Dynamic monopolies for interval graphs with bounded thresholds

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Abstract

For a graph G and an integer-valued threshold function τ on its vertex set, a dynamic monopoly is a set of vertices of G such that iteratively adding to it vertices u of G that have at least $\tau(u)$ neighbors in it eventually yields the vertex set of G. We show that the problem of finding a dynamic monopoly of minimum order can be solved in polynomial time for interval graphs with bounded threshold functions, but is NP-hard for chordal graphs allowing unbounded threshold functions.

Keywords: Dynamic monopoly; target set selection; chordal graph; interval graph

1 Introduction

Dynamic monopolies are a simple model for various types of viral processes in networks [8–10]. Let G be a finite, simple, and undirected graph. A *threshold function* for G is an integer-valued function whose domain contains the vertex set $V(G)$ of G. Let τ be a threshold function for G. For a set D of vertices of G, the *hull* $H_{(G,\tau)}(D)$ *of* D *in* (G,τ) is the set obtained by starting with the empty set, and iteratively adding vertices u to the current set that belong to D or have at least $\tau(u)$ neighbors in the current set as long as possible. The set D is a *dynamic monopoly* or a *target set* of (G,τ) if $H_{(G,\tau)}(D)$ equals $V(G)$, and the minimum order of a dynamic monopoly of (G,τ) is denoted by $\text{dyn}(G,\tau)$.

The parameter $\text{dyn}(G,\tau)$ is computationally hard even when restricted to instances with bounded threshold functions [4, 7, 9, 11]. Efficient algorithms that work for unbounded threshold functions are known for trees [4, 7, 9], block-cactus graphs [5], graphs of bounded treewidth [2], and graphs whose blocks have bounded order [4]. For bounded threshold functions, some more instances become tractable, and $\text{dyn}(G,\tau)$ can be computed efficiently if G is cubic and $\tau = 2$ [1,11] or if G is chordal and $\tau \leq 2$ [4,5]. The latter result relies on the case $t = 2$ of the following theorem.

Theorem 1.1 (Chiang et al. [5]). *If t is a non-negative integer, G is a t-connected chordal graph, and τ is a threshold function for G with $\tau(u) \leq t$ for every vertex u of G, then* $\text{dyn}(G,\tau) \leq t$.

Since this result holds for arbitrary t, it suggests that there might be an efficient algorithm for chordal graphs and bounded threshold functions. In the present paper we show that this is at least true for interval graphs, which form a prominent subclass of chordal graphs.

Theorem 1.2. *Let t be a non-negative integer. For a given interval graph G, and a given threshold function τ for G with $\tau(u) \leq t$ for every vertex u of G, the value of* $\text{dyn}(G, \tau)$ *can be determined in polynomial time.*

It is open [6] whether $\text{dyn}(G, \tau)$ is fixed parameter tractable for instances with bounded threshold functions when parameterized by the distance to interval graphs. Note that Theorem 1.2 would be a consequence of such a fixed parameter tractability.

As our second result we show that dynamic monopolies remain hard for chordal graphs with unbounded threshold functions.

Theorem 1.3. *For a given triple (G, τ, k), where G is a chordal graph, τ is a threshold function for G, and k is a positive integer, it is NP-complete to decide whether* $\text{dyn}(G, \tau) \leq k$.

2 Proofs

Our approach to prove Theorem 1.2 is to construct a sequence $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_k$ of subgraphs of G in such a way that $G_k = G$, and Theorem 1.1 implies that every minimum dynamic monopoly D for (G, τ) intersects a suitable supergraph ∂G_i of each $G_i - V(G_{i-1})$ in at most t vertices. This enables us to apply dynamic programming efficiently calculating partial information for each G_i by emulating the formation of the hull of D within ∂G_i, and exploit previously computed information for G_{i-1}. A notion that is useful in this context is the one of a *cascade* for a dynamic monopoly D of (G, τ), defined as a linear order $u_1 \prec \ldots \prec u_n$ of the vertices of G such that, for every i in $[n]$, either $u_i \in D$ or $u_i \notin D$ and $|N_G(u_i) \cap \{u_j : j \in [i-1]\}| \geq \tau(u_j)$, where $[k]$ denotes the set of positive integers that are less than or equal to some integer k. A cascade encodes the order in which the vertices of G can be added to the hull of D starting with the empty set. Clearly, every dynamic monopoly admits at least one cascade $\prec$. Furthermore, we may assume that $u \prec v$ for every $u \in D$ and every $v \in V(G) \setminus D$.

