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# The chromatic number and switching chromatic number of 2-edge-colored graphs of bounded degree 

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## Extended Abstract

The notion of homomorphisms of 2-edge-colored graphs has already been studied as a way of extending classical results in graph coloring such as Hadwiger's conjecture. Guenin [5] introduced the notion of switching homomorphisms for its relation with a well-known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [6] as it captures a number of well-known conjectures that can be reformulated using the definition of switching homomorphisms. In this extended abstract, we study homomorphisms of 2-edge colored graphs and switching homomorphisms of bounded degree graphs.

A 2-edge-colored graph $G=(V, E, s)$ is a simple graph $(V, E)$ with two kinds of edges: positive and negative edges. The signature $s: E(G) \rightarrow\{-1,+1\}$ assigns to each edge its sign. In the sequel, $\mathcal{D}_{k}$ (resp. $\mathcal{D}_{k}^{c}$ ) denotes the class of 2-edge-colored graphs (resp. connected 2-edge-colored graphs) with maximum degree $k$.

Given two 2-edge-colored graphs $G$ and $H$, the mapping $\varphi: V(G) \rightarrow V(H)$ is a homomorphism if $\varphi$ maps every edge of $G$ to an edge of $H$ with the same sign. This can be seen as coloring the graph $G$ by using the vertices of $H$ as colors. The target graph $H$ gives us the rules that this coloring must obey. If vertices 1 and 2 in $H$ are connected with a positive (resp. negative) edge, then every pair of adjacent vertices in $G$ colored with 1 and 2 must be connected with a positive (resp. negative) edge. The chromatic number $\chi_{2}(G)$ of a 2-edge-colored graph $G$ is the order of a smallest 2-edge-colored graph $H$ such that $G$ admits a homomorphism to $H$. The chromatic number $\chi_{2}(\mathcal{C})$ of a class of 2-edge-colored graphs $\mathcal{C}$ is the maximum of the chromatic numbers of the graphs in the class. This number can be infinite.

2-edge-colored graphs are, in some sense, similar to oriented graphs since a pair of vertices can be adjacent in two different ways in both kinds of graphs: with a positive or a negative edge in the case of 2-edge-colored graphs, with a toward or a backward arc in the oriented case.

The notion of homomorphism of oriented graphs has been introduced by Courcell [3] in 1994 and has been widely studied since then. Due to the similarity above-mentioned, we try to adapt techniques used to study the homomorphisms of oriented graphs of bounded degree to 2-edge-colored graphs of bounded degree. We also study switching homomorphisms of 2-edge-colored graphs in order to obtain results on signed graphs.

Switching a vertex $v$ of a 2-edge-colored graph corresponds to reversing the signs of all edges incident to $v$.

Two 2-edge-colored graphs $G$ and $G^{\prime}$ are switching equivalent if it is possible to turn $G$ into $G^{\prime}$ after some number of switches. We call the classes created by this equivalence relation switching classes (note that switching classes are equivalent to the notion of signed graphs).

Given two 2-edge-colored graphs $G$ and $H$, the mapping $\varphi: V(G) \rightarrow V(H)$ is a switching homomorphism if there is a graph $G^{\prime}$ switching equivalent to $G$ such that $\varphi$ maps every edge of $G^{\prime}$ to an edge of $H$ with the same sign. The switching chromatic number $\chi_{s}(G)$ of a 2-edge-colored graph $G$ is the order of a smallest 2-edge-colored graph $H$ such that $G$ admits a switching homomorphism to $H$.

Table 1 summarizes results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

The first two lines of Table 1 are more or less folklore. Let us explain in the following the difference that exists between the connected case and the non-connected case for the

