# Oriented cliques and colorings of graphs with low maximum degree

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## Abstract

An oriented clique, or oclique, is an oriented graph $G$ such that its oriented chromatic number $\chi_o(G)$ equals its order $|V(G)|$. We disprove a conjecture of Duffy, MacGillivray, and Sopena [Oriented colourings of graphs with maximum degree three and four, *Discrete Mathematics* 342(4) (2019) 959–974] by showing that for maximum degree 4, the maximum order of an oclique is equal to 12. For maximum degree 5, we prove that the maximum order of an oclique is between 16 and 18. In the same paper, Duffy et al. also proved that the oriented chromatic number of *connected* oriented graphs with maximum degree 3 and 4 is at most 9 and 69, respectively. We improve these results by showing that the oriented chromatic number of *non-necessarily connected* oriented graphs with maximum degree 3 (resp. 4) is at most 9 (resp. 26). The bound of 26 actually follows from a general result which determines properties for a target graph to be universal for graphs of bounded maximum degree. This generalization also allows us to get the upper bound of 90 (resp. 306, 1322) for the oriented chromatic number of graphs with maximum degree 5 (resp. 6, 7).

*Keywords:* Oriented graph; Homomorphism; Oriented clique; Bounded degree graph.

## 1. Introduction

Oriented graphs are directed graphs with neither loops nor opposite arcs. Unless otherwise specified, the term *graph* refers to *oriented graph* in the sequel.

For a graph $G$, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For two adjacent vertices $u$ and $v$, we denote by $\overrightarrow{uv}$ the arc from $u$ to $v$, or simply $uv$ whenever its orientation is not relevant (therefore, $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$).

Given two graphs $G$ and $H$, a *homomorphism* from $G$ to $H$ is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs, that is, $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An *oriented k-coloring* of $G$ can be defined as a homomorphism from $G$ to $H$, where $H$ is a graph with $k$ vertices. The existence of such a homomorphism from $G$ to $H$ is denoted by $G \to H$. The vertices of $H$ are called *colors*, and we say that $G$ is $H$-colorable. The *oriented chromatic number* of a graph $G$, denoted by $\chi_o(G)$, is defined as the smallest number of vertices of a graph $H$ such that $G \to H$. If $\mathcal{F}$ is a family of oriented graphs, then $\chi_o(\mathcal{F})$ denotes the maximum of $\chi_o(G)$ over all $G \in \mathcal{F}$.

The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors and the problem of bounding the oriented chromatic number has been investigated for various

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graph classes: outerplanar graphs (with given minimum girth) [18, 20], 2-outerplanar graphs [9, 16], planar graphs (with given minimum girth) [1, 2, 3, 4, 14, 16, 17, 19], graphs with bounded maximum average degree [3, 4], graphs with bounded degree [7, 11, 23], graphs with bounded treewidth [15, 20, 21], Halin graphs [8], graph subdivisions [25]. A survey on the study of oriented colorings has been done by Sopena in 2001 and recently updated [22].

For bounded degree graphs, Kostochka et al. [11] proved as a general bound that graphs with maximum degree $\Delta$ have oriented chromatic number at most $2\Delta^2 2^\Delta$. They also showed that, for every $\Delta$, there exists graphs with maximum degree $\Delta$ and oriented chromatic number at least $2^{\Delta/2}$. For low maximum degrees, specific results are only known for graphs with maximum degree 3 and 4. Sopena [20] proved that graphs with maximum degree 3 have an oriented chromatic number at most 16 and conjectured that any such connected graphs have an oriented chromatic number at most 7. The upper bound was later improved by Sopena and Vignal [23] to 11. Recently, Duffy et al. [7] proved that 9 colors are enough for *connected* graphs with maximum degree 3. They proved in the same paper that *connected* graphs with maximum degree 4 have oriented chromatic number at most 69. Lower bounds are given by Duffy et al. [7] who exhibit a graph with maximum degree 3 (resp. 4) and oriented chromatic number 7 (resp. 11). Note that the above-mentioned conjecture of Sopena is thus best possible. In each of the above cases, the lower bound is achieved by presenting an oclique. An *oclique*, or oriented clique, is an oriented graph $G$ such that $\chi_o(G) = |V(G)|$.

**Theorem 1** ([10]). *An oriented graph is an oclique if and only if any two vertices are connected by a directed path of length 1 or 2.*

In their paper, Duffy et al. proved the following upper bound on maximum size of an oclique with maximum degree $\Delta$.

**Theorem 2** ([7]). *Every oclique with maximum degree $\Delta$ has at most $\left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor$ vertices.*

The theorem gives the upper bound 8 (resp. 13, 18) for $\Delta = 3$ (resp. $\Delta = 4$, $\Delta = 5$). They improved the above general result for $\Delta = 3$ by showing that the largest number of vertices in a subcubic oclique is 7. They also prove that there exists an oclique of size 11 with maximum degree $\Delta = 4$. Moreover they conjectured that the maximum order of an oclique with maximum degree $\Delta = 4$ is 11.

In this paper, we first improve the known upper bounds for graphs with low maximum degree. Secondly, we consider ocliques of maximum degree 4 and 5, and disprove the above-mentioned conjecture of Duffy et al. [7].

We prove in Section 4 that the oriented chromatic number of graphs with maximum degree 3 is at most 9 ($\chi_o(\mathcal{G}_3) \leqslant 9$), that is, we remove the condition of connectivity; see Theorem 8. In Section 5, we prove a general result which determines properties of a target graph to be universal for (non-necessarily connected) graphs of maximum degree $\Delta \geqslant 4$; see Theorem 10. As a consequence of this general result, we obtain that the oriented chromatic number of graphs with maximum degree 4 is at most 26 ($\chi_o(\mathcal{G}_4) \leqslant 26$), substantially decreasing the bound of 69 due to Duffy et al. [7]. We also get that $\chi_o(\mathcal{G}_5) \leqslant 90$, $\chi_o(\mathcal{G}_6) \leqslant 306$, and $\chi_o(\mathcal{G}_7) \leqslant 1322$.

