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# The chromatic number and switching chromatic number of 2-edge-colored graphs of bounded degree

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## EXTENDED ABSTRACT

The notion of homomorphisms of 2-edge-colored graphs has already been studied as a way of extending classical results in graph coloring such as Hadwiger's conjecture. Guenin [5] introduced the notion of switching homomorphisms for its relation with a well-known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [6] as it captures a number of well-known conjectures that can be reformulated using the definition of switching homomorphisms. In this extended abstract, we study homomorphisms of 2-edge colored graphs and switching homomorphisms of bounded degree graphs.

A *2-edge-colored graph*  $G = (V, E, s)$  is a simple graph  $(V, E)$  with two kinds of edges: positive and negative edges. The signature  $s : E(G) \rightarrow \{-1, +1\}$  assigns to each edge its sign. In the sequel,  $\mathcal{D}_k$  (resp.  $\mathcal{D}_k^c$ ) denotes the class of 2-edge-colored graphs (resp. connected 2-edge-colored graphs) with maximum degree  $k$ .

Given two 2-edge-colored graphs  $G$  and  $H$ , the mapping  $\varphi : V(G) \rightarrow V(H)$  is a *homomorphism* if  $\varphi$  maps every edge of  $G$  to an edge of  $H$  with the same sign. This can be seen as coloring the graph  $G$  by using the vertices of  $H$  as colors. The target graph  $H$  gives us the rules that this coloring must obey. If vertices 1 and 2 in  $H$  are connected with a positive (resp. negative) edge, then every pair of adjacent vertices in  $G$  colored with 1 and 2 must be connected with a positive (resp. negative) edge. The *chromatic number*  $\chi_2(G)$  of a 2-edge-colored graph  $G$  is the order of a smallest 2-edge-colored graph  $H$  such that  $G$  admits a homomorphism to  $H$ . The chromatic number  $\chi_2(\mathcal{C})$  of a class of 2-edge-colored graphs  $\mathcal{C}$  is the maximum of the chromatic numbers of the graphs in the class. This number can be infinite.

2-edge-colored graphs are, in some sense, similar to oriented graphs since a pair of vertices can be adjacent in two different ways in both kinds of graphs: with a positive or a negative edge in the case of 2-edge-colored graphs, with a toward or a backward arc in the oriented case.

The notion of homomorphism of oriented graphs has been introduced by Courcell [3] in 1994 and has been widely studied since then. Due to the similarity above-mentioned, we try to adapt techniques used to study the homomorphisms of oriented graphs of bounded degree to 2-edge-colored graphs of bounded degree. We also study switching homomorphisms of 2-edge-colored graphs in order to obtain results on signed graphs.

*Switching* a vertex  $v$  of a 2-edge-colored graph corresponds to reversing the signs of all edges incident to  $v$ .

Two 2-edge-colored graphs  $G$  and  $G'$  are *switching equivalent* if it is possible to turn  $G$  into  $G'$  after some number of switches. We call the classes created by this equivalence relation *switching classes* (note that switching classes are equivalent to the notion of signed graphs).

Given two 2-edge-colored graphs  $G$  and  $H$ , the mapping  $\varphi : V(G) \rightarrow V(H)$  is a *switching homomorphism* if there is a graph  $G'$  switching equivalent to  $G$  such that  $\varphi$  maps every edge of  $G'$  to an edge of  $H$  with the same sign. The *switching chromatic number*  $\chi_s(G)$  of a 2-edge-colored graph  $G$  is the order of a smallest 2-edge-colored graph  $H$  such that  $G$  admits a switching homomorphism to  $H$ .

