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The chromatic number and switching chromatic number of 2-edge-colored graphs of bounded degree

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Extended Abstract

The notion of homomorphisms of 2-edge-colored graphs has already been studied as a way of extending classical results in graph coloring such as Hadwiger's conjecture. Guenin [5] introduced the notion of switching homomorphisms for its relation with a well-known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [6] as it captures a number of well-known conjectures that can be reformulated using the definition of switching homomorphisms. In this extended abstract, we study homomorphisms of 2-edge colored graphs and switching homomorphisms of bounded degree graphs.

A 2-edge-colored graph $G = (V, E, s)$ is a simple graph $(V, E)$ with two kinds of edges: positive and negative edges. The signature $s : E(G) \to \{-1, +1\}$ assigns to each edge its sign. In the sequel, $D_k$ (resp. $D^c_k$) denotes the class of 2-edge-colored graphs (resp. connected 2-edge-colored graphs) with maximum degree $k$.

Given two 2-edge-colored graphs $G$ and $H$, the mapping $\varphi : V(G) \to V(H)$ is a homomorphism if $\varphi$ maps every edge of $G$ to an edge of $H$ with the same sign. This can be seen as coloring the graph $G$ by using the vertices of $H$ as colors. The target graph $H$ gives us the rules that this coloring must obey. If vertices 1 and 2 in $H$ are connected with a positive (resp. negative) edge, then every pair of adjacent vertices in $G$ colored with 1 and 2 must be connected with a positive (resp. negative) edge. The chromatic number $\chi_2(G)$ of a 2-edge-colored graph $G$ is the order of a smallest 2-edge-colored graph $H$ such that $G$ admits a homomorphism to $H$. The chromatic number $\chi_2(C)$ of a class of 2-edge-colored graphs $C$ is the maximum of the chromatic numbers of the graphs in the class. This number can be infinite.

2-edge-colored graphs are, in some sense, similar to oriented graphs since a pair of vertices can be adjacent in two different ways in both kinds of graphs: with a positive or a negative edge in the case of 2-edge-colored graphs, with a toward or a backward arc in the oriented case.

The notion of homomorphism of oriented graphs has been introduced by Courcel [3] in 1994 and has been widely studied since then. Due to the similarity above-mentioned, we try to adapt techniques used to study the homomorphisms of oriented graphs of bounded degree to 2-edge-colored graphs of bounded degree. We also study switching homomorphisms of 2-edge-colored graphs in order to obtain results on signed graphs.

Switching a vertex $v$ of a 2-edge-colored graph corresponds to reversing the signs of all edges incident to $v$.

Two 2-edge-colored graphs $G$ and $G'$ are switching equivalent if it is possible to turn $G$ into $G'$ after some number of switches. We call the classes created by this equivalence relation switching classes (note that switching classes are equivalent to the notion of signed graphs).

Given two 2-edge-colored graphs $G$ and $H$, the mapping $\varphi : V(G) \to V(H)$ is a switching homomorphism if there is a graph $G'$ switching equivalent to $G$ such that $\varphi$ maps every edge of $G'$ to an edge of $H$ with the same sign. The switching chromatic number $\chi_s(G)$ of a 2-edge-colored graph $G$ is the order of a smallest 2-edge-colored graph $H$ such that $G$ admits a switching homomorphism to $H$.

Table 1 summarizes results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

The first two lines of Table 1 are more or less folklore. Let us explain in the following the difference that exists between the connected case and the non-connected case for the...
Table 1: Results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\chi_2(D_k)$</th>
<th>$\chi_2(D_k')$</th>
<th>$\chi_s(D_k)$</th>
<th>$\chi_s(D_k')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\chi_2(D_1) = 3$</td>
<td>$\chi_2(D_1') = 2$</td>
<td>$\chi_s(D_1) = \chi_s(D_1') = 2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\chi_2(D_2) = 6$</td>
<td>$\chi_2(D_2') = 5$</td>
<td>$\chi_s(D_2) = \chi_s(D_2') = 4$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$8 \leq \chi_2(D_3') \leq \chi_2(D_3) \leq 11$</td>
<td>$6 \leq \chi_s(D_3) \leq 7$</td>
<td>$\chi_s(D_3') = 6$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$12 \leq \chi_2(D_4') \leq \chi_2(D_4) \leq 31$</td>
<td>$10 \leq \chi_s(D_4') \leq \chi_s(D_4) \leq 16$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$2^k \leq \chi_2(D_k') \leq \chi_2(D_k) \leq 2^{k+1}(k-1)^2$</td>
<td>$\chi_s(D_k') \leq \chi_s(D_k) \leq 2^{k+1}(k-1)^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The last three lines of Table 1 are dedicated to graphs with maximum degree at least 3.