In Section 6, we disprove the conjecture of Duffy et al. [7] by showing that the maximum order of an oclique maximum degree 4 equals 12 ($\chi_o(\mathcal{G}_4) \geqslant 12$). More precisely, we exhibit an oclique of order 12 and maximum degree 4, and show that there is no such oclique of order at least 13. Similarly in Section 7, we exhibit an oclique of order 16 and maximum degree 5 ($\chi_o(\mathcal{G}_5) \geqslant 16$).

The next two sections will be devoted to define the notation and to present the properties of the target graphs we use to prove our upper bounds.

## 2. Notation

In the remainder of this paper, we use the following notions. For a vertex $v$ of a graph $G$, we denote by $N_G^+(v)$ the set of outgoing neighbors of $v$, by $N_G^-(v)$ the set of incoming neighbors

of $v$ and by $N_G(v) = N_G^+(v) \cup N_G^-(v)$ the set of neighbors of $v$ (subscripts are omitted when the considered graph is clearly identified from the context). The *degree* of a vertex $v$ (resp. *in-degree*, *out-degree*), denoted by $d(v)$ (resp. $d^-(v)$, $d^+(v)$), is the number of its neighbors $|N(v)|$ (resp. incoming neighbors $|N^-(v)|$, outgoing neighbors $|N^+(v)|$). Let $\mathcal{G}_\Delta$ denote the family of oriented graphs with maximum degree $\Delta$. If two graphs $G$ and $H$ are isomorphic, we denote this by $G \cong H$.

## 3. Paley tournaments and Tromp digraphs

In this section, we describe the general construction of graphs that will be used to prove Theorems 8 and 10, and present some of their useful properties.

For a prime power $p \equiv 3 \pmod 4$, the *Paley tournament* $QR_p$ is defined as the graph whose vertices are the elements of the field $\mathbb{F}_p$ and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of $\mathbb{F}_p$. Clearly $QR_p$ is vertex- and arc-transitive.

An *orientation $n$-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{-1, 1\}^n$ of $n$ elements. Let $S = (v_1, v_2, \ldots, v_n)$ be a sequence of $n$ (not necessarily distinct) vertices of a graph $G$. The vertex $u$ is said to be an *$\alpha$-successor of $S$* if for any $i$, $1 \leqslant i \leqslant n$, we have $\overrightarrow{uv_i} \in A(G)$ whenever $\alpha_i = 1$ and $\overrightarrow{v_i u} \in A(G)$ otherwise. A sequence $S = (v_1, v_2, \ldots, v_n)$ of $n$ (not necessarily distinct) vertices of $QR_p$ is said to be *compatible* with an orientation $n$-vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if $\alpha_i = \alpha_j$ whenever $v_i = v_j$ since graphs do not contain opposite arcs. We say that a graph $G$ has *Property $P_{n,k}$* if, for every sequence $S$ of $n$ vertices of $G$ and any compatible orientation $n$-vector $\alpha$, there exist $k$ distinct $\alpha$-successors of $S$. Such Properties $P_{n,k}$ have been extensively used in many papers dealing with oriented coloring.

**Proposition 3.** *The Paley tournament $QR_p$ has Properties $P_{1, \frac{p-1}{2}}$ and $P_{2, \frac{p-3}{4}}$.*

*Proof.* By the vertex-transitivity of $QR_p$, the in-degree of every vertex is equal to its out-degree. This implies that $QR_p$ has Property $P_{1, \frac{p-1}{2}}$.

Let us prove that $QR_p$ has Property $P_{2, \frac{p-3}{4}}$. To do so, by arc-transitivity of $QR_p$, we just have to show that there exist at least $\frac{p-3}{4}$ $\alpha$-successors of the sequence $S = (0, 1)$ for any of the four orientation vector $\alpha \in \{-1, 1\}^2$.

We first need to count the transitive triangles with arcs $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ in $QR_p$. There are $p$ choices for the source vertex $x$ of a transitive triangle. The number of transitive triangles such that $x = 0$ is equal to the number of arcs in $N^+(0)$, that is, $\binom{(p-1)/2}{2} = \frac{(p-1)(p-3)}{8}$. Thus, $QR_p$ contains $\frac{p(p-1)(p-3)}{8}$ transitive triangles.

Considering $\alpha = (+1, +1)$, we can notice that $|N^+(0) \cap N^+(1)|$ is the number of transitive triangles such that $\overrightarrow{xy} = \overrightarrow{01}$. Since $QR_p$ is arc-transitive and contains $\frac{p(p-1)}{2}$ arcs, $|N^+(0) \cap N^+(1)| = \frac{p(p-1)(p-3)/8}{p(p-1)/2} = \frac{p-3}{4}$. Similarly for $\alpha = (-1, -1)$ and $\alpha = (+1, -1)$, considering $\overrightarrow{yz} = \overrightarrow{01}$ gives $|N^-(0) \cap N^-(1)| = \frac{p-3}{4}$ and considering $\overrightarrow{xz} = \overrightarrow{01}$ gives $|N^+(0) \cap N^-(1)| = \frac{p-3}{4}$. Finally for $\alpha = (-1, +1)$, we have $N^-(0) \cap N^+(1) = V(QR_p) \setminus \{0, 1, N^+(0) \cap N^+(1), N^-(0) \cap N^-(1), N^+(0) \cap N^-(1)\}$, so that $|N^-(0) \cap N^+(1)| = p - 2 - \frac{3(p-3)}{4} = \frac{p+1}{4} > \frac{p-3}{4}$. This proves $P_{2, \frac{p-3}{4}}$. $\qquad\square$

Paley tournaments will be used as basic brick to build new graphs as explained below. Tromp (unpublished manuscript) proposed the following construction. Let $G$ be a graph and let $G' \cong G$. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$

- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$

- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$

Figure 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between $u$ and $u'$. Such pairs of vertices will be called *anti-twin vertices*, and we denote by $at(u) = u'$ the anti-twin vertex of $u$.
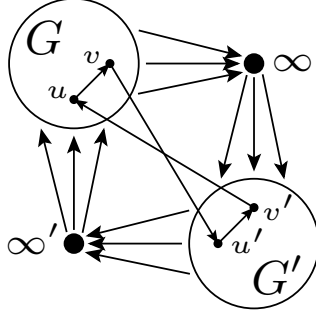
Figure 1: The Tromp graph $Tr(G)$.

