 3-source u of G (indeed, the vertex 8 of T_9 dominates all the vertices of QR_7).

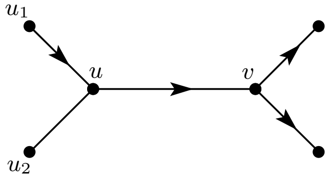


Figure 3: Configuration of Theorem 8.

- We suppose that G is 3-regular and contains no 3-source. Notice that G necessarily contains a vertex v of out-degree two. Let u denote the in-neighbor of v. Since u is not a 3-source, it has an in-neighbor u_1. Let u_2 denote the neighbor of u distinct from u_1 and v (see Figure 3). We consider the graph G' obtained from G by removing the arc $\overrightarrow{uv}$. Since G' is 2-degenerate and contains no 3-source, G' admits a QR_7-coloring φ by Theorem 7.

– If G (or equivalently G') contains the arc $\overrightarrow{uu_2}$, then we necessarily have $\varphi(u_1) \neq \varphi(u_2)$. If $\overrightarrow{\varphi(u_1)\varphi(u_2)} \in A(QR_7)$ (resp. $\overrightarrow{\varphi(u_2)\varphi(u_1)} \in A(QR_7)$), we recolor G' so that $\varphi(u_1) = 1$ (resp. $\varphi(u_1) = 0$) and $\varphi(u_2) = 3$ by the arc-transitivity of QR_7. It can be easily checked that we can extend φ to a T_9-coloring of G by setting $\varphi(u) = 7$ and $\varphi(v) = 8$.

– If G contains $\overrightarrow{u_2u}$, then by the arc-transitivity of QR_7, we can assume that $\{\varphi(u_1), \varphi(u_2)\} \subseteq \{0, 1\}$. Again, φ can be extended to T_9-coloring of G by setting $\varphi(u) = 7$ and $\varphi(v) = 8$.

□

5. Upper bound of the oriented chromatic number of graphs with maximum degree at least 4

In this section, we consider graphs with maximum degree at least 4.

Duffy et al. [7] recently proved that every *connected* graph with maximum degree 4 has an oriented chromatic number at most 69. To prove their result, they consider the case of 3-degenerate graphs with maximum degree 4 and prove that they admit a homomorphism to the Paley tournament QR_{67} on 67 vertices; they then show how to extend such a 67-coloring to *connected* graphs with maximum degree 4 using two more colors, leading to a 69-coloring.

We propose general result which determines properties of a target graph to be universal for graphs of maximum degree $\Delta \geqslant 4$. As for graphs with maximum degree 3 (see Section 4), the condition of connectivity is not needed. As an example, our general result substantially decreases the bound of 69 colors for graphs with maximum 4 due to Duffy et al. [7] to 26 colors.

Theorem 9. *Every $(\Delta - 1)$-degenerate graph with maximum degree $\Delta \geqslant 2$ admits a T-coloring where T is a graph on n vertices with Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,\frac{n(\Delta-2)}{\Delta-1}+1}$.*

Proof. Let G be a minimal counter-example to Theorem 9. By definition, G contains a k-vertex u with $k \leqslant \Delta - 1$. Let $v_1, v_2, \dots, v_k$ be the neighbors of u.