We proceed to the proof of our first result.

Proof of Theorem 1.2. Let t, G, and τ be as in the statement. Clearly, we may assume that G is connected. Let n be the order of G. In linear time [3], we can determine an interval representation $(I(u))_{u \in V(G)}$ of G, that is, two distinct vertices u and v of G are adjacent if and only if the intervals $I(u)$ and $I(v)$ intersect. By applying well-known manipulations, we may assume that each interval $I(u)$ is closed, and that the $2n$ endpoints of the n intervals are all distinct.

Let $x_1 < x_2 < \ldots < x_{2n}$ be the endpoints of the intervals. For each $i \in [2n-1]$, let C_i be the set of vertices u of G with $I_i := [x_i, x_{i+1}] \subseteq I(u)$, and let $c_i = |C_i|$. Since each x_i is either the right endpoint of exactly one interval or the left endpoint of exactly one interval, we have $|c_{i+1} - c_i| = 1$ for every $i \in [2n-1]$.

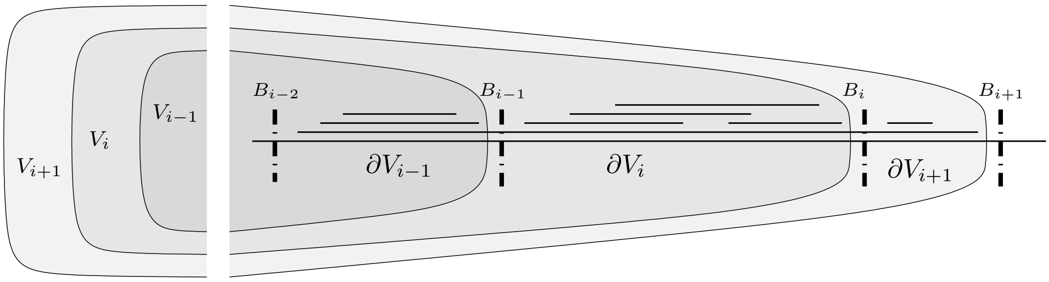

Figure 1: Sets B_i, V_i and ∂V_i on the interval representation of G (for instance, B_i contains all the intervals crossing the corresponding dotted line, ∂V_i contains all the intervals intersecting the zone between B_{i-1} and B_i, and V_i contains all the intervals intersecting the corresponding zone).

Our first claim states a folklore property of interval graphs; we include a proof for the sake of completeness.

Claim 1. *If C is a minimal vertex cut of G, then $C = C_i$ for some $i \in [2n-2] \setminus \{1\}$ with $c_i < \min\{c_{i-1}, c_{i+1}\}$.*

Proof of Claim 1. Clearly, if $i \in [2n-2] \setminus \{1\}$ is such that $c_i < \min\{c_{i-1}, c_{i+1}\}$, then C_i is a minimal vertex cut separating the unique vertex in $C_{i-1} \setminus C_i$ from the unique vertex in $C_{i+1} \setminus C_i$. Conversely, let C be a minimal vertex cut of G. Let u and v be vertices in distinct components of $G - C$. We may assume that the right endpoint $r(u)$ of $I(u)$ is less than the left endpoint $\ell(v)$ of $I(v)$. There are indices i_1 and i_2 such that $[r(u), \ell(v)] = \bigcup_{j=i_1}^{i_2} I_j$. Since $G - C$ contains no path between u and v, there is some index i with $i_1 \leq i \leq i_2$ and $C_i \subseteq C$. Since $G - C_i$ contains no path between u and v, the minimality of C implies $C \subseteq C_i$, and, hence, $C = C_i$. If $i = i_1$, then $c_i < c_{i-1}$, because $I(u)$ ends in i_1. If $i > i_1$ and $c_i > c_{i-1}$, then C_{i-1} is a proper subset of C_i, and also $G - C_{i-1}$ contains no path between u and v, contradicting the minimality of C. Therefore, $c_i < c_{i-1}$, and, by symmetry, also $c_i < c_{i+1}$. □

Let $j_1 < j_2 < \ldots < j_{k-1}$ be the indices i in $[2n-1] \setminus \{1\}$ with $c_i < \min\{c_{i-1}, c_{i+1}, t\}$, and let $j_k = 2n - 1$. For $i \in [k]$, let G_i be the subgraph of G induced by $V_i := C_1 \cup \cdots \cup C_{j_i}$, and let $B_i = C_{j_i}$. Note that B_i contains all vertices in V_i that have a neighbor in $V(G) \setminus V_i$, and that $|B_i| < t$. Let $\partial V_1 = V_1$, and, for $i \in [k] \setminus \{1\}$, let $\partial V_i = (V_i \setminus V_{i-1}) \cup B_{i-1}$. For $i \in [k]$, let ∂G_i be the subgraph of G induced by ∂V_i, cf. Figure 1.