In the following, we apply Tromp's construction to Paley tournaments $QR_p$ which produces graphs with interesting structural properties. First of all, Marshall [12] proved that any $Tr(QR_p)$ is *vertex-transitive* and *arc-transitive*. He also prove that any $Tr(QR_p)$ is *triangle-transitive*, meaning that, given two triangles $u_1u_2u_3$ and $v_1v_2v_3$ of $Tr(QR_p)$ with the same orientation, there exists an automorphism that maps $u_i$ to $v_i$. Secondly, it is possible to derive Properties $P_{n,k}$ for $Tr(QR_p)$ knowing those of $QR_p$ (see Proposition 4). Let us first define the notion of compatible sequence of vertices of $Tr(QR_p)$ with an orientation vector: A sequence $S = (v_1, v_2, \ldots, v_n)$ of $n$ (not necessarily distinct) vertices of $Tr(QR_p)$ is said to be *compatible* with an orientation $n$-vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if for any $i \neq j$, we have $\alpha_i \neq \alpha_j$ whenever $v_i = \text{at}(v_j)$, and $\alpha_i = \alpha_j$ whenever $v_i = v_j$. If the $n$ vertices of $S$ induce an $n$-clique subgraph of $Tr(QR_p)$ (*i.e.* $v_1, v_2, \ldots, v_n$ are pairwise distinct and induce a complete graph), then $S$ is compatible with any orientation $n$-vector since a vertex $u$ and its anti-twin $\text{at}(u)$ cannot belong together to the same clique. The authors already studied properties of $Tr(QR_{19})$ (see [16, Proposition 5]) and their results can be easily generalized to $Tr(QR_p)$:

**Proposition 4.** *[16] If $QR_p$ has Property $P_{n-1,k}$, then $Tr(QR_p)$ has Property $P_{n,k}$.*

Let us now introduce another type of properties. We will say that a graph $G$ has *Property $C_{n,k}$* if, given $n$ vertices $v_1, v_2, \ldots, v_n$ of $G$ that form a clique subgraph, we have $|\bigcup_{1 \leqslant i \leqslant n} N^+(v_i)| \geqslant k$ and $|\bigcup_{1 \leqslant i \leqslant n} N^-(v_i)| \geqslant k$.

**Remark 5.** *Given two integers $n$ and $k$, a graph having Property $C_{n,k}$ has Property $C_{n',k'}$ for any $n'$ and $k'$ such that $n' \geqslant n$ and $k' \leqslant k$.*

**Proposition 6.** *The graph $Tr(QR_p)$ has Properties $C_{2,\frac{3p+1}{2}}$ and $C_{3,\frac{7p+3}{4}}$.*

*Proof.* Recall that $Tr(QR_p)$ is built from two copies of $QR_p$, that will be named $QR_p$ and $QR'_p$ in the following (see Figure 1). In this proof, the in- and out-neighborhood of a vertex $v$ of $Tr(QR_p)$ will be denoted by $N^-(v)$ and $N^+(v)$, while the in- and out-neighborhood of a vertex $v$ in a subgraph $H$ of $Tr(QR_p)$ will be denoted by $N_H^-(v)$ and $N_H^+(v)$.

Note that, given a set of $n$ vertices $v_1, v_2, \ldots, v_n$ of $Tr(QR_p)$ that form a clique subgraph, if $z \in \bigcup_{1 \leqslant i \leqslant n} N^+(v_i)$ then $\text{at}(z) = z' \in \bigcup_{1 \leqslant i \leqslant n} N^-(v_i)$. Thus $|\bigcup_{1 \leqslant i \leqslant n} N^+(v_i)| = |\bigcup_{1 \leqslant i \leqslant n} N^-(v_i)|$.

- Let us first consider Property $C_{2,\frac{3p+1}{2}}$. We have to prove that given two adjacent vertices $x$ and $y$ of $Tr(QR_p)$, we have $|N^+(x) \cup N^+(y)| \geqslant \frac{3p+1}{2}$.

  Since $x$ and $y$ are adjacent, w.l.o.g. $x = 0$ and $y = \infty$ by arc-transitivity of $Tr(QR_p)$. Then $N^+(0) \cup N^+(\infty)$ contains:

  - $N^+(\infty) = \{0', 1', \ldots, (p-1)'\}$ ($p$ vertices);
  - Note that $N^+(0) = N_{QR_p}^+(0) \uplus N_{QR'_p}^+(0) \uplus \{\infty\}$. Since $N_{QR'}^+(0) \subset N^+(\infty)$ is already counted in the previous point, we just consider $N_{QR_p}^+(0)$ (at least $\frac{p-1}{2}$ vertices by Proposition 3) and $\infty$ (1 vertex);

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So that $N^+(\infty) \cup N^+(0)$ contains at least $p + \frac{p-1}{2} + 1 = \frac{3p+1}{2}$ vertices and thus $Tr(QR_p)$ has Property $C_{2, \frac{3p+1}{2}}$.

- Let us now consider Property $C_{3, \frac{7p+3}{4}}$. We have to prove that given three vertices $x$, $y$, and $z$ of $Tr(QR_p)$ that form a triangle, we have $|N^+(x) \cup N^+(y) \cup N^+(z)| \geqslant \frac{7p+3}{4}$.

  We have to consider two cases depending on whether $x, y, z$ form a transitive triangle or $x, y, z$ form a directed triangle. By triangle-transitivity of $Tr(QR_p)$, it suffices to consider the cases $x, y, z = 0, 1, \infty$ (transitive triangle) and $x, y, z = 0, 1', \infty$ (directed triangle).