Suppose first that there exists one arc in the neighborhood of u, and w.l.o.g. assume that $\overrightarrow{v_1v_2}$ is an arc of G. The graph G' obtained from G by removing the arc uv_1 admits a T-coloring φ by minimality of G. Vertex v_1 has at most $\Delta-1$ colored neighbors $w_1, \dots, w_{\Delta-1}$ by φ (u is uncolored). Note that some w_i's may coincide with some v_i's. The sequence $S = (\varphi(w_1), \varphi(w_2), \dots, \varphi(w_{\Delta-1}))$ is compatible w.r.t. the orientations of the arcs w_iv_1 since v_1 and every w_i are colored in G'. We uncolor v_1 and by Property $P_{\Delta-1,\Delta-2}$ of T, we have $\Delta - 2$ available colors for v_1 among which at least one, say color c, is compatible with $\varphi(v_3), \varphi(v_4), \dots, \varphi(v_k)$ w.r.t. to the orientation of the arcs $uv_1, uv_3, uv_4, \dots, uv_k$. Set $\varphi(v_1) = c$. Note that we necessarily have $c \notin \{\varphi(v_2), \mathrm{at}(\varphi(v_2))\}$ since $\overrightarrow{v_1v_2}$ is an arc of G and thus c is compatible with $\varphi(v_2)$. Note also that the sequence $S = (\varphi(v_2), \varphi(v_3), \dots, \varphi(v_k))$ is compatible w.r.t. the orientations of the arcs uv_i, $2 \leqslant i \leqslant k$, since vertices $v_2, v_3, \dots, v_k$ are colored in G'. Therefore, the sequence $S = (\varphi(v_1), \varphi(v_2), \dots, \varphi(v_k))$ is compatible with the orientation of the arcs uv_i, $1 \leqslant i \leqslant k$. By Property $P_{\Delta-1,\Delta-2}$, we have at least $\Delta - 2$ available colors for u. Therefore, φ can be extended to a T-coloring of G, a contradiction.

Assume now that there is no arc in the neighborhood of u. The graph G' obtained from G by removing the vertex u admits a T-coloring φ by minimality of G. Let w_i, $1 \leqslant i \leqslant \Delta - 1$, be the neighbors of v_1 distinct from u. Since v_1 and every w_i are colored in G', the sequence $S = (\varphi(w_1), \varphi(w_2), \dots, \varphi(w_{\Delta-1}))$ is compatible with the orientations of the arcs vw_i and by Property $P_{\Delta-1,\Delta-2}$, we have $\Delta - 2$ available colors for v_1. Note that these $\Delta - 2$ colors (which are vertices of T) form a clique subgraph of T. The same holds for every v_i (i.e. each v_i has $\Delta - 2$ available colors) and since there is no arc in the neighborhood of u, each v_i can be colored independently.

By Property $C_{\Delta-2,\frac{n(\Delta-2)}{\Delta-1}+1}$, given these $\Delta-2$ possible colors for v_1, there are at least $\frac{n(\Delta-2)}{\Delta-1}+1$ choices of colors for u; thus v_1 forbids at most $n - \left(\frac{n(\Delta-2)}{\Delta-1} + 1\right) = \frac{n}{\Delta-1} - 1$ colors for u. By the same argument, each v_i forbids at most $\frac{n}{\Delta-1} - 1$ colors for u. Therefore, u has at most $k(\frac{n}{\Delta-1} - 1) \leqslant (\Delta - 1)(\frac{n}{\Delta-1} - 1) = n - \Delta + 1 < n$ forbidden colors since $\Delta \geqslant 2$. This means that there are at least one available color for u. Therefore, φ can be extended to a T-coloring of G, a contradiction. □

To achieve our bounds on oriented chromatic number, we apply Tromp's construction to Paley tournaments QR_p. Moreover, given a graph $Tr(QR_p)$, we construct the graph $Tr^*(QR_p)$ on $2p+4$ vertices by adding two vertices t_0 and t_1 such that t_0 is a twin vertex of vertex 0 (*i.e.* a vertex with the same neighborhood as vertex 0) and t_1 is a twin vertex of vertex 1. We finally add the arc $\overrightarrow{t_1 t_0}$.

Theorem 10. *Every graph with maximum degree $\Delta \geqslant 4$ admits a $Tr^*(QR_p)$-coloring where $Tr(QR_p)$ is a Tromp graph (built from a Paley tournament QR_p) with Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,\frac{(2p+2)(\Delta-2)}{\Delta-1}+1}$.*

Proof. Since $Tr(QR_p)$ is a subgraph of $Tr^*(QR_p)$, it remains to prove that every connected Δ-regular graph H admits a $Tr^*(QR_p)$-coloring.