Table 1 summarizes results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

The first two lines of Table 1 are more or less folklore. Let us explain in the following the difference that exists between the connected case and the non-connected case for the

	$\chi_2(\mathcal{D}_k)$	$\chi_2(\mathcal{D}_k^c)$	$\chi_s(\mathcal{D}_k)$	$\chi_s(\mathcal{D}_k^c)$
$k = 1$	$\chi_2(\mathcal{D}_k) = 3$	$\chi_2(\mathcal{D}_k^c) = 2$	$\chi_s(\mathcal{D}_k) = \chi_s(\mathcal{D}_k^c) = 2$	
$k = 2$	$\chi_2(\mathcal{D}_k) = 6$	$\chi_2(\mathcal{D}_k^c) = 5$	$\chi_s(\mathcal{D}_k) = \chi_s(\mathcal{D}_k^c) = 4$	
$k = 3$	$8 \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 11$		$6 \leq \chi_s(\mathcal{D}_k) \leq 7$ [1]	$\chi_s(\mathcal{D}_k^c) = 6$ [1]
$k = 4$	$12 \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 31$		$10 \leq \chi_s(\mathcal{D}_k^c) \leq \chi_s(\mathcal{D}_k) \leq 16$	
$k \geq 5$	$2^{\frac{k}{2}} \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 2^{k+1}(k-1)^2$ [4]		$\chi_s(\mathcal{D}_k^c) \leq \chi_s(\mathcal{D}_k) \leq 2^{k+1}(k-1)^2$ [4]	

Table 1: Results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

chromatic number of 2-edge-colored graphs with maximum degree 1 or 2. An edge of a 2-edge-colored graph has chromatic number 2 and thus  $\chi_2(\mathcal{D}_1^c) = 2$ ; however, a 2-edge-colored graph with two non-adjacent edges, one positive and one negative, has chromatic number 3 (the target graph needs a positive and a negative edge, hence at least three vertices) and thus  $\chi_2(\mathcal{D}_1) = 3$ . We therefore have a difference between the chromatic numbers of connected and non-connected 2-edge-colored graphs with maximum degree 1. This difference does not exist for switching homomorphisms since a negative edge can be changed into a positive one after a switch. This difference (and lack thereof for switching homomorphisms) appears also in graphs with maximum degree 2. We have  $\chi_2(\mathcal{D}_2) \geq 6$  since there is no 2-edge-colored graph on 5 vertices that can color all the four graphs depicted in Figure 1. However, every connected 2-edge-colored graph with maximum degree 2 admits a homomorphism to one of the two graphs on 5 vertices depicted in Figure 2 and thus  $\chi_2(\mathcal{D}_2^c) \leq 5$ . In order to color any graph of  $\mathcal{D}_2$ , we need a target graph that contains both graphs depicted in Figure 2 as subgraphs. This is possible with 6 vertices so  $\chi_2(\mathcal{D}_2) = 6$ . We do not know yet if this is also the case for graphs with maximum degree at least 3.

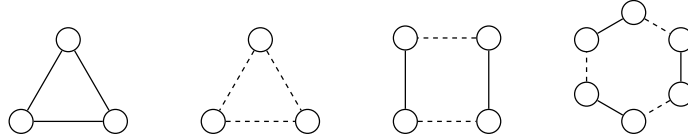


Figure 1: Four examples of 2-edge-colored graphs with chromatic number 5.

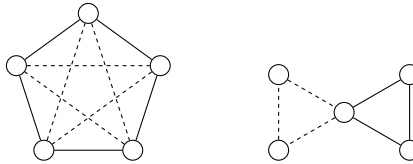


Figure 2: Target graphs for connected 2-edge-colored graphs of maximum degree 2.

The last three lines of Table 1 are dedicated to graphs with maximum degree at least 3. Our main results are the following:

**Theorem 1** *We have:*

- $8 \leq \chi_2(\mathcal{D}_3) \leq 11$ ,
- $12 \leq \chi_2(\mathcal{D}_4) \leq 31$ ,
- $10 \leq \chi_s(\mathcal{D}_4) \leq 16$ .

In order to find an upper bound for a class of graphs, we need to find a target graph that can color every graph in the class. In the case of oriented homomorphisms, oriented graphs that are antiautomorphic,  $K_n$ -transitive for some  $n$ , or that have Property  $P_{n,k}$  for some  $n$  and  $k$  are good candidates. We analogously define these properties in term of 2-edge-colored graphs.

A 2-edge-colored graph  $(V, E, s)$  is said to be *antiautomorphic* if it is isomorphic to  $(V, E, -s)$ .