Claim 2. *For every $i \in [k]$, the graph ∂G_i is either a clique of order less than t or a t-connected graph.*

Proof of Claim 2. Let $i \in [k]$. By definition, there are indices i_1 and i_2 with $i_1 < i_2$ such that $\partial V_i = \bigcup_{j=i_1}^{i_2} C_j$. Now, either $c_j < t$ for every index j with $i_1 \leq j \leq i_2$, which implies that there is an index ℓ with $i_1 < \ell < i_2$ and $c_{i_1} < \ldots < c_{\ell-1} < c_\ell > c_{\ell+1} > \ldots > c_{i_2}$, in which case ∂G_i is a clique of order $c_\ell < t$; or there are indices i_1' and i_2' with $i_1 < i_1' \leq i_2' < i_2$ such that

$c_{i_1} < c_{i_1+1} < \ldots < c_{i'_1}$, $c_j \geq t$ for every index j with $i'_1 \leq j \leq i'_2$, and $c_{i'_2} > c_{i'_2+1} > \ldots > c_{i_2}$, in which case Claim 1 implies that ∂G_i is t-connected. $\square$

As explained above, we apply dynamic programming calculating partial information for each G_i. This information should be rich enough to capture the influence on G_i from outside of G_i of all possible cascades of a minimum dynamic monopoly D of (G, τ). Since the only vertices of G_i with neighbors outside of G_i are in B_i, this leads us to considering a localized version of a cascade that specifies (i) all possible intersections of D with B_i, (ii) all possible orders, in which the elements of B_i appear in a cascade, and (iii) all possible amounts of help that each vertex in B_i receives from outside of G_i when it enters the hull of D. Consequently, for every $i \in [k]$, a *local cascade* for G_i is defined as a triple $(X_i, \prec_i, \rho_i)$, where

(i) X_i is a subset of B_i,

(ii) $\prec_i$ is a linear order on B_i such that $u \prec_i v$ for every $u \in X_i$ and every $v \in B_i \setminus X_i$, and

(iii) $\rho_i : B_i \setminus X \to \{0, 1, \ldots, n\}$.

Since $|B_i| \leq t - 1$, there are $O\Big(2^{t-1}(t-1)!(n+1)^{t-1}\Big)$ local cascades for G_i.

For each local cascade for G_i, we are interested in the minimum number of vertices from $V_i \setminus B_i$ that need to be added to X_i in order to obtain the intersection with V_i of some dynamic monopoly that is compatible with the local cascade. More precisely, for a local cascade $(X_i, \prec_i, \rho_i)$ for G_i, let $\text{dyn}_i(X_i, \prec_i, \rho_i)$ be the minimum order of a subset Y_i of $V_i \setminus B_i$ such that the following conditions hold:

(iv) $|(X_i \cup Y_i) \cap \partial V_j| \leq t$ for every $j \in [i]$.

(v) There is a linear extension $u_1 \prec \ldots \prec u_{n(G_i)}$ of $\prec_i$ to $V(G_i)$ such that $u \prec v$ for every $u \in X_i \cup Y_i$ and every $v \in V_i \setminus (X_i \cup Y_i)$, and, for every j in $[n(G_i)]$,

  (a) either $u_j \in X_i \cup Y_i$,

  (b) or $u_j \notin Y_i \cup B_i$ and $\Big|N_G(u_j) \cap \{u_1, \ldots, u_{j-1}\}\Big| \geq \tau(u_j)$,

  (c) or $u_j \in B_i \setminus X_i$ and $\Big|N_G(u_j) \cap \{u_1, \ldots, u_{j-1}\}\Big| \geq \tau(u_j) - \rho(u_j)$.

If no such set Y_i exists, then $\text{dyn}_i(X_i, \prec_i, \rho_i) = \infty$. Note that (a) and (b) are as in the definition of a cascade, and that (c) incorporates the assumption that u_j has $\rho(u_j)$ neighbors outside of G_i when it enters the hull.