  **Case $x, y, z = 0, 1, \infty$:** Let $A = N^+(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. We clearly have $|N^+(0) \cup N^+(1) \cup N^+(\infty)| = |N^+(0) \cup N^+(\infty)| + |A|$. Since we already know that $|N^+(0) \cup N^+(\infty)| = \frac{3p+1}{2}$ (see the previous point), let us focus on the set $A$. We have $N^+(1) = N^+_{QR_p}(1) \uplus N^+_{QR'_p}(1) \uplus \{\infty\}$. Since $N^+_{QR'_p}(1) \subset N^+(\infty)$ and $\{\infty\} \subset N^+(0)$, we have $A = N^+_{QR_p}(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. Since vertex $\infty$ has no out-neighbor in $QR_p$, we have $A = N^+_{QR_p}(1) \setminus N^+(0)$, that corresponds to the set of out-neighbors of 1 in $QR_p$ which are not out-neighbors of 0. Since $QR_p$ is a tournament, the vertices which are not out-neighbors of a given vertex are the in-neighbors of this vertex. Therefore, the set $A$ corresponds to the set of out-neighbors of 1 in $QR_p$ which are in-neighbors of 0 and thus $A = N^+_{QR_p}(1) \cap N^-(0)$. This set has already been considered in the proof of Proposition 3 where we showed that $|A| = \frac{p+1}{4}$.

  Therefore, the set $N^+(0) \cup N^+(1) \cup N^+(\infty)$ contains $|N^+(0) \cup N^+(\infty)| + |A| \geqslant \frac{3p+1}{2} + \frac{p+1}{4} = \frac{7p+3}{4}$ vertices.

  **Case $x, y, z = 0, 1', \infty$:** Note that $N^+(1') = N^-(1)$. Thus $N^+(0) \cup N^+(1') \cup N^+(\infty) = N^+(0) \cup N^-(1) \cup N^+(\infty)$. Using the same kind of arguments as previous case, we get that $|N^+(0) \cup N^-(1) \cup N^+(\infty)| \geqslant \frac{7p+3}{4}$.

$\square$

## 4. Upper bound of the oriented chromatic number of graphs with maximum degree 3

In this section, we consider graphs with maximum degree 3 and we prove that they all admit a homomorphism to the same target graph on nine vertices.

Duffy et al. [7] proved that every *connected* graph with maximum degree 3 has an oriented chromatic number at most 9. To achieve this bound, they use the Paley tournament $QR_7$ which has vertex set $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r \pmod 7$ for $r \in \{1, 2, 4\}$ (see Figure 2a). Here is a quick sketch of their proof. They first consider 2-degenerated graphs with maximum degree 3 (not necessarily connected) and prove the following:

**Theorem 7.** *[7] Every 2-degenerate graph with maximum degree 3 which does not contain a 3-source adjacent to a 3-sink is $QR_7$-colorable.*

Then, given a *connected* graph $G$ with maximum degree 3, they firstly consider the case where $G$ contains a 3-source. By removing all 3-sources from $G$, we obtain a graph $G'$ that is $QR_7$-colorable by Theorem 7. It is then easy to put back all the 3-sources and color them with a new color 7. Then, subsequently, they consider the case where $G$ does not contain 3-sources. Removing any arc $\overrightarrow{uv}$ from $G$ leads to a graph $G'$ which admits a $QR_7$-coloring $\varphi$ by Theorem 7. To extend $\varphi$ to $G$, it suffices to recolor $u$ and $v$ with two new colors so that $\varphi(u) = 7$ and $\varphi(v) = 8$. This gives that $G$ has an oriented chromatic number at most 9.

The condition of connectivity of $G$ is a necessary condition in their proof. Indeed, given a graph $G$ with maximum degree 3 which is not connected, we need to remove one arc $\overrightarrow{u_i v_i}$ from each 3-regular component $C_i$ of $G$ to get a 2-degenerate graph $G'$ which is $QR_7$-colorable by Theorem 7. However, to extend the coloring to $G$ using two new colors, say color 7 for the $u_i$'s and color 8 for the $v_i$'s, the colorings of each component must agree on the neighbors of each $u_i$'s and on the

neighbors of each $v_i$'s, which is not necessarily the case. This potentially leads to different target graphs on nine vertices for each component. Therefore, even if each component has an oriented chromatic number at most 9, the whole graph may have an oriented chromatic number strictly greater than 9.

In the following, we prove this is not the case by showing that it is possible to color each component with the same target graph $T_9$ on nine vertices whose construction is described below. This implies that the condition of connectivity is no more needed.
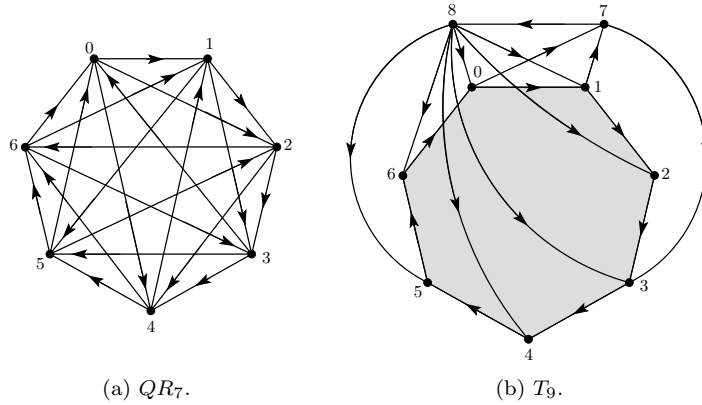


(a) $QR_7$.  (b) $T_9$.

Figure 2: The oriented graphs $QR_7$ and $T_9$.

The oriented graph $T_9$ is obtained from $QR_7$ (see Figure 2a) by adding two vertices labelled 7 and 8, and the arcs $\overrightarrow{07}$, $\overrightarrow{17}$, $\overrightarrow{73}$, $\overrightarrow{78}$, and $\overrightarrow{8i}$ for every $0 \leqslant i \leqslant 6$ (see Figure 2b where the grey part stands for $QR_7$).

We prove the following:

**Theorem 8.** *Every graph with maximum degree 3 admits a $T_9$-coloring and thus $\chi_o(\mathcal{G}_3) \leqslant 9$.*

*Proof.* It is sufficient to show that every connected graph $G$ with maximum degree 3 admits a $T_9$-coloring. We consider the following cases.

- We suppose that $G$ is 2-degenerate or $G$ contains a 3-source. Let $G'$ be the oriented graph obtained from $G$ by removing every 3-source. Since $G'$ is 2-degenerate and contains no 3-source, $G'$ admits a $QR_7$-coloring $\varphi$ by Theorem 7. We extend $\varphi$ to a $T_9$-coloring of $G$ by setting $\varphi(u) = 8$ for every 3-source $u$ of $G$ (indeed, the vertex 8 of $T_9$ dominates all the vertices of $QR_7$).
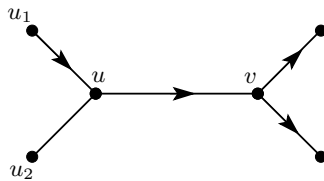


Figure 3: Configuration of Theorem 8.