A 2-edge-colored graph  $G = (V, E, s)$  is said to be  *$K_n$ -transitive* if for every pair of cliques  $\{u_1, u_2, \dots, u_n\}$  and  $\{v_1, v_2, \dots, v_n\}$  in  $G$  such that for all  $i \neq j$ ,  $s(u_i u_j) = s(v_i v_j)$  there exists an automorphism that maps  $u_i$  to  $v_i$  for all  $i$ . For  $n = 1, 2$ , or  $3$ , we say that the graph is *vertex*, *edge*, or *triangle-transitive*, respectively.

A 2-edge-colored graph  $G$  has *Property  $P_{k,n}$*  if for every sequence of  $k$  distinct vertices  $(v_1, v_2, \dots, v_k)$  that induces a clique in  $G$  and for every sign vector  $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \{-1, +1\}^k$  there exist at least  $n$  distinct vertices  $\{u_1, u_2, \dots, u_n\}$  such that  $s(v_i u_j) = \alpha_i$  for  $1 \leq i \leq k$  and  $1 \leq j \leq n$ .

Given an integer  $q \equiv 1 \pmod{4}$ , we consider the family of complete signed Paley graphs  $SP_q$  built from the field of order  $q$  which has the above-mentioned properties. The vertices of  $SP_q$  are the elements of the field of order  $q$  and  $s(uv) = +1$  if  $u - v$  is a square and  $s(uv) = -1$  otherwise.

**Lemma 2 ([7])** *Graph  $SP_q$  is vertex-transitive, edge-transitive, antiautomorphic, and has properties  $P_{1, \frac{q-1}{2}}$  and  $P_{2, \frac{q-5}{4}}$ .*

Let us consider the following operation. Given a 2-edge-colored graph  $G$ , we create the *antitwinned graph* of  $G$  denoted by  $\rho(G)$  as follows. Let  $G^{+1}, G^{-1}$  be two copies of  $G$ . The vertex corresponding to  $v \in V(G)$  in  $G^i$  is denoted by  $v_i$ ,  $V(\rho(G)) = V(G^+) \cup V(G^-)$ ,  $E(\rho(G)) = \{u_i v_j : uv \in E(G), i, j \in \{-1, +1\}\}$  and  $s_{\rho(G)}(u_i v_j) = i \times j \times s_G(u, v)$ .

**Lemma 3 ([2])** *Let  $G$  and  $H$  be two 2-edge-colored graphs. The graph  $G$  admits a homomorphism to  $\rho(H)$  if and only if it admits a switching homomorphism to  $H$ .*

In other words, if a 2-edge-colored graph admits a homomorphism to an antitwinned target graph on  $n$  vertices, then it also admits a switching homomorphism to a target graph on  $\frac{n}{2}$  vertices. The family  $\rho(SP_q)$  also are interesting target graphs (especially for bounding the switching chromatic number since they are antitwinned).

**Lemma 4 ([7])** *The graph  $\rho(SP_q)$  is vertex-transitive, antiautomorphic, and has properties  $P_{1, q-1}$ ,  $P_{2, \frac{q-3}{2}}$ , and  $P_{3, \max(\frac{q-9}{4}, 0)}$ .*

One last family of interesting target graphs are the Tromp-Paley graphs (this construction due to Tromp (unpublished) has been widely used in the case of oriented homomorphisms). Let  $SP_q^+$  be  $SP_q$  with an additional vertex that is connected to every other vertex with a positive edge. The Tromp-Paley graph  $TR(SP_q)$  corresponds to  $\rho(SP_q^+)$ . This construction improves the properties of  $\rho(SP_q)$  at the cost of having two more vertices. Since Tromp-Paley graphs are antitwinned, they are interesting for bounding the switching chromatic number.

