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ABSTRACT

Based on the algorithmic proof of Lovász local lemma due to Moser and Tardos, Dujmović et al. (2016) initiated the use of the so-called *entropy compression method* for graph coloring problems. Then, using the same approach Esperet and Parreau (2013) proved new upper bounds for several chromatic numbers, and explained how that approach could be used for many different coloring problems. Here, we follow this line of research for the particular case of acyclic coloring: we show that every graph with maximum degree $\Delta$ has acyclic chromatic number at most $\frac{3}{2}\Delta^{\frac{4}{3}} + O(\Delta)$.

## 1. Introduction

In the 70s, Lovász introduced the celebrated *Lovász Local Lemma* (LLL for short) to prove results on 3-chromatic hypergraphs [7]. It is a powerful probabilistic method to prove the existence of combinatorial objects satisfying a set of constraints. Since then, this lemma has been used in many occasions. In particular, it is a very efficient tool in graph coloring to provide upper bounds on several chromatic numbers [1,3,10,13,16,17,21,22]. In 2010, Moser and Tardos [23] designed an algorithmic version of LLL by means of the so-called *Entropy Compression Method*. This method seems to be applicable whenever LLL is, with the benefit of providing tighter bounds. Using ideas of Moser and Tardos [23], Grytczuk et al. [15] proposed new approaches in the old field of nonrepetitive sequences [26]. Inspired by these works, Dujmović et al. [6] gave a first application of the entropy compression method in the area of graph colorings (on Thue vertex coloring and some of its game variants). As the approach seems to be extendable to several graph coloring problems, Esperet and Parreau [8] applied that method to acyclic edge-coloring, star vertex-coloring, Thue vertex-coloring, each time improving the best known upper bound or giving very short proofs of known bounds. In their conclusion the authors explain how to apply that approach to any graph coloring problem that can be defined as a coloring where some configurations of colors are forbidden. In the continuity of these works, we prove new upper bounds on the acyclic chromatic number. Note that our contribution also serves as an example of how one should perform the calculations in the general framework described by Esperet and Parreau in order to achieve better bounds. Actually, this way to perform the calculations is similar to the one already used for Thue vertex coloring in [6].

Another contribution of this paper is to show how to improve the bounds provided by the general approach of Esperet and Parreau by using ideas of Dujmović et al.

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A *k-coloring* of a graph $G$ is a proper coloring of $G$ using $k$ colors; a graph admitting a $k$-coloring

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is said to be *k-colorable*. An *acyclic coloring* of a graph $G$ is a proper coloring of $G$ such that $G$ contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. Let $\chi_a(G)$, called the *acyclic chromatic number*, be the smallest integer $k$ such that the graph $G$ admits an acyclic $k$-coloring.

Acyclic coloring was introduced by Grünbaum [14]. In particular, he proved that if the maximum degree $\Delta$ of $G$ is at most 3, then $\chi_a(G) \leq 4$. Acyclic coloring of graphs with small maximum degree has been extensively studied [4,5,9,11,18,19,27–29] and the current knowledge is that graphs with maximum degree $\Delta \leq 4, 5$, and 6, respectively verify $\chi_a(G) \leq 5, 7$, and 11 [4,18,19]. For higher values of the maximum degree, Kostochka and Stocker [19] showed that $\chi_a(G) \leq 1 + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor$. Finally, for large values of the maximum degree, Alon, McDiarmid, and Reed [2] used LLL to prove that every graph with maximum degree $\Delta$ satisfies $\chi_a(G) \leq \lceil 50\Delta^{4/3} \rceil$. Moreover they proved that there exist graphs with maximum degree $\Delta$ for which $\chi_a = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{4/3}}\right)$. Recently, that upper bound was improved twice. It was first improved to $\left\lceil 6.59\Delta^{\frac{4}{3}} + 3.3\Delta \right\rceil$ by Ndreca et al. [24] by using a refined version of LLL, and then to $2.835\Delta^{\frac{4}{3}} + \Delta$ by Sereni and Volec [25] using the entropy compression method.

We improve that upper bound by a constant factor. Our improvement upon Sereni and Volec's bound lies both in our choice of "*Bad Events*" and in the way we performed the calculations.

**Theorem 1.** *Every graph $G$ with maximum degree $\Delta \geq 24$ is such that*

$$\chi_a(G) < \min\left\{\frac{3}{2}\Delta^{\frac{4}{3}} + 5\Delta - 14, \quad \frac{3}{2}\Delta^{\frac{4}{3}} + \Delta + \frac{8\Delta^{\frac{4}{3}}}{\Delta^{\frac{2}{3}} - 4} + 1\right\}.$$

Alon, McDiarmid, and Reed [2] also considered the acyclic chromatic number of graphs having no copy of $K_{2,\gamma+1}$ (the complete bipartite graph with partite sets of size 2 and $\gamma+1$) in which the two vertices in the first class are non-adjacent. Let $\mathcal{K}_\gamma$ be the family of such graphs. Such structure contains many cycles of length 4 which are obstructions to get an upper bound on the acyclic chromatic number linear in $\Delta$. Again using LLL, they proved that every graph $G \in \mathcal{K}_\gamma$ with maximum degree $\Delta$ satisfies $\chi_a(G) \leq \lceil 32\sqrt{\gamma}\Delta \rceil$.

Using similar techniques as for Theorem 1, we obtain:

**Theorem 2.** *Let $\gamma \geq 1$ be an integer and $G \in \mathcal{K}_\gamma$ with maximum degree $\Delta$. We have $\chi_a(G) \leq \left\lceil \Delta\left(1 + \sqrt{2\gamma + 4}\right) \right\rceil$.*

The paper is organized as follows. Section 2 is dedicated to the proof of Theorem 2 (the proof of Theorem 2 will serve as an educational example of the entropy compression method). Theorem 1 will be proved in Section 3.

## 2. Bounding $\chi_a(G)$ for $G \in \mathcal{K}_\gamma$

We prove Theorem 2 by contradiction. Suppose there exists a graph $G \in \mathcal{K}_\gamma$ with maximum degree $\Delta$ such that $\chi_a(G) > \left\lceil \Delta\left(1 + \sqrt{2\gamma + 4}\right) \right\rceil$. We define an algorithm that "tries" to acyclically color $G$ with $\kappa = \left\lceil \Delta\left(1 + \sqrt{2\gamma + 4}\right) \right\rceil$ colors. Let $\prec$ be any total order on the vertices of $G$.

### 2.1. The algorithm

Let $V \in \{1, 2, \ldots, \kappa\}^t$ be a vector of length $t$, for some arbitrarily large $t \gg n = |V(G)|$. Algorithm AcyclicColoringGamma_G (see below) takes the vector $V$ as input and returns a partial acyclic coloring $\varphi : V(G) \to \{\bullet, 1, 2, \ldots, \kappa\}$ of $G$ ($\bullet$ means that the vertex is uncolored) and a text file $R$ that is called a *record* in the remainder of the paper. The acyclic coloring $\varphi$ is necessarily partial since we try to color $G$ with a number of colors less than its acyclic chromatic number. For a given vertex $v$ of $G$, we denote by $N(v)$ the set of neighbors of $v$.

Algorithm AcyclicColoringGamma_G runs as follows. Let $\varphi_i$ be the partial coloring of $G$ after $i$ steps (at the end of the $i$th loop). At Step $i$, we first consider $\varphi_{i-1}$ and we color the smallest (w.r.t. $\prec$) uncolored vertex $v$ with $V[i]$ (line 6 of the algorithm). We then verify whether one of the following types of bad events happens:

Event 1: $G$ contains a monochromatic edge $vu$ for some $u$ (line 8 of the algorithm);
Event $k$: $G$ contains a bicolored cycle of length $2k$ ($v = u_1, u_2, \ldots, u_{2k}$) (line 11 of the algorithm).

If such events happen, then we uncolor some vertices (including $v$) in order that none of these events remains. Since Event 1 is avoided, $\varphi_i$ is a proper coloring and since Event $k$ is avoided, $\varphi_i$ is acyclic. Thus, $\varphi_i$ is clearly a partial acyclic coloring of $G$.