Let $H' = H \setminus \{\overrightarrow{uv}\}$ where $\overrightarrow{uv}$ is any arc of H. By Theorem 9, H' admits a $Tr(QR_p)$-coloring φ. By vertex-transitivity of $Tr(QR_p)$, we may assume that $\varphi(v) = 0$. By Property $P_{\Delta-1,\Delta-2}$ of $Tr(QR_p)$, we have $\Delta-2$ available colors $c_1, c_2, \ldots, c_{\Delta-2}$ for u. Note that, given $1 \leqslant i < j \leqslant \Delta-2$, we necessarily have $c_i \neq \mathrm{at}(c_j)$. We thus recolor u with one of the c_i's so that $\varphi(u) \notin \{0, 0'\}$ since $\Delta \geqslant 4$. Therefore, $\varphi(u)\varphi(v)$ is an arc of $Tr(QR_p)$.

If $\overrightarrow{\varphi(u)\varphi(v)}$ is an arc of $Tr(QR_p)$, then φ is a $Tr(QR_p)$-coloring of H and thus a $Tr^*(QR_p)$-coloring of H. Therefore, $\overrightarrow{\varphi(v)\varphi(u)}$ is an arc of $Tr(QR_p)$. By arc-transitivity of $Tr(QR_p)$, we may assume that $\varphi(u) = 1$. We recolor u and v so that $\varphi(u) = t_1$ and $\varphi(v) = t_0$. Since each t_i is a twin vertex of vertex i in $Tr^*(QR_p)$ and $\overrightarrow{t_1 t_0}$ is an arc $Tr^*(QR_p)$, it is easy to verify that φ is now a $Tr^*(QR_p)$-coloring of H. $\square$

The Properties $P_{n,k}$ of Paley tournaments QR_p can be expressed by a formula for $n \leqslant 2$ (see Proposition 4). For higher values of n, there are no formula and some properties are only known for small values of p. In our case, we are interested in properties of the form $P_{n,n}$. We computed these properties using a computer-check.

Proposition 11. *The smallest Paley tournament with Property $P_{2,2}$ (resp. $P_{3,3}$, $P_{4,4}$, $P_{5,5}$) is QR_{11} (resp. QR_{43}, QR_{151}, QR_{659}).*

Proof. By computer. $\square$

Determining more of such properties would be possible with more computing power and time. Note that Paley tournament QR_{151} has in fact Property $P_{5,6}$ and there exist no smaller Paley tournament with Property $P_{5,5}$.

As a corollary of Propositions 4 and 11, we get the following.

Corollary 12. *The Tromp graph $Tr(QR_{11})$ (resp. $Tr(QR_{43})$, $Tr(QR_{151})$, $Tr(QR_{659})$) has Property $P_{3,2}$ (resp. $P_{4,3}$, $P_{5,4}$, $P_{6,5}$).*

As corollaries of Remark 5, Proposition 6, Theorem 10, and Corollary 12, we obtain the following four results. We give a proof of the last one (Corollary 16), the other three corollaries follow the same arguments.

Corollary 13. *Every graph $G \in \mathcal{G}_4$ admits a $Tr^*(QR_{11})$-coloring. Thus, $\chi_o(\mathcal{G}_4) \leqslant 26$.*

Corollary 14. *Every graph $G \in \mathcal{G}_5$ admits a $Tr^*(QR_{43})$-coloring. Thus, $\chi_o(\mathcal{G}_5) \leqslant 90$.*

Corollary 15. *Every graph $G \in \mathcal{G}_6$ admits a $Tr^*(QR_{151})$-coloring. Thus, $\chi_o(\mathcal{G}_6) \leqslant 306$.*

Corollary 16. *Every graph $G \in \mathcal{G}_7$ admits a $Tr^*(QR_{659})$-coloring. Thus, $\chi_o(\mathcal{G}_7) \leqslant 1322$.*