- We suppose that $G$ is 3-regular and contains no 3-source. Notice that $G$ necessarily contains a vertex $v$ of out-degree two. Let $u$ denote the in-neighbor of $v$. Since $u$ is not a 3-source, it has an in-neighbor $u_1$. Let $u_2$ denote the neighbor of $u$ distinct from $u_1$ and $v$ (see Figure 3). We consider the graph $G'$ obtained from $G$ by removing the arc $\overrightarrow{uv}$. Since $G'$ is 2-degenerate and contains no 3-source, $G'$ admits a $QR_7$-coloring $\varphi$ by Theorem 7.

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- If $G$ (or equivalently $G'$) contains the arc $\overrightarrow{uu_2}$, then we necessarily have $\varphi(u_1) \neq \varphi(u_2)$. If $\overrightarrow{\varphi(u_1)\varphi(u_2)} \in A(QR_7)$ (resp. $\overrightarrow{\varphi(u_2)\varphi(u_1)} \in A(QR_7)$), we recolor $G'$ so that $\varphi(u_1) = 1$ (resp. $\varphi(u_1) = 0$) and $\varphi(u_2) = 3$ by the arc-transitivity of $QR_7$. It can be easily checked that we can extend $\varphi$ to a $T_9$-coloring of $G$ by setting $\varphi(u) = 7$ and $\varphi(v) = 8$.
- If $G$ contains $\overrightarrow{u_2 u}$, then by the arc-transitivity of $QR_7$, we can assume that $\{\varphi(u_1), \varphi(u_2)\} \subseteq \{0,1\}$. Again, $\varphi$ can be extended to $T_9$-coloring of $G$ by setting $\varphi(u) = 7$ and $\varphi(v) = 8$.

$\square$

## 5. Upper bound of the oriented chromatic number of graphs with maximum degree at least 4

In this section, we consider graphs with maximum degree at least 4.

Duffy et al. [7] recently proved that every *connected* graph with maximum degree 4 has an oriented chromatic number at most 69. To prove their result, they consider the case of 3-degenerate graphs with maximum degree 4 and prove that they admit a homomorphism to the Paley tournament $QR_{67}$ on 67 vertices; they then show how to extend such a 67-coloring to *connected* graphs with maximum degree 4 using two more colors, leading to a 69-coloring.

We propose general result which determines properties of a target graph to be universal for graphs of maximum degree $\Delta \geqslant 4$. As for graphs with maximum degree 3 (see Section 4), the condition of connectivity is not needed. As an example, our general result substantially decreases the bound of 69 colors for graphs with maximum 4 due to Duffy et al. [7] to 26 colors.

**Theorem 9.** *Every* $(\Delta - 1)$*-degenerate graph with maximum degree* $\Delta \geqslant 2$ *admits a $T$-coloring where $T$ is a graph on $n$ vertices with Properties $P_{\Delta-1, \Delta-2}$ and $C_{\Delta-2, \frac{n(\Delta-2)}{\Delta-1}+1}$.*

*Proof.* Let $G$ be a minimal counter-example to Theorem 9. By definition, $G$ contains a $k$-vertex $u$ with $k \leqslant \Delta - 1$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of $u$.

Suppose first that there exists one arc in the neighborhood of $u$, and w.l.o.g. assume that $\overrightarrow{v_1 v_2}$ is an arc of $G$. The graph $G'$ obtained from $G$ by removing the arc $uv_1$ admits a $T$-coloring $\varphi$ by minimality of $G$. Vertex $v_1$ has at most $\Delta - 1$ colored neighbors $w_1, \ldots, w_{\Delta-1}$ by $\varphi$ ($u$ is uncolored). Note that some $w_i$'s may coincide with some $v_i$'s. The sequence $S = (\varphi(w_1), \varphi(w_2), \ldots, \varphi(w_{\Delta-1}))$ is compatible w.r.t. the orientations of the arcs $w_i v_1$ since $v_1$ and every $w_i$ are colored in $G'$. We uncolor $v_1$ and by Property $P_{\Delta-1, \Delta-2}$ of $T$, we have $\Delta - 2$ available colors for $v_1$ among which at least one, say color $c$, is compatible with $\varphi(v_3), \varphi(v_4), \ldots, \varphi(v_k)$ w.r.t. to the orientation of the arcs $uv_1, uv_3, uv_4, \ldots, uv_k$. Set $\varphi(v_1) = c$. Note that we necessarily have $c \notin \{\varphi(v_2), \mathrm{at}(\varphi(v_2))\}$ since $\overrightarrow{v_1 v_2}$ is an arc of $G$ and thus $c$ is compatible with $\varphi(v_2)$. Note also that the sequence $S = (\varphi(v_2), \varphi(v_3), \ldots, \varphi(v_k))$ is compatible w.r.t. the orientations of the arcs $uv_i$, $2 \leqslant i \leqslant k$, since vertices $v_2, v_3, \ldots, v_k$ are colored in $G'$. Therefore, the sequence $S = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_k))$ is compatible with the orientation of the arcs $uv_i$, $1 \leqslant i \leqslant k$. By Property $P_{\Delta-1, \Delta-2}$, we have at least $\Delta - 2$ available colors for $u$. Therefore, $\varphi$ can be extended to a $T$-coloring of $G$, a contradiction.

Assume now that there is no arc in the neighborhood of $u$. The graph $G'$ obtained from $G$ by removing the vertex $u$ admits a $T$-coloring $\varphi$ by minimality of $G$. Let $w_i$, $1 \leqslant i \leqslant \Delta - 1$, be the neighbors of $v_1$ distinct from $u$. Since $v_1$ and every $w_i$ are colored in $G'$, the sequence $S = (\varphi(w_1), \varphi(w_2), \ldots, \varphi(w_{\Delta-1}))$ is compatible with the orientations of the arcs $vw_i$ and by Property $P_{\Delta-1, \Delta-2}$, we have $\Delta - 2$ available colors for $v_1$. Note that these $\Delta - 2$ colors (which are vertices of $T$) form a clique subgraph of $T$. The same holds for every $v_i$ (i.e. each $v_i$ has $\Delta - 2$ available colors) and since there is no arc in the neighborhood of $u$, each $v_i$ can be colored independently.