**Proof of Theorem 2.** From each output $(\varphi, R)$ of Algorithm AcyclicColoringGamma_G, one can determine the unique corresponding input $V$ by Lemma 3 (Section 2.2). Thus, the function defined by the algorithm, that maps $V$ to $(\varphi, R)$, is injective. The number of possible inputs $V$ is exactly $\kappa^t$. There are at most $(1+\kappa)^n$ possible partial $\kappa$-colorings $\varphi$ of $G$ (each

**Algorithm 1:** AcyclicColoringGamma_G

**Input** : $V$ (vector of length $t$).
**Output**: $(\varphi, R)$.

**for** *all* $v$ *in* $V(G)$ **do**
  $\varphi(v) \leftarrow \bullet$
$R \leftarrow$ *newfile*()
**for** $i \leftarrow 1$ **to** $t$ **do**
  Let $v$ be the smallest (w.r.t. $\prec$) uncolored vertex of $G$
  $\varphi(v) \leftarrow V[i]$
  Write "`Color \n`" in $R$
  **if** $\varphi(v) = \varphi(u)$ for $u \in N(v)$ **then**
    `// Proper coloring issue`
    $\varphi(v) \leftarrow \bullet$
    Write "`Uncolor, neighbor` $u$ `\n`" in $R$
  **else if** $v$ *belongs to a bicolored cycle of length* $2k$ *($k \geq 2$), say* $(v = u_1, \ldots, u_{2k})$ **then**
    `// Bicolored cycle issue`
    **for** $j \leftarrow 1$ **to** $2k-2$ **do**
      $\varphi(u_j) \leftarrow \bullet$
    Write "`Uncolor,` $2k$`-cycle` $(v = u_1, \ldots, u_{2k})$ `\n`" in $R$
**return** $(\varphi, R)$

vertex is either uncolored or colored with one color from $\{1, 2, \ldots, \kappa\}$) and there are at most $o\left(\kappa^t\right)$ possible records $R$ by Lemma 8 (Section 2.4). Thus, the number of possible outputs $(\varphi, R)$ is $o\left(\kappa^t\right)$, as it is at most $(1+\kappa)^n \times o\left(\kappa^t\right)$ and $t \gg n$. We therefore obtain a contradiction by showing that the number of possible outputs $o\left(\kappa^t\right)$ is strictly smaller than the number of possible inputs $\kappa^t$ when $t$ is chosen large enough. Therefore, assuming the existence of a counterexample $G$ leads us to a contradiction. That concludes the proof of Theorem 2. □

### 2.2. One-to-one correspondence

Recall that $\varphi_i$ denotes the partial acyclic coloring obtained after $i$ steps. Let us denote by $\overline{\varphi}_i \subset V(G)$ the set of vertices that are colored in $\varphi_i$. Let also $v_i$, $R_i$, and $V_i$ respectively denote the current vertex $v$ of the $i$th step, the record $R$ after $i$ steps, and the input vector $V$ restricted to its $i$ first elements. Observe that as $\varphi_i$ is a partial acyclic $\kappa$-coloring of $G$ and as $G$ is not acyclically $\kappa$-colorable, we have that $\overline{\varphi}_i \subsetneq V(G)$ and thus $v_{i+1}$ is well defined. That also implies that $R$ has $t$ "`Color`" lines. Finally observe that $R_i$ corresponds to the lines of $R$ before the $(i+1)$th "`Color`" line.

**Lemma 3.** *Given $(\varphi_i, R_i)$, one can recover the unique corresponding $V_i$.*

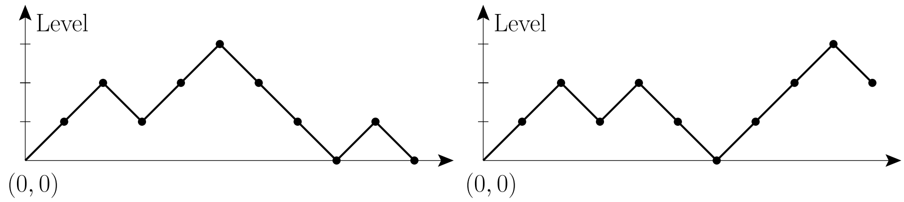
**Proof.** By induction on $i$. Trivially, $V_0$ (which is empty) can be recovered from $(\varphi_0, R_0)$. Consider now $(\varphi_i, R_i)$ and let us try to recover $V_i$. By induction, it is sufficient to recover $R_{i-1}$, $\varphi_{i-1}$, and $V[i]$. As observed before, to recover $R_{i-1}$ from $R_i$ it is sufficient to consider the lines before the last (i.e. the $i$th) "`Color`" line. Then reading $R_{i-1}$, one can easily recover $\overline{\varphi}_{i-1}$ and deduce $v_i$. Note that in the $i$th step we wrote one or two lines in the record: exactly one "`Color`" line followed by either nothing, or one "`Uncolor, neighbor`" line, or one "`Uncolor,`" $2k$"`-cycle`" line (there cannot be an "`Uncolor,`" $2k$"`-cycle`" line following an "`Uncolor, neighbor`" line, as $v$ would be uncolored by the algorithm before considering bicolored cycles passing through $v$). Let us consider these three cases separately.

- If Step $i$ was a color step alone, then $V[i] = \varphi_i(v_i)$ and $\varphi_{i-1}$ is obtained from $\varphi_i$ by uncoloring $v_i$.
- If the last line of $R_i$ is "`Uncolor, neighbor`" $u$, then $V[i] = \varphi_i(u)$ and $\varphi_{i-1} = \varphi_i$.
- If the last line of $R_i$ is "`Uncolor,`" $2k$"`-cycle`" $(u_1, \ldots, u_{2k})$, then $V[i] = \varphi_i(u_{2k-1})$ and $\varphi_{i-1}$ is obtained from $\varphi_i$ by coloring the vertices $u_j$ for $2 \leq j \leq 2k-2$ (which were uncolored in $\varphi_i$), in such a way that $\varphi_{i-1}(u_j)$ equals $\varphi_i(u_{2k-1})$ if $j$ is odd, or equals $\varphi_i(u_{2k})$ otherwise. Note that this is possible because in the $i$th loop, the algorithm uncolored neither $u_{2k-1}$ nor $u_{2k}$.

That concludes the proof of the lemma. □

### 2.3. Records and annotated Dyck words

Since Algorithm AcyclicColoringGamma_G fails to color $G$, the record $R$ has exactly $t$ "`Color`" lines (i.e. the algorithm consumes the whole input vector). It contains also "`Uncolor`" lines of different types: "`neighbor`" (type 1), 4"`-cycle`"




(a) A Dyck path. (b) A generalized Dyck path.

**Fig. 1.** Examples of (generalized) Dyck paths.

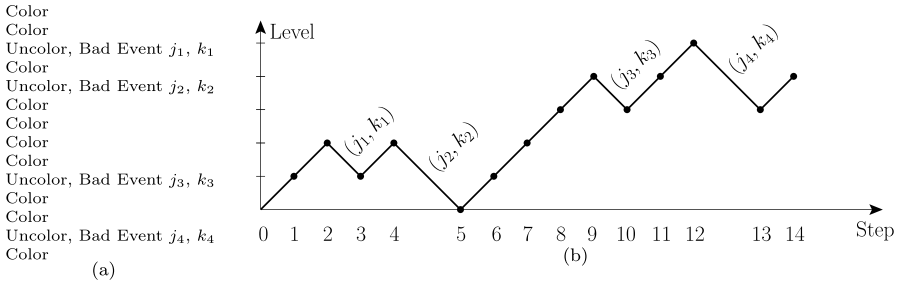



**Fig. 2.** (a) A record and (b) its corresponding annotated generalized Dyck path.

(type 2), 6"-cycle" (type 3), ..., $n$"-cycle" (type $\frac{n}{2}$). Let $\mathcal{T} = \left\{1, 2, \ldots, \frac{n}{2}\right\}$ be the set of bad event types. Denote by $s_j$ the number of previously colored vertices that are uncolored when a bad event of type $j$ occurs. Observe that:

- For every "Uncolor, neighbor" step, the algorithm uncolors 1 previously colored vertex. Hence set $s_1 = 1$.
- For every "Uncolor," $2k$"-cycle" step, where the cycle has length $2k$, the algorithm uncolors $2k - 2$ previously colored vertices. Hence set $s_k = 2k - 2$ for $2 \le k \le \lfloor n/2 \rfloor$.