Proof. Let $\Delta = 7$. By Proposition 6, the graph $Tr(QR_{659})$ has Property $C_{3,\frac{7p+3}{4}} = C_{3,1154}$. By Remark 5, it thus has Property $C_{5,1101} = C_{\Delta-2,\frac{(2p+2)(\Delta-2)}{\Delta-1}+1}$. By Corollary 12, it also has Property $P_{6,5} = P_{\Delta-1,\Delta-2}$. The graph $Tr(QR_{659})$ verifies the hypothesis of Theorem 10, and thus every graph with maximum degree 7 admits a $Tr^*(QR_{659})$-coloring. $\square$

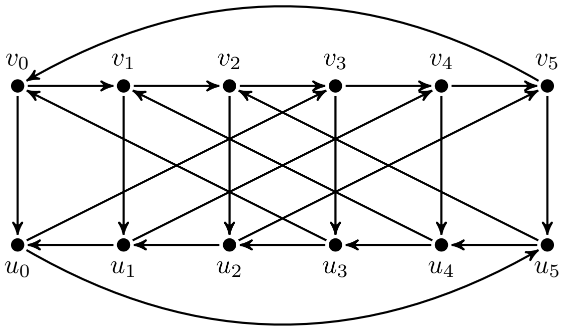


Figure 4: 4-regular oclique on 12 vertices.

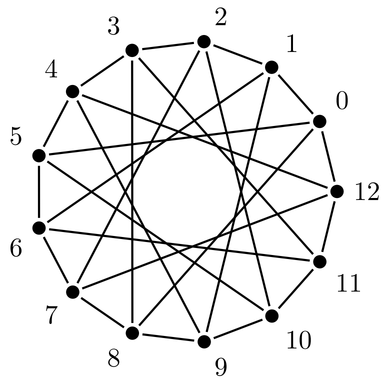


Figure 5: Unique triangle-free 4-regular graph with 13 vertices and diameter 2.

6. Oclique with maximum degree 4

Lemma 17. *There is an oclique of order 12 with maximum degree* $\Delta = 4$.

Proof. Consider the oriented graph presented on Figure 4. The graph consists of two directed cycles: $(v_0, v_1, v_2, v_3, v_4, v_5)$ and $(u_5, u_4, u_3, u_2, u_1, u_0)$. These two cycles are connected by the arcs $\overrightarrow{v_i u_i}$ and $\overrightarrow{u_{(i+3) \bmod 6} v_i}$ for every $0 \leqslant i \leqslant 5$. The graph is vertex-transitive. By Theorem 1, it is enough to find a directed 2-path between v_0 and every vertex not adjacent to v_0:

$v_0 \to v_1 \to v_2$, $v_0 \to u_0 \to v_3$, $v_4 \to v_5 \to v_0$, $v_0 \to v_1 \to u_1$, $u_2 \to v_5 \to v_0$, $u_4 \to u_3 \to v_0$, $v_0 \to u_0 \to u_5$. □

Lemma 18. *If G is an oclique with maximum degree 4 and order 13 then:*

(a) *every vertex has in-degree and out-degree equal to 2,*

(b) *every two vertices are connected by only one directed path of length at most 2.*

Proof. Each vertex $x \in V(G)$ is the midpoint of $d^-(x) \cdot d^+(x)$ directed paths of length 2, so we have

$$\sum_{x \in V(G)} d^-(x) \cdot d^+(x)$$

directed paths of length 2 in the graph. This sum achieves the maximum value of $4 \cdot 13 = 52$ if and only if $d^-(x) = d^+(x) = 2$ for every vertex x. The number of directed paths of length 1 is at most $\frac{4 \cdot 13}{2} = 26$. On the other hand we have $\binom{13}{2} = 52 + 26$ pairs of vertices. Hence, if every pair of vertices is connected by a path of length 1 or two, then every vertex has in-degree and out-degree equal to 2, and every two vertices are connected by only one directed path of length at most 2. □

Corollary 19. *The underlying graph of an oclique on* 13 *vertices and maximum degree* 4 *is triangle-free.*

Let G^{13} be the graph $(\mathbb{Z}_{13}, E)$, where $(u,v) \in E \iff v-u = a \pmod{13}$ for $a \in \{-1,1,-5,5\}$, see Figure 5. Meringer [13] describes a method to generate regular graphs of given girth. Using this generator, we can list the 4-regular graphs[1] on 13 vertices with girth at least 4 (triangle-free). Among them, only G^{13} has diameter 2. Hence, an oclique on 13 vertices and maximum degree 4 must be an orientation of G^{13}.