|  | $\chi_{2}\left(\mathcal{D}_{k}\right)$ | $\chi_{2}\left(\mathcal{D}_{k}^{c}\right)$ | $\chi_{s}\left(\mathcal{D}_{k}\right)$ | $\chi_{s}\left(\mathcal{D}_{k}^{c}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| $k=1$ | $\chi_{2}\left(\mathcal{D}_{k}\right)=3$ | $\chi_{2}\left(\mathcal{D}_{k}^{c}\right)=2$ | $\chi_{s}\left(\mathcal{D}_{k}\right)=\chi_{s}\left(\mathcal{D}_{k}^{c}\right)=2$ |  |
| $k=2$ | $\chi_{2}\left(\mathcal{D}_{k}\right)=6$ | $\chi_{2}\left(\mathcal{D}_{k}^{c}\right)=5$ | $\chi_{s}\left(\mathcal{D}_{k}\right)=\chi_{s}\left(\mathcal{D}_{k}^{c}\right)=4$ |  |
| $k=3$ | $8 \leq \chi_{2}\left(\mathcal{D}_{k}^{c}\right) \leq \chi_{2}\left(\mathcal{D}_{k}\right) \leq 11$ |  | $6 \leq \chi_{s}\left(\mathcal{D}_{k}\right) \leq 7[1]$ | $\chi_{s}\left(\mathcal{D}_{k}^{c}\right)=6[1]$ |
| $k=4$ | $12 \leq \chi_{2}\left(\mathcal{D}_{k}^{c}\right) \leq \chi_{2}\left(\mathcal{D}_{k}\right) \leq 31$ |  | $10 \leq \chi_{s}\left(\mathcal{D}_{k}^{c}\right) \leq \chi_{s}\left(\mathcal{D}_{k}\right) \leq 16$ |  |
| $k \geq 5$ | $2^{\frac{k}{2}} \leq \chi_{2}\left(\mathcal{D}_{k}^{c}\right) \leq \chi_{2}\left(\mathcal{D}_{k}\right) \leq 2^{k+1}(k-1)^{2}[4]$ | $\chi_{s}\left(\mathcal{D}_{k}^{c}\right) \leq \chi_{s}\left(\mathcal{D}_{k}\right) \leq 2^{k+1}(k-1)^{2}[4]$ |  |  |

Table 1: Results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.
chromatic number of 2-edge-colored graphs with maximum degree 1 or 2 . An edge of a 2 -edge-colored graph has chromatic number 2 and thus $\chi_{2}\left(\mathcal{D}_{1}^{c}\right)=2$; however, a 2-edge-colored graph with two non-adjacent edges, one positive and one negative, has chromatic number 3 (the target graph needs a positive and a negative edge, hence at least three vertices) and thus $\chi_{2}\left(\mathcal{D}_{1}\right)=3$. We therefore have a difference between the chromatic numbers of connected and non-connected 2 -edge-colored graphs with maximum degree 1 . This difference does not exist for switching homomorphisms since a negative edge can be changed into a positive one after a switch. This difference (and lack thereof for switching homomorphisms) appears also in graphs with maximum degree 2 . We have $\chi_{2}\left(\mathcal{D}_{2}\right) \geq 6$ since there is no 2-edge-colored graph on 5 vertices that can color all the four graphs depicted in Figure 1. However, every connected 2-edge-colored graph with maximum degree 2 admits a homomorphism to one of the two graphs on 5 vertices depicted in Figure 2 and thus $\chi_{2}\left(\mathcal{D}_{2}^{c}\right) \leq 5$. In order to color any graph of $\mathcal{D}_{2}$, we need a target graph that contains both graphs depicted in Figure 2 as subgraphs. This is possible with 6 vertices so $\chi_{2}\left(\mathcal{D}_{2}\right)=6$. We do not know yet if this is also the case for graphs with maximum degree at least 3 .


Figure 1: Four examples of 2-edge-colored graphs with chromatic number 5.



Figure 2: Target graphs for connected 2-edge-colored graphs of maximum degree 2.

The last three lines of Table 1 are dedicated to graphs with maximum degree at least 3 . Our main results are the following:

Theorem 1 We have:

- $8 \leq \chi_{2}\left(\mathcal{D}_{3}\right) \leq 11$,
- $12 \leq \chi_{2}\left(\mathcal{D}_{4}\right) \leq 31$,
- $10 \leq \chi_{s}\left(\mathcal{D}_{4}\right) \leq 16$.

In order to find an upper bound for a class of graphs, we need to find a target graph that can color every graph in the class. In the case of oriented homomorphisms, oriented graphs that are antiautomorphic, $K_{n}$-transitive for some $n$, or that have Property $P_{n, k}$ for some $n$ and $k$ are good candidates. We analogously define these properties in term of 2 -edge-colored graphs.