To compute the total number of possible records, let us compute how many different entries, denoted $C_j$, an "Uncolor" step of type $j$ can produce in the record. Observe that:

- An "Uncolor, neighbor" line can produce $\Delta$ different entries in the record, according to a neighbor of $v$ (the vertex just colored by the algorithm) that shares the same color. Hence set $C_1 = \Delta$.
- An "Uncolor," $2k$"-cycle" line involving a cycle of length $2k$ can produce as many different entries in the record as the number of $2k$-cycles going through $v$. That number of entries is at most $\frac{1}{2}\gamma\Delta^{2k-2}$ according to Lemma 3.2 of [2]. Hence set $C_k = \frac{1}{2}\gamma\Delta^{2k-2}$ for $2 \le k \le \lfloor n/2 \rfloor$.

Hence, a record $R$ is a sequence of "Color" and "Uncolor" lines, where each "Uncolor" line is associated to an occurrence of a bad event of type $j$ ($j \in \mathcal{T}$). As there are at most $C_j$ such bad events, one can number them from 1 to $C_j$. We will refer to $(j, k)$-*event* for a bad event of type $j$ with the $k$th entry ($j \in \mathcal{T}$, $k \in [\![1, C_j]\!]$).

A *Dyck path* is defined as a lattice path consisting of up-steps $\nearrow$ and down-steps $\searrow$ which start at $(0, 0)$, ends on the $x$-axis and does not go below the $x$-axis. In the following, we will also consider *generalized* Dyck paths these are subpaths of Dyck path that start at $(0, 0)$ but not necessarily ending on the $x$-axis. The *level* of a generalized Dyck path is defined as the number of up-steps minus the number of down-steps. Note that a generalized Dyck path of level 0 is a Dyck path. The *size* of a generalized Dyck path is its number of up-steps. An example of a Dyck path of size five is given in Fig. 1(a) and an example of a generalized Dyck path of size six and level two is given in Fig. 1(b).

Observe that a record $R$ can be seen as a generalized Dyck path where:

- each up-step corresponds to a "Color" line;
- each descent (maximal sequence of consecutive down-steps) of length $\ell$ is annotated with a couple $(j, k)$ and corresponds to a $(j, k)$-event where $\ell = s_j$, that is the number of vertices uncolored after a bad event of type $j$.

Such a generalized Dyck path is said to be *annotated* with parameters $((s_j, C_j))_{j\in\mathcal{T}}$ for a set $\mathcal{T}$. Observe Fig. 2 which gives an example of such an annotated generalized Dyck path where $s_{j_1} = 1$, $s_{j_2} = 2$, $s_{j_3} = 1$, and $s_{j_4} = 2$. From now on, the term *record* refers to both a record produced by Algorithm AcyclicColoringGamma_G and its corresponding annotated generalized Dyck path.

At a given step, it is clear that the level of the record corresponds to the number of colored vertices in $G$ (for example, at Step 8 of Fig. 2, the graph $G$ has 3 colored vertices). Thus the ending level of the record should be between 0 and $n$. The following theorem allows us to bound the number of annotated Dyck paths.

**Theorem 4.** *The number of annotated generalized Dyck paths of size $t$ with parameters$((s_j, C_j))_{j\in\mathcal{T}}$ for any set $\mathcal{T}$ is at most $o\left(\left(\frac{Q(x)}{x}\right)^t\right)$ where $Q(x) = 1 + \sum_{j\in\mathcal{T}} C_j x^{s_j}$ for any $x \in ]0, 1]$.*

In practice, our aim is to minimize the value of $\frac{Q(x)}{x}$. Observe that:

**Remark 5.** In Theorem 4, the minimum value of $\frac{Q(x)}{x}$ is as follows:

- If $s_j = 1$ for all $j \in \mathcal{T}$, then the minimum is reached for $x = 1$ and $\dfrac{Q(x)}{x} = 1 + \sum\limits_{j\in\mathcal{T}} C_j$.
- Otherwise, the minimum is reached for the unique positive root of the polynomial $P(x) = -1 + \sum\limits_{j\in\mathcal{T}} (s_j - 1)C_j x^{s_j}$.

To prove Theorem 4, we will make use of the *smooth implicit-function schema*[1] (SIFS for short) of Meir and Moon [20] (see also Flajolet and Sedgewick's book [12, Section VII.4.1]). This *schema*, presented in Definition 6, defines a set of analytic functions whose Taylor coefficients can be finely approximated.

**Definition 6** (*Smooth Implicit-Function Schema [12, Definition VII.4, p. 467]*)**.** Let $A(y)$ be a function analytic at 0, $A(y) = \sum_{t\geq 0} a_t y^t$, with $a_0 = 0$ and $a_t \geq 0$. The function is said to belong to the *smooth implicit-function schema* if there exists a bivariate function $G(y, z)$ such that $A(y) = G(y, A(y))$, where $G(y, z)$ satisfy the following conditions:

(a) $G(y, z) = \sum_{m,n\geq 0} g_{m,n} y^m z^n$ is analytic in a domain $|y| < R$ and $|z| < S$, for some $R, S > 0$.
(b) The coefficients of $G$ satisfy

$$g_{m,n} \geq 0,\ g_{0,0} = 0,\ g_{0,1} \neq 1,$$
$$g_{m,n} > 0 \text{ for some } m \geq 0 \text{ and some } n \geq 2.$$

(c) There exist two numbers $r$ and $s$, such that $0 < r < R$ and $0 < s < S$, satisfying the system of equations[2]

$$G(r, s) = s,\ G_z(r, s) = 1, \quad \text{with } r < R,\ s < S$$

which is called the *characteristic system*.

**Theorem 7** (*Meir and Moon [20], [12, Theorem VII.3, p. 468]*)**.** *Let $A(y)$ belong to the smooth implicit-function schema defined by $G(y, z)$ with $(r, s)$ the positive solution of the characteristic system. Then, $A(y)$ converges at $y = r$, where it has a square-root singularity,*

$$\lim_{y\to r} A(y) = s - \gamma\sqrt{1 - \frac{y}{r}} + O\left(1 - \frac{y}{r}\right), \quad \textit{with} \ \ \gamma = \sqrt{\frac{2rG_y(r, s)}{G_{zz}(r, s)}},$$

*the expansion being valid in a $\Delta$-domain. In addition, if $A(y)$ is aperiodic, then $r$ is the unique dominant singularity of $A$ and the coefficients satisfy*

$$\lim_{t\to\infty} [y^t]A(y) = \frac{\gamma}{2\sqrt{\pi t^3}} r^{-t}\left(1 + O(t^{-1})\right).$$

**Proof of Theorem 4.** Let $\mathcal{R}$ be the set of annotated generalized Dyck paths with parameters $((s_j, C_j))_{j\in\mathcal{T}}$ and let $\mathcal{B} \subseteq \mathcal{R}$ be the subclass of annotated Dyck paths ending at level 0. In the following, those are just called *annotated Dyck paths* as they are not generalized.

Let us define the generating functions of $\mathcal{R}$ and $\mathcal{B}$ as $R(y) = \sum_{t\geq 0} r_t y^t$ and $B(y) = \sum_{t\geq 0} b_t y^t$ where $r_t$ (resp. $b_t$) denotes the number of elements of size $t$ in $\mathcal{R}$ (resp. $\mathcal{B}$). Our aim is to prove that $r_t = o(\lambda^t)$ for $\lambda = \min\limits_{0<x\leq 1} \dfrac{Q(x)}{x}$.