Lemma 20. *No orientation of G^{13} is an oclique.*

Proof. Suppose for a contradiction, that there is an orientation of G^{13} which is an oclique. Vertex 0 is connected to vertex 2 only by one path of length at most 2, namely 0–1–2. Without loss of generality assume that those edges are oriented: $0 \to 1 \to 2$. Furthermore, there is only one path going from 1 to 3, namely the one going through vertex 2. Hence, there is the arc $2 \to 3$ and so on, we show that there is the cycle $0 \to 1 \to 2 \to \ldots \to 12 \to 0$. Edges which are not oriented so far also form the cycle $0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8$. By Lemma 18a, this cycle must be oriented in one direction. So, if $y = x + 5 \bmod 13$, we have either every arc $\overrightarrow{xy}$ or every arc $\overrightarrow{yx}$. However, the orientation with the arcs $\overrightarrow{yx}$ is isomorphic to the orientation with the arcs $\overrightarrow{xy}$ via the mapping $x \mapsto 5x \bmod 13$. So we only consider the orientation with the arcs $\overrightarrow{xy}$. Then we have two directed paths between 0 and 6, namely via 1 and via 5. This contradicts Lemma 18b. $\square$

Using Lemma 18 and Lemma 20 we have:

Theorem 21. *The maximum order of an oclique with maximum degree* 4 *is* 12.

Corollary 22. $\chi_o(\mathcal{G}_4) \geqslant 12$.

7. Oclique with maximum degree 5

Lemma 23. *There is an oclique with maximum degree* 5 *and order* 16.

Proof. Consider the graph presented on Fig. 6. The graph is constructed from the oriented 4-regular oclique G on 12 vertices presented on Fig. 4 by adding four vertices x_1, x_2, x_3, x_4. These vertices form the cycle $x_1 \to x_4 \to x_3 \to x_2 \to x_1$ and there are arcs:

- $\overrightarrow{x_2v_0}$, $\overrightarrow{x_2v_2}$, $\overrightarrow{x_2v_4}$,
- $\overrightarrow{x_4v_1}$, $\overrightarrow{x_4v_3}$, $\overrightarrow{x_4v_5}$,
- $\overrightarrow{u_0x_1}$, $\overrightarrow{u_2x_1}$, $\overrightarrow{u_4x_1}$,
- $\overrightarrow{u_1x_3}$, $\overrightarrow{u_3x_3}$, $\overrightarrow{u_5x_3}$.

We will show that any two vertices in G are connected by a path of length at most 2. Since new vertices form the oriented cycle C_4, there are paths between these vertices. It is easy to find all other directed paths. For example, paths connecting x_2 and all other vertices are:

- $x_2 \to v_i$ for $i \in \{0,2,4\}$,
- $x_2 \to v_i \to v_{i+1 \bmod 6}$ for $i \in \{1,3,5\}$,
- $x_2 \to v_i \to u_i$ for $i \in \{0,2,4\}$,
- $u_i \to x_3 \to x_2$ for $i \in \{1,3,5\}$.

$\square$

From Lemma 23 and Theorem 2 we have:

Corollary 24. *The maximum order of oclique with maximum degree* 5 *is between* 16 *and* 18.

Corollary 25. $\chi_o(\mathcal{G}_5) \geqslant 16$.