Our main results are the following:

**Theorem 1** We have:
- $8 \leq \chi_2(D_3) \leq 11$,
- $12 \leq \chi_2(D_4) \leq 31$,
- $10 \leq \chi_s(D_4) \leq 16$. 

Figure 1: Four examples of 2-edge-colored graphs with chromatic number 5.

Figure 2: Target graphs for connected 2-edge-colored graphs of maximum degree 2.
In order to find an upper bound for a class of graphs, we need to find a target graph that can color every graph in the class. In the case of oriented homomorphisms, oriented graphs that are antiautomorphic, $K_n$-transitive for some $n$, or that have Property $P_{n,k}$ for some $n$ and $k$ are good candidates. We analogously define these properties in term of 2-edge-colored graphs.

A 2-edge-colored graph $(V, E, s)$ is said to be antiautomorphic if it is isomorphic to $(V, E, -s)$.

A 2-edge-colored graph $G = (V, E, s)$ is said to be $K_n$-transitive if for every pair of cliques $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ in $G$ such that for all $i \neq j$, $s(u_i u_j) = s(v_i v_j)$ there exists an automorphism that maps $u_i$ to $v_i$ for all $i$. For $n = 1, 2, 3$, or $4$, we say that the graph is vertex, edge, or triangle-transitive, respectively.

A 2-edge-colored graph $G$ has Property $P_{k,n}$ if for every sequence of $k$ distinct vertices $(v_1, v_2, \ldots, v_k)$ that induces a clique in $G$ and for every sign vector $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{-1, +1\}^k$ there exist at least $n$ distinct vertices $\{u_1, u_2, \ldots, u_n\}$ such that $s(v_i u_j) = \alpha_i$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Given an integer $q \equiv 1 \ (\text{mod} \ 4)$, we consider the family of complete signed Paley graphs $SP_q$ built from the field of order $q$ which has the above-mentioned properties. The vertices of $SP_q$ are the elements of the field of order $q$ and $s(uv) = +1$ if $u-v$ is a square and $s(uv) = -1$ otherwise.

**Lemma 2** ([7]) Graph $SP_q$ is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1, \frac{q-1}{2}}$ and $P_{2, \frac{q+1}{2}}$.

Let us consider the following operation. Given a 2-edge-colored graph $G$, we create the antitwinned graph of $G$ denoted by $\rho(G)$ as follows. Let $G^+, G^-$ be two copies of $G$. The vertex corresponding to $v \in V(G)$ in $G^i$ is denoted by $v_i$, $V(\rho(G)) = V(G^+) \cup V(G^-)$, $E(\rho(G)) = \{u_i v_j : uv \in E(G), \ i, j \in \{-1, +1\}\}$ and $s_{\rho(G)}(u_i v_j) = i \times j \times s_G(u, v)$.

**Lemma 3** ([2]) Let $G$ and $H$ be two 2-edge-colored graphs. The graph $G$ admits a homomorphism to $\rho(H)$ if and only if it admits a switching homomorphism to $H$.

In other words, if a 2-edge-colored graph admits a homomorphism to an antitwinned target graph on $n$ vertices, then it also admits a switching homomorphism to a target graph on $\frac{n}{2}$ vertices. The family $\rho(SP_q)$ also are interesting target graphs (especially for bounding the switching chromatic number since they are antitwinned).

**Lemma 4** ([7]) The graph $\rho(SP_q)$ is vertex-transitive, antiautomorphic, and has properties $P_{1, q-1}, P_{2, \frac{q+1}{2}},$ and $P_{3, \max(\frac{q+1}{4}, 0)}$.