Let $\mathcal{R}_\ell \subseteq \mathcal{R}$ be the set of annotated generalized Dyck paths of $\mathcal{R}$ ending at level $\ell$. An annotated generalized Dyck path $R \in \mathcal{R}_\ell$ can be split into $\ell$ up-steps (which correspond to the last up-steps between level $i$ and $i + 1$, for each $0 \leq i \leq \ell - 1$) and $\ell + 1$ annotated Dyck paths $\{B_1, B_2, \ldots, B_{\ell+1}\} \subseteq \mathcal{B}$ (see Fig. 3). Hence, the generating function of $\mathcal{R}_\ell$ satisfies $R_\ell(y) = y^\ell B(y)^{\ell+1}$. Therefore,

$$R(y) = \sum_{0\leq\ell\leq n} R_\ell(y) = \sum_{0\leq\ell\leq n} y^\ell B(y)^{\ell+1} \tag{1}$$

[1] The Proposition IV.5 of Flajolet–Sedgewick's book [12], used for example by Dujmović et al. [6], is not sufficient here as it would only provide us the radius of convergence of $R(y)$. This is not sufficient to deduce the $o(\ )$ of our theorem. Theorem VI.6 of [12] is also not applicable here as we do not have $B(y) = y\phi(B(y))$ for some $\phi$, and as $B(y)$ can be periodic.

[2] $G_y$ (resp. $G_z$) denotes the derivative of $G$ with respect to its first (resp. second) variable.

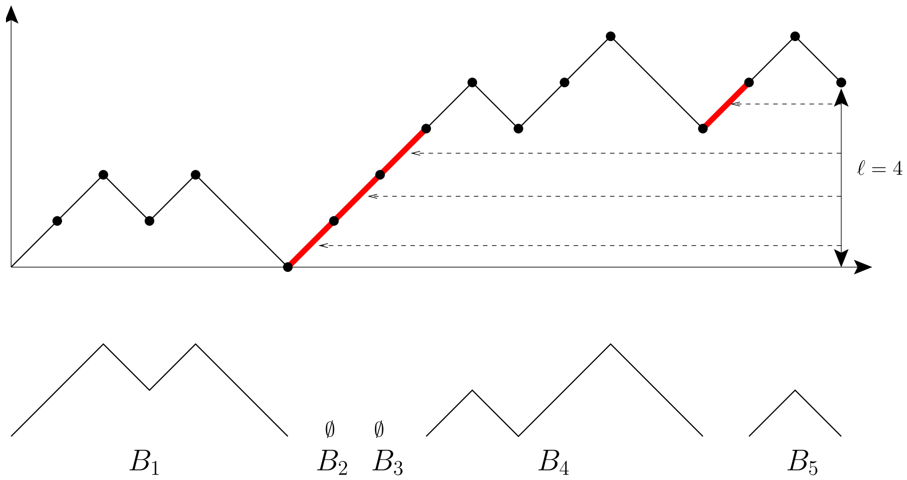



**Fig. 3.** Splitting a generalized Dyck path of level $\ell$ into $\ell+1$ Dyck paths $\{B_1, B_2, \ldots, B_{\ell+1}\}$ and $\ell$ up-steps (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let $\mathcal{B}_j \subseteq \mathcal{B}$ be the set of annotated Dyck paths of $\mathcal{B}$ ending with a descent annotated $(j, k)$ for some $k$ (note that $k$ may take $C_j$ distinct possible values by definition). Therefore, an annotated Dyck path $R \in \mathcal{B}_j$ ends with a last up-step and a last descent of length $s_j$. The subpath $R'$ obtained from $R$ by removing the last up-step and the last descent belongs to $\mathcal{R}_{s_j-1}$. Hence, the generating function of $\mathcal{B}_j$ is $B_j(y) = R_{s_j-1}(y) \times yC_j = y^{s_j} C_j B(y)^{s_j}$. Therefore, since an annotated Dyck path $R \in \mathcal{B}$ is either empty (i.e. of size 0) or ends with a descent annotated $(j, k)$, we have:

$$B(y) = 1 + \sum_{j\in\mathcal{T}} B_j(y) = 1 + \sum_{j\in\mathcal{T}} y^{s_j} C_j B(y)^{s_j} \tag{2}$$

If $s_j = 1$ for all $j \in \mathcal{T}$, we have $\frac{Q(x)}{x} = \frac{1}{x} + \sum_{j\in\mathcal{T}} C_j$ by definition. In that case, $\lambda = 1 + \sum_{j\in\mathcal{T}} C_j$ (the minimum is reached for $x = 1$). Moreover, $b_t = \left(\sum_{j\in\mathcal{T}} C_j\right)^t$ by Eq. (2). Thus, $b_t = (\lambda - 1)^t$. It follows that $r_t = \sum_{0\le\ell\le n} \binom{t}{\ell} (\lambda-1)^{t-\ell}$ by Eq. (1) (observe that $[y^i]B(y)^j = \binom{i+j-1}{j-i}(\lambda-1)^i$, where $[y^i]B(y)^j$ stands for the coefficient of the $y^i$ monomial in $B(y)^j$). Finally, $r_t < (n+1)t^{n+1} (\lambda-1)^t$ and therefore $r_t = o(\lambda^t)$.

From now on, we consider the case where $s_j \ge 2$ for some $j \in \mathcal{T}$. As observed by Esperet and Parreau [8, Lemma 6], there is a constant $C$ (depending only on the lengths of the descents) such that $r_t \le b_{t+C}$. It suffices hence to show that $b_t = o(\lambda^t)$.

Function $B(y)$ does not satisfy the SIFS (see Definition 6) and we thus introduce the function $A(y)$ defined by $A(y) = B(y^{\frac{1}{d}}) - 1$ where $d = \gcd\{s_j \mid j \in \mathcal{T}\}$. We prove in the following that $A(y)$ satisfies the SIFS. Note that the size of any annotated Dyck path of $\mathcal{B}$ is a multiple of $d$. Therefore, we have:

$$B(y) = \sum_{t\ge0} b_t y^t = \sum_{t \text{ multiple of } d} b_t y^t.$$

Thus $B(y^{\frac{1}{d}}) = 1 + \sum_{t\ge1} b_{dt} y^t$. Hence $A(y) = \sum_{t\ge0} a_t y^t$ with $a_0 = 0$ and $a_t = b_{dt}$ for $t \ge 1$. Thus $A(y)$ is analytic at 0, $a_0 = 0$, and $a_t \ge 0$ for all $t \ge 0$. Furthermore, note that for any sufficiently large $t$, the integer $dt$ can be written as a sum which summands belong to $\{s_j \mid j \in \mathcal{T}\}$. Hence $a_t = b_{dt} > 0$ for any sufficiently large $t$. It follows that $A(y)$ is aperiodic.[3] By Eq. (2), we have $A(y) = G(y, A(y))$ for the bivariate function $G$ defined by

$$G(y,z) = \sum_{j\in\mathcal{T}} C_j y^{s_j/d} (z+1)^{s_j}.$$

Observe that

$$G(y,z) = \sum_{j\in\mathcal{T}} \sum_{0\le i\le s_j} \binom{s_j}{i} C_j y^{s_j/d} z^i,$$

and hence $G(y, z)$ is a bivariate power series satisfying the following conditions:

[3] *Aperiodic* is used in the usual sense of Definition IV.5 of Flajolet–Sedgewick's book [12]. Equivalently, there exist three indices $i < j < k$ such that $a_i a_j a_k \neq 0$ and $\gcd(j-i, k-i) = 1$.

($a$) $G(y,z)$ is analytic in the domain $|y| < +\infty$ and $|z| < +\infty$.
($b$) Setting $G(y,z) = \sum_{m,n\geq 0} g_{m,n} y^m z^n$, the coefficients of $G$ satisfy $g_{m,n} \geq 0$, $g_{0,0} = 0$, $g_{0,1} = 0$, and $g_{\frac{s_j}{d}, s_j} > 0$ for the $j \in \mathcal{T}$ such that $s_j \geq 2$.
($c$) There exist two positive numbers $r$ and $s$ satisfying the system of equations[4]

$$G(r,s) = s \;\text{ and }\; G_z(r,s) = 1.$$

Indeed, by setting $X = r^{1/d}(s+1)$, these two equations respectively become

$$\sum_{j\in\mathcal{T}} C_j X^{s_j} = s \;\text{ and }\; \sum_{j\in\mathcal{T}} s_j C_j X^{s_j} = s+1.$$

By subtracting the first one from the second one, we obtain that $X$ is the unique positive root of $P(x)$ (see Remark 5). The first equation hence clearly defines $s$. In this first equation, adding 1 to both sides, and then multiplying them both by $r^{1/d}$, one obtains that $r = \left(\frac{X}{Q(X)}\right)^d$.