By Property $C_{\Delta-2, \frac{n(\Delta-2)}{\Delta-1}+1}$, given these $\Delta-2$ possible colors for $v_1$, there are at least $\frac{n(\Delta-2)}{\Delta-1}+1$ choices of colors for $u$; thus $v_1$ forbids at most $n - \left(\frac{n(\Delta-2)}{\Delta-1}+1\right) = \frac{n}{\Delta-1} - 1$ colors for $u$. By the same argument, each $v_i$ forbids at most $\frac{n}{\Delta-1} - 1$ colors for $u$. Therefore, $u$ has at most $k(\frac{n}{\Delta-1} - 1) \leqslant (\Delta-1)(\frac{n}{\Delta-1} - 1) = n - \Delta + 1 < n$ forbidden colors since $\Delta \geqslant 2$. This means that there are at least one available color for $u$. Therefore, $\varphi$ can be extended to a $T$-coloring of $G$, a contradiction. $\square$

To achieve our bounds on oriented chromatic number, we apply Tromp's construction to Paley tournaments $QR_p$. Moreover, given a graph $Tr(QR_p)$, we construct the graph $Tr^*(QR_p)$ on $2p+4$ vertices by adding two vertices $t_0$ and $t_1$ such that $t_0$ is a twin vertex of vertex 0 (*i.e.* a vertex with the same neighborhood as vertex 0) and $t_1$ is a twin vertex of vertex 1. We finally add the arc $\overrightarrow{t_1 t_0}$.

**Theorem 10.** *Every graph with maximum degree $\Delta \geqslant 4$ admits a $Tr^*(QR_p)$-coloring where $Tr(QR_p)$ is a Tromp graph (built from a Paley tournament $QR_p$) with Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,\frac{(2p+2)(\Delta-2)}{\Delta-1}+1}$.*

*Proof.* Since $Tr(QR_p)$ is a subgraph of $Tr^*(QR_p)$, it remains to prove that every connected $\Delta$-regular graph $H$ admits a $Tr^*(QR_p)$-coloring.

Let $H' = H \setminus \{\overrightarrow{uv}\}$ where $\overrightarrow{uv}$ is any arc of $H$. By Theorem 9, $H'$ admits a $Tr(QR_p)$-coloring $\varphi$. By vertex-transitivity of $Tr(QR_p)$, we may assume that $\varphi(v) = 0$. By Property $P_{\Delta-1,\Delta-2}$ of $Tr(QR_p)$, we have $\Delta - 2$ available colors $c_1, c_2, \ldots, c_{\Delta-2}$ for $u$. Note that, given $1 \leqslant i < j \leqslant \Delta - 2$, we necessarily have $c_i \neq at(c_j)$. We thus recolor $u$ with one of the $c_i$'s so that $\varphi(u) \notin \{0, 0'\}$ since $\Delta \geqslant 4$. Therefore, $\varphi(u)\varphi(v)$ is an arc of $Tr(QR_p)$.

If $\overrightarrow{\varphi(u)\varphi(v)}$ is an arc of $Tr(QR_p)$, then $\varphi$ is a $Tr(QR_p)$-coloring of $H$ and thus a $Tr^*(QR_p)$-coloring of $H$. Therefore, $\overrightarrow{\varphi(v)\varphi(u)}$ is an arc of $Tr(QR_p)$. By arc-transitivity of $Tr(QR_p)$, we may assume that $\varphi(u) = 1$. We recolor $u$ and $v$ so that $\varphi(u) = t_1$ and $\varphi(v) = t_0$. Since each $t_i$ is a twin vertex of vertex $i$ in $Tr^*(QR_p)$ and $\overrightarrow{t_1 t_0}$ is an arc $Tr^*(QR_p)$, it is easy to verify that $\varphi$ is now a $Tr^*(QR_p)$-coloring of $H$. $\qquad\square$

The Properties $P_{n,k}$ of Paley tournaments $QR_p$ can be expressed by a formula for $n \leqslant 2$ (see Proposition 4). For higher values of $n$, there are no formula and some properties are only known for small values of $p$. In our case, we are interested in properties of the form $P_{n,n}$. We computed these properties using a computer-check.

**Proposition 11.** *The smallest Paley tournament with Property $P_{2,2}$ (resp. $P_{3,3}$, $P_{4,4}$, $P_{5,5}$) is $QR_{11}$ (resp. $QR_{43}$, $QR_{151}$, $QR_{659}$).*

*Proof.* By computer. $\qquad\square$

Determining more of such properties would be possible with more computing power and time. Note that Paley tournament $QR_{151}$ has in fact Property $P_{5,6}$ and there exist no smaller Paley tournament with Property $P_{5,5}$.

As a corollary of Propositions 4 and 11, we get the following.

**Corollary 12.** *The Tromp graph $Tr(QR_{11})$ (resp. $Tr(QR_{43})$, $Tr(QR_{151})$, $Tr(QR_{659})$) has Property $P_{3,2}$ (resp. $P_{4,3}$, $P_{5,4}$, $P_{6,5}$).*

As corollaries of Remark 5, Proposition 6, Theorem 10, and Corollary 12, we obtain the following four results. We give a proof of the last one (Corollary 16), the other three corollaries follow the same arguments.