A 2-edge-colored graph $(V, E, s)$ is said to be antiautomorphic if it is isomorphic to $(V, E,-s)$.

A 2-edge-colored graph $G=(V, E, s)$ is said to be $K_{n}$-transitive if for every pair of cliques $\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ in $G$ such that for all $i \neq j, s\left(u_{i} u_{j}\right)=s\left(v_{i} v_{j}\right)$ there exists an automorphism that maps $u_{i}$ to $v_{i}$ for all $i$. For $n=1,2$, or 3 , we say that the graph is vertex, edge, or triangle-transitive, respectively.

A 2-edge-colored graph $G$ has Property $P_{k, n}$ if for every sequence of $k$ distinct vertices $\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ that induces a clique in $G$ and for every sign vector $\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}\right) \in\{-1,+1\}^{k}$ there exist at least $n$ distinct vertices $\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ such that $s\left(v_{i} u_{j}\right)=\alpha_{i}$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Given an integer $q \equiv 1(\bmod 4)$, we consider the family of complete signed Paley graphs $S P_{q}$ built from the field of order $q$ which has the above-mentioned properties. The vertices of $S P_{q}$ are the elements of the field of order $q$ and $s(u v)=+1$ if $u-v$ is a square and $s(u v)=-1$ otherwise.

Lemma 2 ([7]) Graph $S P_{q}$ is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1, \frac{q-1}{2}}$ and $P_{2, \frac{q-5}{4}}$.

Let us consider the following operation. Given a 2 -edge-colored graph $G$, we create the antitwinned graph of $G$ denoted by $\rho(G)$ as follows. Let $G^{+1}, G^{-1}$ be two copies of $G$. The vertex corresponding to $v \in V(G)$ in $G^{i}$ is denoted by $v_{i}, V(\rho(G))=V\left(G^{+}\right) \cup V\left(G^{-}\right)$, $E(\rho(G))=\left\{u_{i} v_{j}: u v \in E(G), i, j \in\{-1,+1\}\right\}$ and $s_{\rho(G)}\left(u_{i} v_{j}\right)=i \times j \times s_{G}(u, v)$.
Lemma 3 ([2]) Let $G$ and $H$ be two 2-edge-colored graphs. The graph $G$ admits a homomorphism to $\rho(H)$ if and only if it admits a switching homomorphism to $H$.

In other words, if a 2-edge-colored graph admits a homomorphism to an antitwinned target graph on $n$ vertices, then it also admits a switching homomorphism to a target graph on $\frac{n}{2}$ vertices. The family $\rho\left(S P_{q}\right)$ also are interesting target graphs (especially for bounding the switching chromatic number since they are antitwinned).

Lemma 4 ([7]) The graph $\rho\left(S P_{q}\right)$ is vertex-transitive, antiautomorphic, and has properties $P_{1, q-1}, P_{2, \frac{q-3}{2}}$, and $P_{3, \max \left(\frac{q-9}{4}, 0\right)}$.

One last family of interesting target graphs are the Tromp-Paley graphs (this construction due to Tromp (unpublished) has been widely used in the case of oriented homomorphisms). Let $S P_{q}^{+}$be $S P_{q}$ with an additional vertex that is connected to every other vertex with a positive edge. The Tromp-Paley graph $T R\left(S P_{q}\right)$ corresponds to $\rho\left(S P_{q}^{+}\right)$. This construction improves the properties of $\rho\left(S P_{q}\right)$ at the cost of having two more vertices. Since Tromp-Paley graphs are antitwinned, they are interesting for bounding the switching chromatic number.

Lemma 5 ([7]) TR(SP $P_{q}$ is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1, q}, P_{2, \frac{q-1}{2}}$, and $P_{3, \frac{q-5}{4}}$.

