r-hued coloring of planar graphs with girth at least 8

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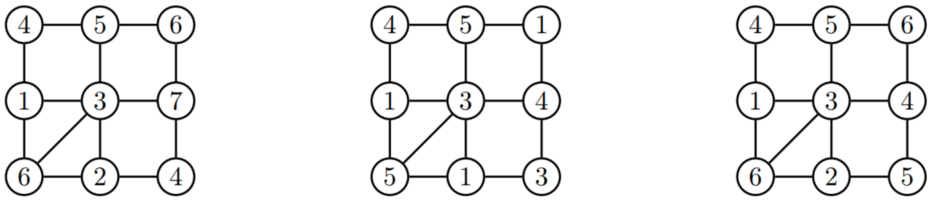
Extended Abstract

In this paper, we consider simple undirected graphs without loops. A *proper k-coloring* of the vertices of a graph $G = (V, E)$ is an assignment of colors from 1 to k such that no two adjacent vertices have the same color. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k so that G has a proper k-coloring. In 1969, Kramer and Kramer introduced the notion of *2-distance k-coloring* [23] which is a proper k-coloring such that no pair of vertices at distance 2 have the same color. The *2-distance chromatic number* of G, denoted $\chi^2(G)$, is the smallest integer k so that G has a 2-distance k-coloring. An example of 2-distance coloring is given in Figure 1a.



(a) A 2-distance 7-coloring.

(b) A 2-hued 5-coloring that is not 2-distance.

(c) A 5-hued 6-coloring which is also 2-distance.

Figure 1: A graph G with $\chi^2(G) = 6$ and $\chi(G) = 3$

Note that for any graph G with maximum degree Δ, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex v with degree Δ, and v itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta(G)^2 + 1$. Moreover, this bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. The Moore graphs of type (3,2) and of type (7,2) are the Petersen graph and the Hoffman-Singleton graph respectively.

In this paper, we focus on *planar graphs* which are graphs that can be drawn in the plane without crossing the edges. One motivation to study this class of graphs is the following famous conjecture stating an upper bound which is linear in Δ:

Conjecture 1 (Wegner, 1977 [31]) *Let G be a planar graph with maximum degree Δ. Then,*

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner showed that the upper bounds of the conjecture are tight. For instance when $\Delta \geq 8$ consider the Wegner graph obtained as follows : take a triangle xyz, multiply each edge $\frac{3}{2}\Delta$ times, subdivide them once, then add an edge between x and y. The case $\Delta \leq 3$ of Conjecture 1 was proved independently by Thomassen [29] and by Hartke *et al.* [17]. For $\Delta \geq 8$, Havet *et al.* [18] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$.

The coefficient before Δ becomes 1 when the *girth*, the length of a shortest cycle, of the graph increases. Extensive researches have been done in this case, and many results have taken the following form: *every planar graph G of girth $g \geq g_0$ and $\Delta(G) \geq \Delta_0$ satisfies $\chi^2(G) \leq \Delta + c(g_0, \Delta_0)$, where $c(g_0, \Delta_0)$ is a constant depending only on g_0 and Δ_0.* Table 1 shows all known such results on the 2-distance chromatic number of planar graphs with fixed girth, up to our own knowledge.

$g_0\backslash\chi^2(G)$	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3	X			$\Delta = 3$[29, 17]				
4	X							
5	X	$\Delta \geq 10^7$[1]	$\Delta \geq 339$[15]	$\Delta \geq 312$[14]	$\Delta \geq 15$[8]	$\Delta \geq 12$[7]	$\Delta \neq 7, 8$[14]	all Δ[30]
6	X	$\Delta \geq 17$[4]	$\Delta \geq 9$[7]		all Δ[10]			
7	$\Delta \geq 16$[19]			$\Delta = 4$[12]				
8	$\Delta \geq 10$[19] $\Delta \geq 9$ (Corollary 4)		$\Delta = 5$[6]					
9	$\Delta \geq 8$[3]	$\Delta = 5$[6]	$\Delta = 3$[13]					
10	$\Delta \geq 6$[19]							
11		$\Delta = 4$[12]						
12	$\Delta = 5$[19]	$\Delta = 3$[5]						
13								
14	$\Delta \geq 4$[2]							
...								
22	$\Delta = 3$[19]							

Table 1: The latest results with a coefficient 1 before Δ in the upper bound of χ^2.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows : "*every planar graph G of girth at least 7 and of Δ at least 16 satisfies* $\chi^2(G) \leq \Delta + 1$". The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs G with $\chi^2(G) = \Delta + 2$ for arbitrarily large Δ[16]. The lack of results for $g \geq 4$ is due to the fact that the Wegner graph without xy has girth 4, and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$ for all Δ.