**Lemma 5 ([7])**  *$TR(SP_q)$  is vertex-transitive, edge-transitive, antiautomorphic, and has properties  $P_{1, q}$ ,  $P_{2, \frac{q-1}{2}}$ , and  $P_{3, \frac{q-5}{4}}$ .*

Bensmail et al. [1] recently proved that every 2-edge-colored graph with maximum degree 3 except the all positive and all negative  $K_4$  admits a homomorphism to  $TR(SP_5)$ , hence  $\chi_2(\mathcal{D}_3^c) \leq 12$ , and  $\chi_s(\mathcal{D}_3^c) \leq 6$  by Lemma 3. In the non-connected case, we can easily get  $\chi_2(\mathcal{D}_3) \leq 14$  and thus  $\chi_s(\mathcal{D}_3) \leq 7$  by Lemma 3 (it is possible to create an all positive  $K_4$  and an all negative  $K_4$  in  $TR(SP_5)$  by adding two vertices). Their proof uses a computer to show that a minimal counter-example cannot contain some configurations and then concludes by using the properties of  $TR(SP_5)$ . Theorem 1 improves the upper bound of 14 to 11.

Let us give a sketch of proof of the first result of Theorem 1, namely  $\chi_2(\mathcal{D}_3) \leq 11$ .

Consider the graph  $SP_9^*$  obtained from  $SP_9$  by adding two new vertices  $0'$  and  $1'$  as follows. Take the two vertices 0 and 1 of  $SP_9$  (note that  $s(01) = +1$ ), and link  $0'$  and  $1'$  to the vertices of  $SP_9$  in the same way as 0 and 1 are, respectively; add an edge  $0'1'$  with  $s(0'1') = -1$ ; finally we add edges  $00'$  and  $11'$  with  $s(00') = -1$  and  $s(11') = +1$ . We will prove that every graph from  $\mathcal{D}_3$  admits a homomorphism to  $SP_9^*$ .

We first show that every connected 2-degenerate 2-edge-colored graph with maximum degree 3 admits a homomorphism to  $SP_9$  by using its structural properties given by Lemma 2 (a unique exception exists and is separately treated).

Let  $G$  be a connected 3-regular 2-edge-colored graph. If  $G$  is all positive, then we color it using an all positive  $K_4$  that  $SP_9^*$  contains as a subgraph. Assume now that  $G$  is not all positive. Let  $uv$  be a negative edge of  $G$ . We remove  $uv$  from  $G$  to create a new graph  $G'$ . Graph  $G'$  is 2-degenerate so it admits a homomorphism  $\varphi'$  to  $SP_9$ . If  $s(\varphi'(u)\varphi'(v)) = -1$ , then  $\varphi'$  is also a homomorphism from  $G$  to  $SP_9$ .

If  $s(\varphi'(u)\varphi'(v)) = +1$ , then by edge-transitivity of  $SP_9$  we can recolor the vertices of  $G'$  such that  $\varphi'(u) = 0$  and  $\varphi'(v) = 1$ . We can then extend  $\varphi'$  to a homomorphism  $\varphi$  of  $G$  to  $SP_9^*$  by recoloring  $u$  and  $v$  such that  $\varphi(u) = 0'$  and  $\varphi(v) = 1'$  since  $s(0'1') = -1$ .

Finally, if  $\varphi'(u) = \varphi'(v)$ , then by vertex-transitivity of  $SP_9$  we can recolor the vertices of  $G'$  such that  $\varphi'(u) = \varphi'(v) = 0$ . We can then extend  $\varphi'$  to a homomorphism  $\varphi$  of  $G$  to  $SP_9^*$  by recoloring  $v$  such that  $\varphi(v) = 0'$  since  $s(00') = -1$ .

We have proven that every graph in  $\mathcal{D}_3^c$  admits a homomorphism to  $SP_9^*$  which means that  $SP_9^*$  is universal for  $\mathcal{D}_3$ . This concludes the proof.

To prove the two other upper bounds of Theorem 1, we use the same method on target graphs  $SP_{29}$  and  $TR(SP_{13})$ .

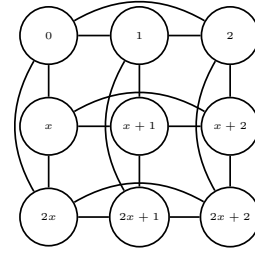


Figure 3: The graph  $SP_9$ , non-edges are negative edges

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