By definition, we have $G = G_k$, and $|B_k| = 1$, which implies that there are exactly two local cascades $(X_k, \prec_k, \rho_k)$ for G_k with $\rho_k(u) = 0$ for every $u \in B_k \setminus X_k$; these are the local cascades $(B_k, \emptyset, 0)$ and $(\emptyset, \emptyset, 0)$.

Claim 3. $\text{dyn}(G, \tau) = \min\Big\{1 + \text{dyn}_k(B_k, \emptyset, 0), 0 + \text{dyn}_k(\emptyset, \emptyset, 0)\Big\}$.

Proof of Claim 3. Let D be a dynamic monopoly of (G, τ) of order $\text{dyn}(G, \tau)$.

Our first goal is to show that (iv) holds for $i = k$, $X_k = D \cap B_k$, and $Y_k = D \setminus X_k$. Suppose, for contradiction, that $|D \cap \partial V_j| > t$ for some $j \in [k]$. Clearly, ∂G_j can not be a clique of size less than t in this case. Therefore, by Claim 2, ∂G_j is t-connected, and, by Theorem 1.1, there is a dynamic monopoly D_j of $(\partial G_j, \tau)$ of size at most t. Now, $(D \setminus \partial V_j) \cup D_j$ is a dynamic monopoly of (G, τ) of order less than D, which is a contradiction. Hence, (iv) holds.

Let $u_1 \prec \cdots \prec u_n$ be a cascade for D. Since this cascade is a linear extension of the trivial linear order on the one-element set B_k, we obtain (v) with $\rho_k(u) = 0$ for every $u \in B_k \setminus X_k$. This implies $|X_k| + \text{dyn}_k(X_k, \emptyset, 0) \leq |X_k| + |Y_k| = \text{dyn}(G, \tau)$.

Conversely, let $X_k \subseteq B_k$ be such that $\min\Big\{1 + \text{dyn}_k(B_k, \emptyset, 0), 0 + \text{dyn}_k(\emptyset, \emptyset, 0)\Big\}$ equals $|X_k| + \text{dyn}_k(X_k, \emptyset, 0)$. If Y_k is as in the definition of $\text{dyn}_k(X_k, \emptyset, 0)$, then (v) and $\rho_k = 0$ imply that $X_k \cup Y_k$ is a dynamic monopoly of (G, τ), which implies $\text{dyn}(G, \tau) \leq |X_k| + |Y_k| = |X_k| + \text{dyn}_k(X_k, \emptyset, 0)$. □

Our next two claims imply that the values $\text{dyn}_i(X_i, \prec_i, \rho_i)$ can be determined recursively in polynomial time.

Claim 4. *For every local cascade $(X_1, \prec_1, \rho_1)$ for G_1, the value $\text{dyn}_1(X_1, \prec_1, \rho_1)$ can be computed in polynomial time.*

Proof of Claim 4. Let $v_1 \prec_1 \ldots \prec_1 v_p$ be the linear order $\prec_1$ on B_1. Since $V_1 = \partial V_1$, every subset Y_1 of $V_1 \setminus B_1$ satisfying condition (iv) has at most $t - |X_1|$ elements, which implies that there are only $O(n^t)$ candidates for Y_1. For each such set Y_1, condition (v) holds if and only if

(b′) $B_1 \cup Y_1$ is a dynamic monopoly of (G_1, τ), and

(c′) for every i in $[p]$ with $v_i \in B_1 \setminus X_1$, the hull of the set

$$\Big\{v_j : j \in [i-1]\Big\} \cup X_1 \cup Y_1$$

in $\Big(G_1 - \Big\{v_j : j \in [p] \setminus [i-1]\Big\}, \tau\Big)$ contains at least $\tau(v_i) - \rho(v_i)$ many neighbors of v_i.

In fact, if there is a linear extension $u_1 \prec \ldots \prec u_{n(G_i)}$ of $\prec_1$ satisfying (v), then (a) and (b) imply (b′), and (c) implies (c′). Conversely, if (b′) and (c′) hold, then concatenating cascades for the p hulls considered in (c′) for i from 1 up to p, and removing all but the first appearance of each vertex in the resulting sequence, yields a linear order satisfying (v). Since (b′) and (c′) can be checked efficiently for the polynomially many candidates for Y_1, the claim follows. □

Claim 5. *For every $i \in [k] \setminus \{1\}$ and every local cascade $(X_i, \prec_i, \rho_i)$ for G_i, given the values $\text{dyn}_{i-1}(X_{i-1}, \prec_{i-1}, \rho_{i-1})$ for all local cascades $(X_{i-1}, \prec_{i-1}, \rho_{i-1})$ for G_{i-1}, the value $\text{dyn}_i(X_i, \prec_i, \rho_i)$ can be computed in polynomial time.*