Bensmail et al. [1] recently proved that every 2-edge-colored graph with maximum degree 3 except the all positive and all negative $K_{4}$ admits a homomorphism to $T R\left(S P_{5}\right)$, hence $\chi_{2}\left(\mathcal{D}_{3}^{c}\right) \leq 12$, and $\chi_{s}\left(\mathcal{D}_{3}^{c}\right) \leq 6$ by Lemma 3 . In the non-connected case, we can easily get $\chi_{2}\left(\mathcal{D}_{3}\right) \leq 14$ and thus $\chi_{s}\left(\mathcal{D}_{3}\right) \leq 7$ by Lemma 3 (it is possible to create an all positive $K_{4}$ and an all negative $K_{4}$ in $T R\left(S P_{5}\right)$ by adding two vertices). Their proof uses a computer to show that a minimal counter-example cannot contain some configurations and then concludes by using the properties of $T R\left(S P_{5}\right)$. Theorem 1 improves the upper bound of 14 to 11 .

Let us give a sketch of proof of the first result of Theorem 1 , namely $\chi_{2}\left(\mathcal{D}_{3}\right) \leq 11$.

Consider the graph $S P_{9}^{*}$ obtained from $S P_{9}$ by adding two new vertices $0^{\prime}$ and $1^{\prime}$ as follows. Take the two vertices 0 and 1 of $S P_{9}$ (note that $s(01)=+1$ ), and link $0^{\prime}$ and $1^{\prime}$ to the vertices of $S P_{9}$ in the same way as 0 and 1 are, respectively; add an edge $0^{\prime} 1^{\prime}$ with $s\left(0^{\prime} 1^{\prime}\right)=-1$; finally we add edges $00^{\prime}$ and $11^{\prime}$ with $s\left(00^{\prime}\right)=-1$ and $s\left(11^{\prime}\right)=+1$. We will prove that every graph from $\mathcal{D}_{3}$ admits a homomorphism to $S P_{9}^{*}$.

We first show that every connected 2-degenerate 2-edge-colored graph with maximum degree 3 admits a homomorphism to $S P_{9}$ by using its structural properties given by Lemma 2 (a unique exception exists and is separately treated).

Let $G$ be a connected 3-regular 2-edge-colored graph. If $G$ is all positive, then we color it using an all positive $K_{4}$ that $S P_{9}^{*}$ contains as a subgraph. Assume now that $G$ is not all positive. Let $u v$ be a negative edge of $G$. We remove $u v$ from $G$ to create a new graph $G^{\prime}$. Graph $G^{\prime}$ is 2-degenerate so it admits a homomorphism $\varphi^{\prime}$ to $S P_{9}$. If $s\left(\varphi^{\prime}(u) \varphi^{\prime}(v)\right)=-1$, then $\varphi^{\prime}$ is also a homomorphism from $G$ to $S P_{9}$.

If $s\left(\varphi^{\prime}(u) \varphi^{\prime}(v)\right)=+1$, then by edge-transitivity of $S P_{9}$ we can recolor the vertices of $G^{\prime}$ such that $\varphi^{\prime}(u)=0$ and $\varphi^{\prime}(v)=1$. We can then extend $\varphi^{\prime}$ to a homomorphism $\varphi$ of $G$ to $S P_{9}^{*}$ by recoloring $u$ and $v$ such that $\varphi(u)=0^{\prime}$ and $\varphi(v)=1^{\prime}$ since $s\left(0^{\prime} 1^{\prime}\right)=-1$.

Finally, if $\varphi^{\prime}(u)=\varphi^{\prime}(v)$, then by vertex-transitivity of $S P_{9}$ we can recolor the vertices of $G^{\prime}$ such that $\varphi^{\prime}(u)=\varphi^{\prime}(v)=0$. We can then extend $\varphi^{\prime}$ to a homomorphism $\varphi$ of $G$ to $S P_{9}^{*}$ by recoloring $v$ such that $\varphi(v)=0^{\prime}$ since $s\left(00^{\prime}\right)=-1$.

We have proven that every graph in $\mathcal{D}_{3}^{c}$ admits a homomorphism to $S P_{9}^{*}$ which means that $S P_{9}^{*}$ is universal for $\mathcal{D}_{3}$. This concludes the proof.

To prove the two other upper bounds of Theorem 1, we use the same method on target graphs $S P_{29}$ and $T R\left(S P_{13}\right)$.

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