The "2-distance" condition in 2-distance colorings requires that vertices at distance at most two have different colors. In other words, all neighbors of the same vertex must have different colors. Recently, this condition was generalized and the notion of r-hued coloring was introduced [26]. Let $r, k \geq 1$ be two integers. An *r-hued k-coloring* of the vertices of G is a proper k-coloring of the vertices, such that all vertices are r-hued. A vertex is r-hued if the number of colors in its neighborhood $N_G(v) = \{x | xv \in E\}$ is at least $\min\{d_G(v), r\}$. The *r-hued chromatic number* of G, denoted $\chi_r(G)$, is the smallest integer k so that G has an r-hued k-coloring. It is indeed a generalization of 2-distance colorings which correspond to the case $r \geq \Delta$, as all vertices in the same neighborhood will have different colors. More generally, its link to proper coloring and 2-distance coloring resides in the following equation:

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi^2(G) \qquad (1)$$

Examples of r-hued colorings are given in Figure 1b and Figure 1c.

Similar to the 2-distance chromatic number, the r-hued chromatic number is linear in r when it comes to planar graphs. In 2014, Song *et al.* proposed a generalization of Conjecture 1:

Conjecture 2 (Song *et al.*, 2014 [27]) *Let G be a planar graph. Then,*

$$\chi_r(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2, \\ r+5, & \text{if } 3 \leq r \leq 7, \\ \lfloor \frac{3r}{2} \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

Note that Conjecture 2 implies Conjecture 1 except for the case $r = 3$. Moreover, the only extremal known examples reaching the upper bounds of Conjecture 2 are the same as for Conjecture 1. It is less clear what would be the expected upper bound when $r < \Delta$. In 2018, Song and Lai [28] proved that, if $r \geq 8$, then every planar graph G verifies $\chi_r(G) \leq 2r + 16$. Similar to 2-distance coloring, the coefficient before r in this upper bound becomes 1 for graphs with a higher girth. Table 2 shows all known results of the following form: *let r and r_0 be integers such that $r \geq r_0$, all planar graph G of girth $g(G) \geq g_0$ satisfies $\chi_r(G) \leq r + c(g_0, r_0)$, where $c(g_0, r_0)$ is a constant depending only on g_0 and r_0.* The result from the "9" line and "$r+1$" column reads "for $r \geq 8$, all planar graph G of girth at least 9 satisfies $\chi_r(G) \leq r+1$".

Since $r + 1$ is a trivial lower bound for χ_r, we study the class of planar graphs verifying $\chi_r = r + 1$ and show the following:

$g_0 \backslash \chi_r(G)$	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+6$	$r+7$	$\dots$	$r+10$
3		$r=2$[20][1]	$r=2$[20]	$r=2$[22]			$r=3$[25]		
4									
5					$r \geq 15$[8]				all r[8]
6					$r \geq 3$[24]				
7		$r=2$[22]		$r=3$[21]					
8	$r \geq 9$ (Theorem 3)								
9	$r \geq 8$[9]		$r=3$[21]						
10	$r \geq 6$[9]								
11									
12	$r \geq 5$[9]								
13									
14		$r=3$[11]							

Table 2: The latest results with a coefficient 1 before r in the upper bound of χ_r.

Theorem 3 *If G is a planar graph with $g(G) \geq 8$, then $\chi_r(G) = r+1$ for $r \geq 9$.*

Our proof uses the discharging method and exploits planarity arguments.

For $r \geq \Delta$, Theorem 3 gives an improvement of a result on 2-distance coloring published in [19] (see Table 1):

Corollary 4 *If G is a planar graph with $g(G) \geq 8$ and $\Delta(G) \geq 9$, then $\chi^2(G) = \Delta(G)+1$.*

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