Proof of Claim 5. By definition, we have $B_i \cap V_{i-1} \subseteq B_{i-1}$. Therefore, the two sets $B'_{i-1} = B_i \cap V_{i-1}$ and $B''_{i-1} = B_{i-1} \setminus B'_{i-1}$ partition the set B_{i-1}. Let $X'_{i-1} = X_i \cap B_{i-1}$. Note that $B'_{i-1} = B_i \cap B_{i-1}$, $X'_{i-1} \subseteq B'_{i-1}$, and $B''_{i-1} = B_{i-1} \setminus B_i$, cf. Figure 2.

Our approach to determine $\text{dyn}_i(X_i, \prec_i, \rho_i)$ relies on considering all candidates for the two intersections — later referred to as X''_{i-1} and ∂Y_i — of a set Y_i as in the definition of $\text{dyn}_i(X_i, \prec_i, \rho_i)$ with the two sets B''_{i-1} and $\partial V_i \setminus (B_i \cup B_{i-1})$. By (iv), these two intersections may contain a total of at most $t - |X_i|$ vertices. In order to exploit the given values $\text{dyn}_{i-1}(X_{i-1}, \prec_{i-1}, \rho_{i-1})$,

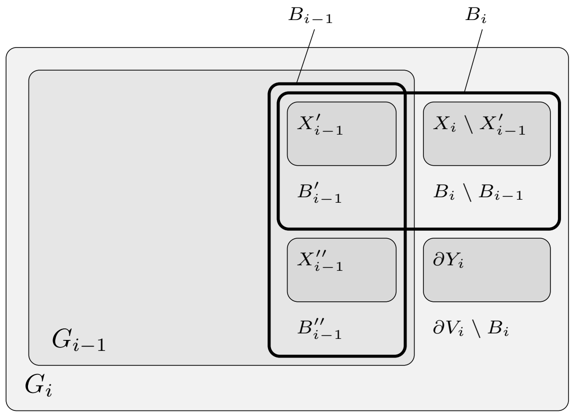

Figure 2: G_i and relevant subsets of V_i.

we decouple ∂G_i from $G_i - B_i$, which leads us to consider all candidates for an extension $\prec_{(i-1,i)}$ of $\prec_i$ to $B_{i-1} \cup B_i$ specifying a possible order in which the vertices in $B_{i-1} \cup B_i$ appear in a cascade. Fixing the triple $\left(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\right)$, we specify that $Y_i \cup X_i$ intersects B_{i-1} in the set $X_{i-1} := X'_{i-1} \cup X''_{i-1}$, and that $\prec_{(i-1,i)}$ contains a linear order $\prec_{i-1}$ on B_{i-1}, which means that we can emulate the formation of the hull within G_i just by working within ∂G_i. We fix ∂Y_i in order to determine the right choice for ρ_{i-1}.

Formally, let $\mathcal{Y}$ be the set of all triples $\left(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\right)$, where

- X''_{i-1} is a subset of B''_{i-1},
- ∂Y_i is a subset of $\partial V_i \setminus (B_i \cup B_{i-1})$,
- $\left|X''_{i-1} \cup \partial Y_i\right| \leq t - |X_i|$, and
- $\prec_{(i-1,i)}$ is a linear extension of $\prec_i$ to $B_{i-1} \cup B_i$ such that $u \prec_{(i-1,i)} v$ for every $u \in X_i \cup X''_{i-1}$ and every $v \in \left(B_{i-1} \cup B_i\right) \setminus \left(X_i \cup X''_{i-1}\right)$.

Note that $\mathcal{Y}$ contains $O\left(2^{t-1} n^t (2t-2)!\right)$ elements.

We now explain how to choose ρ_{i-1} given an element of $\mathcal{Y}$.

Let $\left(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\right)$ be an element of $\mathcal{Y}$.

Let $v_1 \prec_{(i-1,i)} \ldots \prec_{(i-1,i)} v_p$ be the linear order $\prec_{(i-1,i)}$ on $B_{i-1} \cup B_i$.

For every j in $[p]$ with $v_j \in (B_i \cup B_{i-1}) \setminus (X_i \cup X''_{i-1})$, let h_j be the number of neighbors of v_j in the hull of the set

$$\left\{v_\ell : \ell \in [j-1]\right\} \cup X_i \cup X''_{i-1} \cup \partial Y_i$$

in $\left(\partial G_i - \left\{v_\ell : \ell \in [p] \setminus [j-1]\right\}, \tau\right)$.