Hence $A(y) = \sum_{t\geq 0} a_t y^t$ satisfies a smooth implicit-function schema with characteristic system $(r,s)$. By Theorem 7, we have that $a_t = O\left(t^{-\frac{3}{2}} r^{-t}\right)$. It follows that $a_t = o\left(r^{-t}\right)$ and $b_t = o\left(r^{-t/d}\right) = o\left(\left(\frac{Q(X)}{X}\right)^t\right)$. As $X$ is the unique positive root of $P(x)$, that completes the proof. □

### 2.4. Number of records

Let us now bound the number of possible records.

**Lemma 8.** *Algorithm* AcyclicColoringGamma_G *produces at most* $o\left(\kappa^t\right)$ *distinct records R.*

**Proof.** By Theorem 4, AcyclicColoringGamma_G produces at most $o\left(\left(\frac{Q(x)}{x}\right)^t\right)$ possible records, where $Q(x) = 1 + \sum_{j\in\mathcal{T}} C_j x^{s_j}$ and $x \in ]0,1]$, with the following parameters: $\mathcal{T} = \left\{1, 2, \dots, \frac{n}{2}\right\}$, $C_1 = \Delta$, $s_1 = 1$, $C_k = \frac{1}{2}\gamma\Delta^{2k-2}$ and $s_k = 2k-2$ for $2 \leq k \leq 2k-2$. We thus have:

$$Q(x) = 1 + \Delta x + \sum_{2\leq i\leq \frac{n}{2}} \frac{1}{2}\gamma\Delta^{2i-2}x^{2i-2} < 1 + \Delta x + \frac{\gamma\Delta^2x^2}{2-2\Delta^2x^2} \qquad \text{for } x < \frac{1}{\Delta}$$

Setting $X = \frac{1}{\Delta}\sqrt{\frac{2}{\gamma+2}}$, we have:

$$\frac{Q(X)}{X} < \Delta\sqrt{\frac{\gamma+2}{2}}\left(1 + \sqrt{\frac{2}{\gamma+2}} + 1\right) = \Delta\left(1 + \sqrt{2\gamma+4}\right) \leq \kappa$$

Since $\gamma \geq 1$, then $\frac{2}{\gamma+2} < 1$ and thus we have $0 < X < \frac{1}{\Delta} \leq 1$. Therefore, Algorithm AcyclicColoringGamma_G produces at most $o(\kappa^t)$ different records. That completes the proof. □

## 3. Bounding $\chi_a(G)$ for any $G$

To prove Theorem 1, we prove that, given a graph $G$ with maximum degree $\Delta$, we have $\chi_a(G) < \frac{3}{2}\Delta^{\frac{4}{3}} + 5\Delta - 14$ for $\Delta \geq 24$ in Section 3.1 and that $\chi_a(G) < \frac{3}{2}\Delta^{\frac{4}{3}} + \Delta + \frac{8\Delta^{\frac{4}{3}}}{\Delta^{\frac{2}{3}}-4} + 1$ for $\Delta \geq 9$ in Section 3.2.

The proof is made by contradiction. Suppose there exists a graph $G$ with maximum degree $\Delta$ which is a counterexample to Theorem 1. Define a total order $\prec$ on the vertices of $G$. Let $N(u)$ and $N^2(u)$ be respectively the set of neighbors and distance-two vertices of $u$. For each pair of non-adjacent vertices $u$ and $v$, let $N(u,v) = N(u) \cap N(v)$, and let $\deg(u,v) = |N(u,v)|$. For each vertex $u$ of $G$, let the order $\prec_u$ on $N^2(u)$ be a strict partial order such that $v \prec_u w$ if $\deg(u,v) < \deg(u,w)$. A couple of vertices $(u,v)$ with $v \in N^2(u)$ is *special* if there are less than $\alpha\Delta^{\frac{4}{3}}$ ($\alpha$ is a constant to be set later) vertices $w$ such that $v \prec_u w$. That is, $(u,v)$ is special if and only if, $v$ is one of the $\alpha\Delta^{4/3}$ highest elements of $\prec_u$ (see Fig. 4). Note that a couple $(u,v)$ can be special even if the couple $(v,u)$ is non-special. Let us denote $S(u) \subseteq N^2(u)$ the set of vertices $v$ such that $(u,v)$ is special. By definition, $|S(u)| = \min\left\{\alpha\Delta^{\frac{4}{3}}, |N^2(u)|\right\}$.

[4] $G_z$ denotes the derivative of $G$ with respect to its second variable.

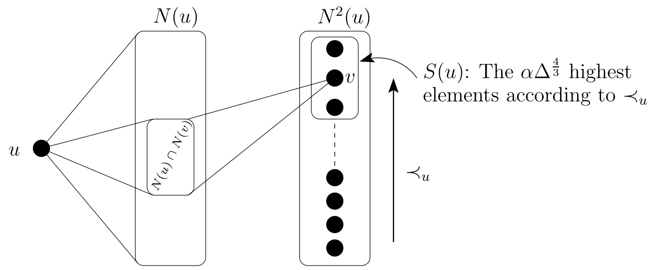



**Fig. 4.** Example of a special couple $(u, v)$.

### 3.1. First upper bound

By contradiction hypothesis, $\chi_a(G) \geq \frac{3}{2}\Delta^{\frac{4}{3}} + 5\Delta - 14$. Let $\kappa$ be the greatest integer such that $\kappa < \frac{3}{2}\Delta^{\frac{4}{3}} + 5\Delta - 14$ (i.e. $\kappa = \left\lceil \frac{3}{2}\Delta^{\frac{4}{3}} + 5\Delta - 15 \right\rceil$).

*The algorithm.* Let $V \in \{1, 2, \ldots, \kappa\}^t$ be a vector of length $t$. Algorithm AcyclicColoring_G (see on the following page) takes the vector $V$ as input and returns a partial acyclic coloring $\varphi : V(G) \to \{\bullet, 1, 2, \ldots, \kappa\}$ of $G$ (recall that $\bullet$ means that the vertex is uncolored) and a record $R$.

**Algorithm 2:** AcyclicColoring_G

**Input** : $V$ (vector of length $t$).
**Output**: $(\varphi, R)$.

1 **for** *all* $v$ *in* $V(G)$ **do**
2     $\varphi(v) \leftarrow \bullet$
3 $R \leftarrow$ *newfile*()
4 **for** $i \leftarrow 1$ **to** $t$ **do**
5     Let $v$ be the smallest (w.r.t. $\prec$) uncolored vertex of $G$
6     $\varphi(v) \leftarrow V[i]$
7     Write "`Color \n`" in $R$
8     **if** $\varphi(v) = \varphi(u)$ for $u \in N(v)$ **then**
        `// Proper coloring issue`
9         $\varphi(v) \leftarrow \bullet$
10         Write "`Uncolor, neighbor` $u$ `\n`" in $R$
11     **else if** $\varphi(v) = \varphi(u)$ for $u \in S(v)$ **then**
        `// Special couple issue`
12         $\varphi(v) \leftarrow \bullet$
13         Write "`Uncolor, special` $u$ `\n`" in $R$
14     **else if** $v$ *belongs to a bicolored cycle of length* 4 $(v = u_1, u_2, u_3, u_4)$ **then**
        `// Bicolored cycle issue`
15         $\varphi(v) \leftarrow \bullet$
16         $\varphi(u_2) \leftarrow \bullet$
17         Write "`Uncolor, cycle` $(u_1, u_2, u_3, u_4)$ `\n`" in $R$
18     **else if** $v$ *belongs to a bicolored path of length* 6 $(u_1, u_2, u_3, u_4, u_5, u_6)$ *with* $u_2 = v$ *and* $u_1 \prec u_3$ **then**
        `// Bicolored path issue`
19         $\varphi(u_1) \leftarrow \bullet$
20         $\varphi(v) \leftarrow \bullet$
21         $\varphi(u_3) \leftarrow \bullet$
22         $\varphi(u_4) \leftarrow \bullet$
23         Write "`Uncolor, path` $(u_1, u_2, u_3, u_4, u_5, u_6)$ `\n`" in $R$
24 **return** $(\varphi, R)$