[1]There are 31 such graphs which can be found at http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html

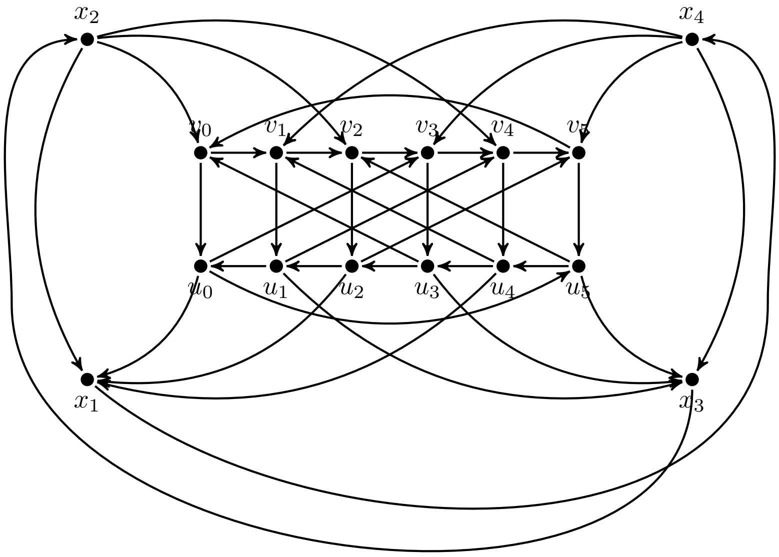


Figure 6: 5-regular oclique on 16 vertices.

8. Discussion

It is worth noticing that similarly as in Lemma 18, one can prove the following:

Lemma 26. *Suppose that G is an oclique of maximum degree $\Delta \geqslant 2$ and order*

$$\left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor.$$

Then:

- *Every two vertices are connected by only one directed path of length at most 2.*
- *For every vertex v:*
  - *either $d^-(v) = \lfloor \frac{\Delta}{2} \rfloor$ and $d^+(v) = \lceil \frac{\Delta}{2} \rceil$,*
  - *or $d^+(v) = \lfloor \frac{\Delta}{2} \rfloor$ and $d^-(v) = \lceil \frac{\Delta}{2} \rceil$.*

We can also obtain a lower bound confirming that the maximum order of oclique is quadratic in terms of Δ.

Lemma 27. *For every Δ, there exists an oclique of maximum degree Δ and order $\frac{\Delta^2}{7} + O(\Delta)$.*

Proof. The postage stamp problem[2] asks, given integers h and k, for a set of integers $V = \{v_1, v_2, \ldots, v_k\}$ that maximizes the smallest integer $PSP(h,k)$ that cannot be written as the sum of at most h (not necessarily distinct) elements of V. In the context of ocliques, we are interested in the case of $h = 2$ stamps, which corresponds to directed paths of length at most 2. Using a set V corresponding to the value of $PSP(2,k)$, we construct the circulant oriented graph G_k on $n = 2PSP(2,k) - 1$ vertices $g_0, \ldots, g_{n-1}$ and the arcs $\overrightarrow{g_i g_{i+v_j}}$ such that $0 \leqslant i < n$, $1 \leqslant j \leqslant k$, and indices are taken modulo n. By the properties of V, there exists a directed path of length at most 2 in G_k from g_0 to every vertex g_i with $1 \leqslant i \leqslant PSP(2,k) - 1$. Now since G_k is circular, G_k is an oclique. Also, G_k is Δ-regular with $\Delta = 2k$. We use the bound $PSP(2,k) \geqslant \frac{2}{7}k^2 + O(k)$ [24] to obtain $|V(G_k)| = 2PSP(2,k) - 1 \geqslant 2 \times \frac{2}{7}k^2 + O(k) = \frac{\Delta^2}{7} + O(\Delta)$. □

Finally, Duffy proves in a recent preprint [6] that the oriented chromatic number of a connected cubic graph is at most 8.

[2] https://en.wikipedia.org/wiki/Postage_stamp_problem

Acknowledgements

We thank Kevin P. Costello for telling us that the key notion in Lemma 27 is called the postage stamp problem in the literature.

References

[1] O. V. Borodin and A. O. Ivanova. An oriented 7-colouring of planar graphs with girth at least 7. *Sib. Electron. Math. Reports*, 2:222–229, 2005.