One last family of interesting target graphs are the Tromp-Paley graphs (this construction due to Tromp (unpublished) has been widely used in the case of oriented homomorphisms). Let $SP_\pm$ be $SP_q$ with an additional vertex that is connected to every other vertex with a positive edge. The Tromp-Paley graph $TR(SP_q)$ corresponds to $\rho(SP_\pm)$. This construction improves the properties of $\rho(SP_q)$ at the cost of having two more vertices. Since Tromp-Paley graphs are antitwinned, they are interesting for bounding the switching chromatic number.

**Lemma 5** ([7]) $TR(SP_q)$ is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1, q}, P_{2, \frac{q+1}{2}},$ and $P_{3, \frac{q+3}{2}}$.

Bensmail et al. [1] recently proved that every 2-edge-colored graph with maximum degree 3 except the all positive and all negative $K_4$ admits a homomorphism to $TR(SP_3)$, hence $\chi_2(D_3) \leq 12$, and $\chi_4(D_3) \leq 6$ by Lemma 3. In the non-connected case, we can easily get $\chi_2(D_3) \leq 14$ and thus $\chi_4(D_3) \leq 7$ by Lemma 3 (it is possible to create an all positive $K_4$ and an all negative $K_4$ in $TR(SP_3)$ by adding two vertices). Their proof uses a computer to show that a minimal counter-example cannot contain some configurations and then concludes by using the properties of $TR(SP_3)$. Theorem 1 improves the upper bound of 14 to 11.

Let us give a sketch of proof of the first result of Theorem 1, namely $\chi_2(D_3) \leq 11$. 

Consider the graph $SP_9^*$ obtained from $SP_9$ by adding two new vertices $0'$ and $1'$ as follows. Take the two vertices 0 and 1 of $SP_9$ (note that $s(01) = +1$), and link $0'$ and $1'$ to the vertices of $SP_9$ in the same way as 0 and 1 are, respectively; add an edge $0'1'$ with $s(0'1') = -1$; finally we add edges 00' and 11' with $s(00') = -1$ and $s(11') = +1$. We will prove that every graph from $D_3$ admits a homomorphism to $SP_9^*$.

We first show that every connected 2-degenerate 2-edge-colored graph with maximum degree 3 admits a homomorphism to $SP_9$ by using its structural properties given by Lemma 2 (a unique exception exists and is separately treated).

Let $G$ be a connected 3-regular 2-edge-colored graph. If $G$ is all positive, then we color it using an all positive $K_4$ that $SP_9^*$ contains as a subgraph. Assume now that $G$ is not all positive. Let $uv$ be a negative edge of $G$. We remove $uv$ from $G$ to create a new graph $G'$. Graph $G'$ is 2-degenerate so it admits a homomorphism $\varphi'$ to $SP_9$. If $s(\varphi'(u)\varphi'(v)) = -1$, then $\varphi'$ is also a homomorphism from $G$ to $SP_9$.

If $s(\varphi'(u)\varphi'(v)) = +1$, then by edge-transitivity of $SP_9$ we can recolor the vertices of $G'$ such that $\varphi'(u) = 0$ and $\varphi'(v) = 1$. We can then extend $\varphi'$ to a homomorphism $\varphi$ of $G$ to $SP_9^*$ by recoloring $u$ and $v$ such that $\varphi(u) = 0'$ and $\varphi(v) = 1'$ since $s(0'1') = -1$.

Finally, if $\varphi'(u) = \varphi'(v)$, then by vertex-transitivity of $SP_9$ we can recolor the vertices of $G'$ such that $\varphi'(u) = \varphi'(v) = 0$. We can then extend $\varphi'$ to a homomorphism $\varphi$ of $G$ to $SP_9^*$ by recoloring $v$ such that $\varphi(v) = 0'$ since $s(00') = -1$.

We have proven that every graph in $D_3$ admits a homomorphism to $SP_9^*$ which means that $SP_9^*$ is universal for $D_3$. This concludes the proof.

To prove the two other upper bounds of Theorem 1, we use the same method on target graphs $SP_9$ and $TR(SP_{13})$.

References