Algorithm AcyclicColoring_G runs as follows. Let $\varphi_i$ be the partial coloring of $G$ after $i$ steps (at the end of the $i$th loop). At Step $i$, we first consider $\varphi_{i-1}$ and we color the smallest (w.r.t. $\prec$) uncolored vertex $v$ with $V[i]$ (line 6 of the algorithm). We then verify whether one of the following types of bad events happens:

Event $N$ (for neighbor): $G$ contains a monochromatic edge $vu$ for some $u$ (line 8 of the algorithm);
Event $S$ (for special): $G$ contains a special couple $(v, u)$ with $u$ and $v$ having the same color (line 11 of the algorithm);
Event $C$ (for cycle): $G$ contains a bicolored cycle of length 4 $(v = u_1, u_2, u_3, u_4)$ (line 14 of the algorithm);
Event $P$ (for path): $G$ contains a bicolored path of length 6 $(u_1, u_2, u_3, u_4, u_5, u_6)$ with $u_2 = v$ and $u_1 \prec u_3$ (line 18 of the algorithm).

If such events happen, then we modify the coloring (i.e. we uncolor some vertices as mentioned in Algorithm AcyclicColoring_G) in order that none of these events remains. Note that at some Step $i$, for $u$ and $v$ two vertices of $G$ such that $(u, v)$ is a special couple but $(v, u)$ is not, we may have $\varphi(u) = \varphi(v)$; this means that $u$ has been colored before $v$. Clearly, $\varphi_i$ is a partial acyclic coloring of $G$. Indeed, since Event $N$ is avoided, $\varphi_i$ is a proper coloring; since Events $C$ and $P$ are avoided, $\varphi_i$ is acyclic.

**Proof of Theorem 1.** As in the proof of Theorem 2, we prove that the function defined by AcyclicColoring_G, that maps $V$ to $(\varphi, R)$, is injective (see Lemma 9). The number of possible inputs $V$ is exactly $\kappa^t$. The number of possible partial acyclic $\kappa$-colorings $\varphi$ of $G$ is $(1+\kappa)^n$ (each vertex is either uncolored or colored with one color from $\{1, 2, \ldots, \kappa\}$) and the number of possible records $R$ is $o\left(\kappa^t\right)$ (see Lemma 10). Thus, the number of possible outputs $(\varphi, R)$ is $o\left(\kappa^t\right)$, as it is at most $(1+\kappa)^n \times o\left(\kappa^t\right)$ and $t \gg n$. A contradiction is then obtained by showing that the number of possible outputs $o\left(\kappa^t\right)$ is strictly smaller than the number of possible inputs $\kappa^t$ when $t$ is chosen large enough. □

*Algorithm analysis.* Recall that $\varphi_i$, $v_i$, $R_i$, and $V_i$ respectively denote the partial acyclic coloring obtained after $i$ steps, the current vertex $v$ of the $i$th step, the record $R$ after $i$ steps, and the input vector $V$ restricted to its $i$ first elements.

We first show that the function defined by AcyclicColoring_G is injective.

**Lemma 9.** *Given $(\varphi_i, R_i)$, one can recover the unique corresponding $V_i$.*

**Proof.** First note that, at each step of Algorithm AcyclicColoring_G, a "`Color`" line possibly followed by an "`Uncolor`" line is appended to $R$. We will say that a step which only appends a "`Color`" line is a *color step*, and a step which appends a "`Color`" line followed by an "`Uncolor`" line is an *uncolor step*. Therefore, by looking at the last line of $R$, we know whether the last step was a color step or an uncolor step.

We first prove by induction on $i$ that $R_i$ uniquely determines the set of colored vertices at Step $i$ (i.e. $\overline{\varphi_i}$). Observe that $R_1$ necessarily contains only one line which is "`Color`"; then $v_1$ is the unique colored vertex (which is the first element w.r.t. $\prec$). Assume now that $i \geq 2$. By induction hypothesis, $R_{i-1}$ (obtained from $R_i$ by removing the last line if Step $i$ was a color step or by removing the two last lines if Step $i$ was an uncolor step) uniquely determines the set of colored vertices at Step $i-1$. Then at Step $i$, the smallest uncolored vertex of $G$ is colored. If one of Events $N$, $S$, $C$ or $P$ happens, then the last line of $R_i$ is an "`Uncolor`" line whose indicates which vertices are uncolored. Therefore, $R_i$ uniquely determines the set of colored vertices at Step $i$.

Let us now prove by induction that the pair $(\varphi_i, R_i)$ permits to recover $V_i$. At Step 1, $(\varphi_1, R_1)$ clearly permits to recover $V_1$: indeed, $v_1$ is the unique colored vertex and thus $V[1] = \varphi_1(v_1)$. Assume now that $i \geq 2$. The record $R_{i-1}$ gives us the set of colored vertices at Step $i-1$, and thus we know what is the smallest uncolored vertex $v$ at the beginning of Step $i$. Consider the following two cases:

- If Step $i$ was a color step, then $\varphi_{i-1}$ is obtained from $\varphi_i$ in such a way that $\varphi_{i-1}(u) = \varphi_i(u)$ for all $u \neq v$ and $\varphi_{i-1}(v) = \bullet$. By induction hypothesis, $(\varphi_{i-1}, R_{i-1})$ permits to recover $V_{i-1}$ and $V[i] = \varphi_i(v)$.
- If Step $i$ was an uncolor step, then the last line of $R_i$ allows us to determine the set of uncolored vertices at Step $i$ and therefore, we can deduce $\varphi_{i-1}$. Then by induction hypothesis, $(\varphi_{i-1}, R_{i-1})$ permits to recover $V_{i-1}$. We obtain $V[i]$ by considering the following cases:
  - If the last line is of the form "`Uncolor, neighbor`" $u$, then $V[i] = \varphi_i(u)$.
  - If the last line is of the form "`Uncolor, special`" $u$, then $V[i] = \varphi_i(u)$.
  - If the last line is of the form "`Uncolor, cycle`" $(u_1, u_2, u_3, u_4)$, then $V[i] = \varphi_i(u_3)$.
  - If the last line is of the form "`Uncolor, path`" $(u_1, u_2, u_3, u_4, u_5, u_6)$, then $V[i] = \varphi_i(u_6)$.

That completes the proof. □

**Lemma 10.** *Algorithm* AcyclicColoring_G *produces at most $o(\kappa^t)$ distinct records.*

**Proof.** As Algorithm AcyclicColoring_G fails to color $G$, the record $R$ has exactly $t$ "`Color`" steps. It contains also "`Uncolor`" lines of different types: "`neighbor`" (type $N$), "`special`" (type $S$), "`cycle`" (type $C$), and "`path`" (type $P$). Let $\mathcal{T} = \{N, S, C, P\}$ be the set of bad event types. Let denote $s_j$ the number of uncolored vertices when a bad event of type $j$ occurs. Note that each "`Uncolor`" step of type "`neighbor`" (resp. "`special`", "`cycle`", and "`path`") uncolors 1 (resp. 1, 2, 4) previously colored vertex. Hence set $s_N = 1$, $s_S = 1$, $s_C = 2$, and $s_P = 4$.