[2] O. V. Borodin and A. O. Ivanova. An oriented colouring of planar graphs with girth at least 4. *Sib. Electron. Math. Reports*, 2:239–249, 2005.

[3] O. V. Borodin, A. O. Ivanova, and A. V. Kostochka. Oriented 5-coloring of sparse plane graphs. *J. Applied and Industrial Mathematics*, 1(1):9–17, 2007.

[4] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud, and É. Sopena. On the maximum average degree and the oriented chromatic number of a graph. *Discrete Math.*, 206:77–89, 1999.

[5] B. Courcelle. The monadic second order-logic of graphs \textscVI : on several representations of graphs by relational structures. *Discrete Appl. Math.*, 54:117–149, 1994.

[6] C. Duffy. A note on colourings of connected oriented cubic graphs. arXiv e-print 1908.02883, Aug. 2019.

[7] C. Duffy, G. MacGillivray, and É. Sopena. Oriented colourings of graphs with maximum degree three and four. *Discrete Mathematics*, 342(4):959–974, Apr. 2019.

[8] J. Dybizbański and A. Szepietowski. The oriented chromatic number of Halin graphs. *Inform. Process. Lett.*, 114(1–2):45–49, 2014.

[9] L. Esperet and P. Ochem. Oriented coloring of 2-outerplanar graphs. *Inform. Process. Lett.*, 101(5):215–219, 2007.

[10] W. F. Klostermeyer and G. MacGillivray. Analogs of Cliques for Oriented Coloring. *Discussiones Mathematicae Graph Theory*, 24:373–388, 2004.

[11] A. V. Kostochka, É. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. *J. Graph Theory*, 24:331–340, 1997.

[12] T. H. Marshall. Homomorphism bounds for oriented planar graphs. *J. Graph Theory*, 55(3):175–190, 2007.

[13] M. Meringer. Fast generation of regular graphs and construction of cages. *Journal of Graph Theory*, 30(2):137–146, 1999.

[14] P. Ochem. Oriented Colorings of Triangle-free Planar Graphs. *Inform. Process. Lett.*, 92:71–76, 2004.

[15] P. Ochem and A. Pinlou. Oriented colorings of partial 2-trees. *Inform. Process. Lett.*, 108(2):82–86, 2008.

[16] P. Ochem and A. Pinlou. Oriented Coloring of Triangle-Free Planar Graphs and 2-Outerplanar Graphs. *Graphs and Combinatorics*, 30(2):439–453, Mar. 2014.

[17] A. Pinlou. An oriented coloring of planar graphs with girth at least five. *Discrete Math.*, 309(8):2108–2118, 2009.

[18] A. Pinlou and É. Sopena. Oriented vertex and arc colorings of outerplanar graphs. *Inform. Process. Lett.*, 100(3):97–104, 2006.

[19] A. Raspaud and É. Sopena. Good and semi-strong colorings of oriented planar graphs. *Inform. Process. Lett.*, 51(4):171–174, 1994.

[20] É. Sopena. The chromatic number of oriented graphs. *J. Graph Theory*, 25:191–205, 1997.

[21] É. Sopena. Oriented graph coloring. *Discrete Math.*, 229(1-3):359–369, 2001.

[22] É. Sopena. Homomorphisms and colourings of oriented graphs: An updated survey. *Discrete Mathematics*, 339(7):1993–2005, 2016.

[23] É. Sopena and L. Vignal. A note on the chromatic number of graphs with maximum degree three. Technical Report RR-1125-96, LaBRI, Universit\'e Bordeaux 1, 1996.

[24] A. Von Mrose. Untere Schranken für die Reichweiten von Extremalbasen fester Ordnung. *Abh. Math. Semin. Univ. Hambg.*, 48(1):118–124, 1979.

[25] D. R. Wood. Acyclic, Star and Oriented Colourings of Graph Subdivisions. *Discrete Math. Theoret. Comput. Sci.*, 7(1):37–50, 2005.