**Corollary 13.** *Every graph $G \in \mathcal{G}_4$ admits a $Tr^*(QR_{11})$-coloring. Thus, $\chi_o(\mathcal{G}_4) \leqslant 26$.*

**Corollary 14.** *Every graph $G \in \mathcal{G}_5$ admits a $Tr^*(QR_{43})$-coloring. Thus, $\chi_o(\mathcal{G}_5) \leqslant 90$.*

**Corollary 15.** *Every graph $G \in \mathcal{G}_6$ admits a $Tr^*(QR_{151})$-coloring. Thus, $\chi_o(\mathcal{G}_6) \leqslant 306$.*

**Corollary 16.** *Every graph $G \in \mathcal{G}_7$ admits a $Tr^*(QR_{659})$-coloring. Thus, $\chi_o(\mathcal{G}_7) \leqslant 1322$.*

*Proof.* Let $\Delta = 7$. By Proposition 6, the graph $Tr(QR_{659})$ has Property $C_{3,\frac{7p+3}{4}} = C_{3,1154}$. By Remark 5, it thus has Property $C_{5,1101} = C_{\Delta-2,\frac{(2p+2)(\Delta-2)}{\Delta-1}+1}$. By Corollary 12, it also has Property $P_{6,5} = P_{\Delta-1,\Delta-2}$. The graph $Tr(QR_{659})$ verifies the hypothesis of Theorem 10, and thus every graph with maximum degree 7 admits a $Tr^*(QR_{659})$-coloring. $\qquad\square$
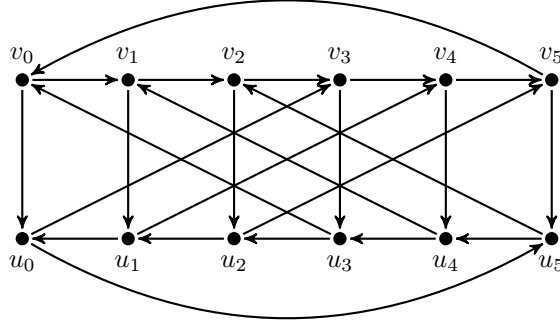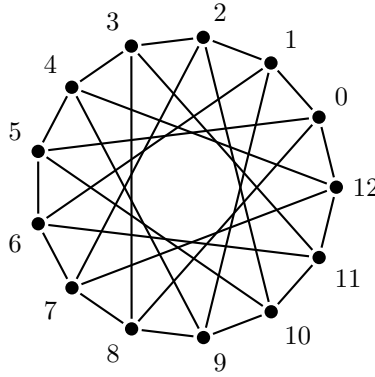
Figure 4: 4-regular oclique on 12 vertices.



Figure 5: Unique triangle-free 4-regular graph with 13 vertices and diameter 2.

## 6. Oclique with maximum degree 4

**Lemma 17.** *There is an oclique of order 12 with maximum degree $\Delta = 4$.*

*Proof.* Consider the oriented graph presented on Figure 4. The graph consists of two directed cycles: $(v_0, v_1, v_2, v_3, v_4, v_5)$ and $(u_5, u_4, u_3, u_2, u_1, u_0)$. These two cycles are connected by the arcs $\overrightarrow{v_i u_i}$ and $\overrightarrow{u_{(i+3) \bmod 6} v_i}$ for every $0 \leqslant i \leqslant 5$. The graph is vertex-transitive. By Theorem 1, it is enough to find a directed 2-path between $v_0$ and every vertex not adjacent to $v_0$:

$v_0 \to v_1 \to v_2$, $v_0 \to u_0 \to v_3$, $v_4 \to v_5 \to v_0$, $v_0 \to v_1 \to u_1$, $u_2 \to v_5 \to v_0$, $u_4 \to u_3 \to v_0$, $v_0 \to u_0 \to u_5$. □

**Lemma 18.** *If $G$ is an oclique with maximum degree 4 and order 13 then:*

(a) *every vertex has in-degree and out-degree equal to 2,*

(b) *every two vertices are connected by only one directed path of length at most 2.*

*Proof.* Each vertex $x \in V(G)$ is the midpoint of $d^-(x) \cdot d^+(x)$ directed paths of length 2, so we have

$$\sum_{x \in V(G)} d^-(x) \cdot d^+(x)$$

directed paths of length 2 in the graph. This sum achieves the maximum value of $4 \cdot 13 = 52$ if and only if $d^-(x) = d^+(x) = 2$ for every vertex $x$. The number of directed paths of length 1 is at most $\frac{4 \cdot 13}{2} = 26$. On the other hand we have $\binom{13}{2} = 52 + 26$ pairs of vertices. Hence, if every pair of vertices is connected by a path of length 1 or two, then every vertex has in-degree and out-degree equal to 2, and every two vertices are connected by only one directed path of length at most 2. □

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**Corollary 19.** *The underlying graph of an oclique on* 13 *vertices and maximum degree* 4 *is triangle-free.*

Let $G^{13}$ be the graph $(\mathbb{Z}_{13}, E)$, where $(u, v) \in E \iff v - u = a \pmod{13}$ for $a \in \{-1, 1, -5, 5\}$, see Figure 5. Meringer [13] describes a method to generate regular graphs of given girth. Using this generator, we can list the 4-regular graphs[1] on 13 vertices with girth at least 4 (triangle-free). Among them, only $G^{13}$ has diameter 2. Hence, an oclique on 13 vertices and maximum degree 4 must be an orientation of $G^{13}$.

**Lemma 20.** *No orientation of $G^{13}$ is an oclique.*

*Proof.* Suppose for a contradiction, that there is an orientation of $G^{13}$ which is an oclique. Vertex 0 is connected to vertex 2 only by one path of length at most 2, namely 0–1–2. Without loss of generality assume that those edges are oriented: $0 \to 1 \to 2$. Furthermore, there is only one path going from 1 to 3, namely the one going through vertex 2. Hence, there is the arc $2 \to 3$ and so on, we show that there is the cycle $0 \to 1 \to 2 \to ... \to 12 \to 0$. Edges which are not oriented so far also form the cycle $0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8$. By Lemma 18a, this cycle must be oriented in one direction. So, if $y = x + 5 \mod 13$, we have either every arc $\overrightarrow{xy}$ or every arc $\overrightarrow{yx}$. However, the orientation with the arcs $\overrightarrow{yx}$ is isomorphic to the orientation with the arcs $\overrightarrow{xy}$ via the mapping $x \mapsto 5x \mod 13$. So we only consider the orientation with the arcs $\overrightarrow{xy}$. Then we have two directed paths between 0 and 6, namely via 1 and via 5. This contradicts Lemma 18b. $\square$

Using Lemma 18 and Lemma 20 we have:

**Theorem 21.** *The maximum order of an oclique with maximum degree* 4 *is* 12.

**Corollary 22.** $\chi_o(\mathcal{G}_4) \geqslant 12.$

## 7. Oclique with maximum degree 5

**Lemma 23.** *There is an oclique with maximum degree* 5 *and order* 16.

*Proof.* Consider the graph presented on Fig. 6. The graph is constructed from the oriented 4-regular oclique $G$ on 12 vertices presented on Fig. 4 by adding four vertices $x_1, x_2, x_3, x_4$. These vertices form the cycle $x_1 \to x_4 \to x_3 \to x_2 \to x_1$ and there are arcs:

- $\overrightarrow{x_2 v_0}, \overrightarrow{x_2 v_2}, \overrightarrow{x_2 v_4},$
- $\overrightarrow{x_4 v_1}, \overrightarrow{x_4 v_3}, \overrightarrow{x_4 v_5},$
- $\overrightarrow{u_0 x_1}, \overrightarrow{u_2 x_1}, \overrightarrow{u_4 x_1},$
- $\overrightarrow{u_1 x_3}, \overrightarrow{u_3 x_3}, \overrightarrow{u_5 x_3}.$

We will show that any two vertices in $G$ are connected by a path of length at most 2. Since new vertices form the oriented cycle $C_4$, there are paths between these vertices. It is easy to find all other directed paths. For example, paths connecting $x_2$ and all other vertices are:

- $x_2 \to v_i$ for $i \in \{0, 2, 4\}$,
- $x_2 \to v_i \to v_{i+1 \mod 6}$ for $i \in \{1, 3, 5\}$,
- $x_2 \to v_i \to u_i$ for $i \in \{0, 2, 4\}$,
- $u_i \to x_3 \to x_2$ for $i \in \{1, 3, 5\}$.

$\square$

From Lemma 23 and Theorem 2 we have:

**Corollary 24.** *The maximum order of oclique with maximum degree* 5 *is between* 16 *and* 18.

**Corollary 25.** $\chi_o(\mathcal{G}_5) \geqslant 16.$

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[1] There are 31 such graphs which can be found at http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html
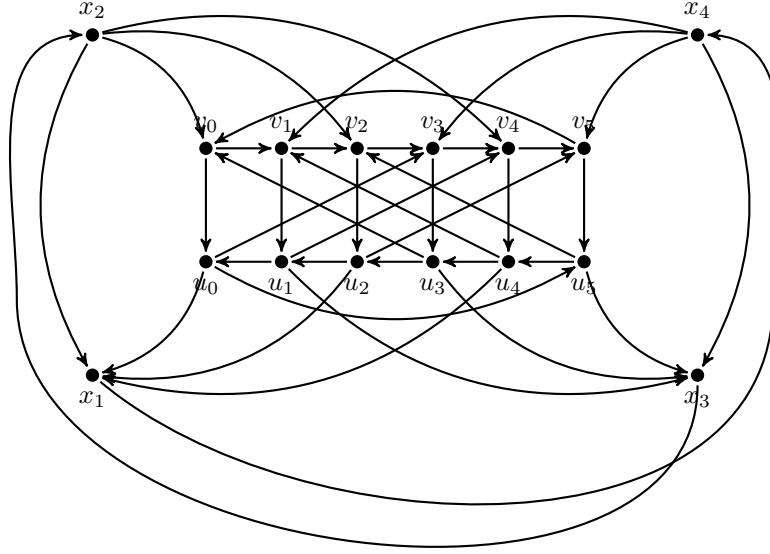
Figure 6: 5-regular oclique on 16 vertices.

## 8. Discussion

It is worth noticing that similarly as in Lemma 18, one can prove the following:

**Lemma 26.** *Suppose that $G$ is an oclique of maximum degree $\Delta \geqslant 2$ and order*

$$\left\lfloor \frac{(\Delta+1)^2 + 1}{2} \right\rfloor.$$

*Then:*

- *Every two vertices are connected by only one directed path of length at most 2.*

- *For every vertex $v$:*
    - *either $d^-(v) = \lfloor \frac{\Delta}{2} \rfloor$ and $d^+(v) = \lceil \frac{\Delta}{2} \rceil$,*
    - *or $d^+(v) = \lfloor \frac{\Delta}{2} \rfloor$ and $d^-(v) = \lceil \frac{\Delta}{2} \rceil$.*

We can also obtain a lower bound confirming that the maximum order of oclique is quadratic in terms of $\Delta$.

**Lemma 27.** *For every $\Delta$, there exists an oclique of maximum degree $\Delta$ and order $\frac{\Delta^2}{7} + O(\Delta)$.*

*Proof.* The postage stamp problem[2] asks, given integers $h$ and $k$, for a set of integers $V = \{v_1, v_2, \ldots, v_k\}$ that maximizes the smallest integer $PSP(h, k)$ that cannot be written as the sum of at most $h$ (not necessarily distinct) elements of $V$. In the context of ocliques, we are interested in the case of $h = 2$ stamps, which corresponds to directed paths of length at most 2. Using a set $V$ corresponding to the value of $PSP(2, k)$, we construct the circulant oriented graph $G_k$ on $n = 2PSP(2, k) - 1$ vertices $g_0, \ldots, g_{n-1}$ and the arcs $\overrightarrow{g_i g_{i+v_j}}$ such that $0 \leqslant i < n$, $1 \leqslant j \leqslant k$, and indices are taken modulo $n$. By the properties of $V$, there exists a directed path of length at most 2 in $G_k$ from $g_0$ to every vertex $g_i$ with $1 \leqslant i \leqslant PSP(2, k) - 1$. Now since $G_k$ is circular, $G_k$ is an oclique. Also, $G_k$ is $\Delta$-regular with $\Delta = 2k$. We use the bound $PSP(2, k) \geqslant \frac{2}{7}k^2 + O(k)$ [24] to obtain $|V(G_k)| = 2PSP(2, k) - 1 \geqslant 2 \times \frac{2}{7}k^2 + O(k) = \frac{\Delta^2}{7} + O(\Delta)$. $\qquad\square$

Finally, Duffy proves in a recent preprint [6] that the oriented chromatic number of a connected cubic graph is at most 8.

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## Acknowledgements

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