To compute the total number of possible records, let us compute how many different entries, denoted $C_j$, an "`Uncolor`" step of type $j$ can produce in the record. By considering vertex $v$ in AcyclicColoring_G, observe that:

- An "Uncolor" step of type "neighbor" can produce $\varDelta$ different entries in the record, according to a neighbor of $v$ that shares the same color; hence let $C_N = \varDelta$.
- An "Uncolor" step of type "special" can produce $|S(v)| \leq \alpha\varDelta^{\frac{4}{3}}$ different entries in the record, according to the vertex $u \in S(v)$ that shares the same color; hence let $C_S = \alpha\varDelta^{\frac{4}{3}}$.
- An "Uncolor" step of type "cycle" can produce as many different entries in the record as the number of induced 4-cycles going through $v$ and avoiding $S(v)$ (it is sufficient to consider *induced* 4-cycles since bicolored 4-cycles with a chord would have been considered as Event $N$). We do not consider bicolored 4-cycles going through $v$ and some vertex $u \in S(v)$, since we would have an "Uncolor, special" $u$ step instead. Hence this number of entries is bounded by $\frac{\varDelta^{\frac{8}{3}}}{8\alpha}$ according to Claim 11, and thus let $C_C = \frac{\varDelta^{\frac{8}{3}}}{8\alpha}$.

**Claim 11.** *Given a graph $G$ with maximum degree $\varDelta$, for any vertex $v$ of $G$, there are at most $\frac{\varDelta^{\frac{8}{3}}}{8\alpha}$ induced 4-cycles going through $v$ and avoiding $S(v)$.*

**Proof.** There are at most $\varDelta^2$ edges between $N(v)$ and $N^2(v)$. Let $d$ be an integer such that $\deg(v, u) \geq d$ if and only if $u \in S(v)$. Therefore, there are at least $d|S(v)|$ edges between $N(v)$ and $S(v)$. Thus there are at most $\varDelta^2 - d\alpha\varDelta^{\frac{4}{3}}$ edges between $N(v)$ and $\overline{S}(v) = N^2(v) \setminus S(v)$, and

$$\sum_{u\in\overline{S}(v)} \deg(v, u) \leq \varDelta^2 - d\alpha\varDelta^{\frac{4}{3}} \tag{3}$$

One can see that the set of induced 4-cycles passing through $v$ and through some vertex $u \in N^2(v)$ is in bijection with the pairs of edges $\{ux, uy\}$ with $x \neq y$ and $\{x, y\} \subseteq N(v, u)$. Thus there are $\binom{\deg(v,u)}{2}$ such cycles. Summing over all vertices in $\overline{S}(v)$, we can thus conclude that this is less than the following value $K = \frac{1}{2}\sum_{u\in\overline{S}(v)} \deg(v, u)^2$. As this function is quadratic in $\deg(v, u)$, and as here $\deg(v, u) \leq d$, Eq. (3) implies that $K \leq K(d)$ for $K(d) = \frac{1}{2}(\varDelta^2 - d\alpha\varDelta^{\frac{4}{3}})d$. By simple calculation one can see that the polynomial $K(d)$ is maximal for $d = \frac{\varDelta^{\frac{2}{3}}}{2\alpha}$ and we thus have that $K \leq K\left(\frac{\varDelta^{\frac{2}{3}}}{2\alpha}\right) = \frac{\varDelta^{\frac{8}{3}}}{8\alpha}$. That concludes the proof of the claim. □

- An "Uncolor" step of type "path" can produce as many different entries in the record as the number of 6-paths $P = (u_1, u_2, u_3, u_4, u_5, u_6)$ with $u_2 = v$ and $u_1 \prec u_3$. Hence this number of entries is bounded by $\frac{1}{2}\varDelta(\varDelta - 1)^4$ according to Claim 12, and thus let $C_P = \frac{1}{2}\varDelta(\varDelta - 1)^4$.

**Claim 12.** *Given a graph $G$ with maximum degree $\varDelta$, for any vertex $v$ of $G$, there are at most $\frac{1}{2}\varDelta(\varDelta - 1)^4$ paths $(u_1, u_2, u_3, u_4, u_5, u_6)$ of length 6 with $u_2 = v$ and $u_1 \prec u_3$.*

**Proof.** Given vertex $v$, there are $\binom{\varDelta}{2}$ possibilities to choose $u_1$ and $u_3$, and then $\varDelta - 1$ candidates for being vertex $u_{i+1}$ once $u_i$ is known ($i \geq 3$). That clearly leads to the given upper bound. □

We complete the proof by means of Theorem 4. Let us consider the following polynomial $Q(x)$:

$$\begin{aligned} Q(x) &= 1 + \sum_{i\in\mathcal{T}} C_i x^{s_i} \\ &= 1 + C_N x^{s_N} + C_S x^{s_S} + C_C x^{s_C} + C_P x^{s_P} \\ &= 1 + \varDelta x + \alpha\varDelta^{\frac{4}{3}} x + \frac{\varDelta^{\frac{8}{3}}}{8\alpha} x^2 + \frac{1}{2}\varDelta(\varDelta - 1)^4 x^4. \end{aligned}$$

Setting $X = \frac{2\sqrt{2\alpha}}{\varDelta^{\frac{4}{3}}}$, we have:

$$\frac{Q(X)}{X} = \left(\frac{1}{\sqrt{2\alpha}} + \alpha\right)\varDelta^{\frac{4}{3}} + \left(8\alpha^{\frac{3}{2}}\sqrt{2} + 1\right)\varDelta - 32\alpha^{\frac{3}{2}}\sqrt{2} + \frac{8\alpha^{\frac{3}{2}}\sqrt{2}}{\varDelta}\left(6 - \frac{4}{\varDelta} + \frac{1}{\varDelta^2}\right). \tag{4}$$

In order to minimize $\frac{1}{\sqrt{2\alpha}} + \alpha$, we set $\alpha = \frac{1}{2}$, giving $X = \frac{2}{\varDelta^{\frac{4}{3}}}$ and we obtain:

$$\frac{Q(X)}{X} = \frac{3}{2}\varDelta^{\frac{4}{3}} + 5\varDelta - 16 + \frac{24}{\varDelta} - \frac{16}{\varDelta^2} + \frac{4}{\varDelta^3} < \frac{3}{2}\varDelta^{\frac{4}{3}} + 5\varDelta - 15 \leq \kappa \text{ as soon as } \varDelta \geq 24.$$

Since $0 < X \leq 1$ for $\varDelta \geq 24$, Algorithm AcyclicColoring_G produces at most $o(\kappa^t)$ different records by Theorem 4. That completes the proof. □

**Remark 13.** For small values of $\varDelta$, note that setting $\alpha = \frac{1}{2}$ is not optimal. Indeed the best choice of $\alpha$ is the value minimizing the right term of Eq. (4). For example, for $\varDelta = 27$, setting $\alpha = 0.225$ leads us to 194 colors instead of 242,

**Table 1**
Optimal values of $\alpha$ for some given $\Delta$.

| $\Delta$ | 27 | 28 | 29 | 30 | 100 | 1000 | 10000 | 100000 | 1000000 |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.225 | 0.225 | 0.226 | 0.226 | 0.25 | 0.32 | 0.384 | 0.434 | 0.465 |

already improving on Kostochka and Stocker's bound $1 + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor = 197$. Actually one can observe in Table 1 that the optimal value of $\alpha$ (for a given $\Delta$) converges to $\frac{1}{2}$ rather slowly.

### 3.2. A better upper bound for large value of $\Delta$

The choice of the bad event types is important and considering two different sets of bad event types (insuring the acyclic coloring property) may lead to different bounds. In the previous subsection, we considered four bad event types that insure a coloring to be acyclic. In this subsection, we consider another set of bad event types which leads to a better upper bound for large value of $\Delta$.

Algorithm AcyclicColoring-V2_G is a variant of Algorithm AcyclicColoring_G based on the following set of three bad events:

Event $N$: $G$ contains a monochromatic edge $vu$ for some $u$ (line 8 of the algorithm);
Event $S$: $G$ contains a special couple $(v, u)$ with $u$ and $v$ having the same color (line 11 of the algorithm);
Event $k$: $G$ contains a bicolored cycle of length $2k$ $(u_1, u_2, u_3, \ldots, u_{2k})$ for $k \geq 2$ with $v = u_2$ and $u_1 \prec u_3$ (line 14 of the algorithm).

**Algorithm 3:** AcyclicColoring-V2_G

**Input** : $V$ (vector of length $t$).
**Output**: $(\varphi, R)$.

1 **for** *all* $v$ *in* $V(G)$ **do**
2     $\varphi(v) \leftarrow \bullet$
3 $R \leftarrow newfile()$
4 **for** $i \leftarrow 1$ **to** $t$ **do**
5     Let $v$ be the smallest (w.r.t. $\prec$) uncolored vertex of $G$
6     $\varphi(v) \leftarrow V[i]$
7     Write `"Color \n"` in $R$
8     **if** $\varphi(v) = \varphi(u)$ for $u \in N(v)$ **then**
        `// Proper coloring issue`
9         $\varphi(v) \leftarrow \bullet$
10         Write `"Uncolor, neighbor` $u$ `\n"` in $R$
11     **else if** $\varphi(v) = \varphi(u)$ for $u \in S(v)$ **then**
        `// Special couple issue`
12         $\varphi(v) \leftarrow \bullet$
13         Write `"Uncolor, special` $u$ `\n"` in $R$
14     **else if** *$v$ belongs to a bicolored cycle of length $2k$ ($k \geq 2$), say $(u_1, u_2 = v, u_3, \ldots, u_{2k})$ with $u_1 \prec u_3$* **then**
        `// Bicolored cycle issue`
15         **for** $j \leftarrow 1$ **to** $2k-2$ **do**
16             $\varphi(u_j) \leftarrow \bullet$
17         Write `"Uncolor, cycle` $(u_1, \ldots, u_{2k})$ `\n"` in $R$
18 **return** $(\varphi, R)$

This leads to the following upper bound when $\Delta \geq 9$:

$$\chi_a(G) < \frac{3}{2}\Delta^{\frac{4}{3}} + \Delta + \frac{8\Delta^{\frac{4}{3}}}{\Delta^{\frac{2}{3}} - 4} + 1.$$

Let $\kappa$ be the greatest integer such that $\kappa < \frac{3}{2}\Delta^{\frac{4}{3}} + \Delta + \frac{8\Delta^{\frac{4}{3}}}{\Delta^{\frac{2}{3}}-4} + 1$ and let $\alpha = \frac{1}{2}$. We now briefly sketch the proof. Let $\mathcal{T} = \left\{N, S, 2, 3, 4, \ldots, \frac{n}{2}\right\}$ be the set of bad event types. Note that each "`Uncolor`" step of type "`neighbor`" (resp. "`special`" and $2k$"`-cycle`") uncolors 1 (resp. 1, $2k-2$) previously colored vertex. Hence set $s_N = 1$, $s_S = 1$ and $s_k = 2k-2$ for $k \geq 2$.

By considering $v$ in Algorithm AcyclicColoring-V2_G, observe that:

- An "Uncolor" step of type "neighbor" can produce $\Delta$ different entries in the record. Set $C_N = \Delta$.
- An "Uncolor" step of type "special" can produce $|S(v)| \leq \frac{1}{2}\Delta^{\frac{4}{3}}$ different entries in the record, according to the vertex $u \in S(v)$ that shares the same color. Set $C_S = \frac{1}{2}\Delta^{\frac{4}{3}}$.
- Now consider cycles of length $2k$, $k \geq 2$. For cycles of length 4, there are at most $\frac{1}{4}\Delta^{\frac{8}{3}}$ induced 4-cycles going through $v$ and avoiding $S(v)$ (see Claim 11); we set $C_2 = \frac{1}{4}\Delta^{\frac{8}{3}}$.

Let $k \geq 3$. Let us upper bound the number of $2k$-cycles going through $v$ that may be bicolored. To do so, we count the number of $2k$-cycles $(u_1, u_2, u_3, \ldots, u_{2k})$ with $u_2 = v$, $u_1 \prec u_3$ such that $(u_1, u_{2k-1})$ or $(u_{2k-1}, u_1)$ is not special (if both $(u_1, u_{2k-1})$ and $(u_{2k-1}, u_1)$ are special, then $u_1$ and $u_{2k-1}$ cannot receive the same color). There are at most $\Delta^{2k-\frac{4}{3}}$ such cycles according to Claim 14. We set $C_k = \Delta^{2k-\frac{4}{3}}$.

**Claim 14.** *For $k \geq 3$, there are at most $\Delta^{2k-\frac{4}{3}}$ $2k$-cycles $(u_1, u_2, u_3, \ldots, u_{2k})$ going through $v$ with $v = u_2$ and $u_1 \prec u_3$ such that $(u_1, u_{2k-1})$ or $(u_{2k-1}, u_1)$ is not special.*

**Proof.** As $u_1 \prec u_3$, given $v$, there are $\binom{\Delta}{2}$ possible $(u_1, u_3)$. Then knowing $u_i$, there are at most $\Delta$ possible choices for $u_{i+1}$, $3 \leq i \leq 2k-2$. Now let $(r, s)$ be a non-special couple being either $(u_1, u_{2k-1})$ or $(u_{2k-1}, u_1)$. Hence $s \in N^2(r)\backslash S(r)$. Let $d$ be the highest value of $\deg(r, u)$ for $u \in N^2(r)\backslash S(r)$. Therefore, there are at least $d|S(r)|$ edges between $N(r)$ and $S(r)$, and so at most $\Delta^2 - \frac{d}{2}\Delta^{\frac{4}{3}}$ edges between $N(r)$ and $N^2(r) \setminus S(r)$. It follows that $d$ is at most $2\Delta^{\frac{2}{3}}$. Hence, there are at most $2\Delta^{\frac{2}{3}}$ possible choices for $u_{2k}$. That leads to the given upper bound. $\square$

Let us consider the following polynomial $Q(x)$:

$$\begin{aligned}
Q(x) &= 1 + \sum_{i \in \mathcal{T}} C_i x^{s_i} \\
&= 1 + C_N x^{s_N} + C_S x^{s_S} + C_2 x^{s_2} + \sum_{k \geq 3}^{\lfloor n/2 \rfloor} C_k x^{s_k} \\
&= 1 + \Delta x + \frac{1}{2}\Delta^{\frac{4}{3}} x + \frac{1}{4}\Delta^{\frac{8}{3}} x^2 + \sum_{k \geq 3}^{\lfloor n/2 \rfloor} \Delta^{2k-\frac{4}{3}} x^{2k-2} \\
&< 1 + \Delta x + \frac{1}{2}\Delta^{\frac{4}{3}} x + \frac{1}{4}\Delta^{\frac{8}{3}} x^2 + \frac{\Delta^{\frac{14}{3}} x^4}{1 - \Delta^2 x^2} \qquad \text{for } x < \frac{1}{\Delta}.
\end{aligned}$$

Setting $X = \frac{2}{\Delta^{\frac{4}{3}}}$, we have $X \leq \frac{1}{\Delta}$ as soon as $\Delta \geq 9$ and thus:

$$\frac{Q(X)}{X} < \frac{3}{2}\Delta^{\frac{4}{3}} + \Delta + \frac{8\Delta^{\frac{4}{3}}}{\Delta^{\frac{2}{3}} - 4} \leq \kappa.$$

Algorithm AcyclicColoring-V2_G produces at most $o(\kappa^t)$ different records by Theorem 4. That completes the sketch of the proof.

## 4. Extension to list-coloring

Given a graph $G$ and a list assignment $L(v)$ of colors for every vertex $v$ of $G$, we say that $G$ admits an $L$-coloring if there is a vertex-coloring such that every vertex $v$ receives its color from its own list $L(v)$. A graph is *k-choosable* if it is $L$-colorable for any list assignment $L$ such that $|L(v)| \geq k$ for every $v$. The minimum integer $k$ such that $G$ is $k$-choosable is called the *choice number* of $G$. The usual coloring is a particular case of $L$-coloring (all the lists are equal) and thus the choice number is an upper bound on the chromatic number.

Until now, our methods were developed for usual colorings (i.e. without lists). Every algorithm takes a vector of colors $V$ as input and, at each Step $i$, a vertex $v$ is colored with color $V[i]$. It is easy to slightly modify our procedure to extend all the previous results to list-coloring. To do so, the input vector $V$ is no longer a vector of colors but a vector of indices. Then, at each Step $i$, the current vertex $v$ is colored with the $V[i]^{\text{th}}$ color of $L(v)$. Therefore, Theorems 1 and 2 extend to list-coloring.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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