If $B_i \cup B_{i-1} \cup \partial Y_i$ is not a dynamic monopoly of $(\partial G_i, \tau)$ or if $h_j < \tau(v_j) - \rho_i(v_j)$ for some j in $[p]$ with $v_j \in B_i \setminus (X_i \cup B_{i-1})$, then let $f\left(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\right) = \infty$. Note that these two cases correspond to violations of the conditions (b$'$) and (c$'$) in the proof of Claim 4, that is, in these cases there is no set Y_i as in the definition of $\text{dyn}_i(X_i, \prec_i, \rho_i)$, and, consequently, $\text{dyn}_i(X_i, \prec_i, \rho_i) = \infty$.

Now, we may assume that $B_i \cup B_{i-1} \cup \partial Y_i$ is a dynamic monopoly of $(\partial G_i, \tau)$ and that $h_j \geq \tau(v_j) - \rho_i(v_j)$ for every j in $[p]$ with $v_j \in B_i \setminus (X_i \cup B_{i-1})$. In this case, let $f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big)$ equal

$$|\partial Y_i| + |X''_{i-1}| + \mathrm{dyn}_{i-1}\Big(\Big(X'_{i-1} \cup X''_{i-1}\Big), \prec_{i-1}, \rho_{i-1}\Big),$$

where

- $\prec_{i-1}$ is the restriction of $\prec_{(i-1,i)}$ to B_{i-1},
- $\rho_{i-1}(v_j) = \rho_i(v_j) + h_j$ for every j in $[p]$ with $v_j \in B'_{i-1} \setminus X'_{i-1}$, and
- $\rho_{i-1}(v_j) = h_j$ for every j in $[p]$ with $v_j \in B''_{i-1} \setminus X''_{i-1}$.

Note that also in this case $f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big)$ can be ∞. Note furthermore that, for every $v_j \in B'_{i-1} \setminus X'_{i-1}$, the value of $\rho_{i-1}(v_j)$ has a contributing term $\rho_i(v_j)$ quantifying the help from outside of V_i as well as a contributing term h_j quantifying the help from outside of V_{i-1} but from inside of V_i. For every $v_j \in B''_{i-1} \setminus X''_{i-1}$, there is no help from outside of V_i, that is, the first term disappears. In view of the above explanation, it now follows easily that the best choice within $\mathcal{Y}$ yields $\mathrm{dyn}_i(X_i, \prec_i, \rho_i)$, that is,

$$\mathrm{dyn}_i(X_i, \prec_i, \rho_i) = \min\Big\{f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big) : \Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big) \in \mathcal{Y}\Big\}. \tag{1}$$

In fact, if Y_i is as in the definition of $\mathrm{dyn}_i(X_i, \prec_i, \rho_i)$, and $\prec$ is as in (v) for that set, then

$$\begin{aligned}
\mathrm{dyn}_i(X_i, \prec_i, \rho_i) &= |Y_i| \\
&= |\partial Y_i| + |X''_{i-1}| + |Y_{i-1}| \\
&\geq |\partial Y_i| + |X''_{i-1}| + \mathrm{dyn}_{i-1}\Big(\Big(X'_{i-1} \cup X''_{i-1}\Big), \prec_{i-1}, \rho_{i-1}\Big) \\
&= f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big),
\end{aligned}$$

where $\partial Y_i = Y_i \cap (\partial V_i \setminus B_i)$, $X''_{i-1} = Y \cap B''_{i-1}$, $Y_{i-1} = Y_i \cap (V_{i-1} \setminus B_{i-1})$, $X'_{i-1} = X_i \cap B'_{i-1}$, and $\prec_{i-1}$ is the restriction of $\prec$ to B_{i-1}, where the inequality follows because the set Y_{i-1} satisfies the conditions in the definition of $\mathrm{dyn}_{i-1}\Big(\Big(X'_{i-1} \cup X''_{i-1}\Big), \prec_{i-1}, \rho_{i-1}\Big)$.

Conversely, if $\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big)$ is in $\mathcal{Y}$, and the set Y_{i-1} is as in the definition of $\mathrm{dyn}_{i-1}\Big(\Big(X'_{i-1} \cup X''_{i-1}\Big), \prec_{i-1}, \rho_{i-1}\Big)$, then the set $Y_i = Y_{i-1} \cup X''_{i-1} \cup \partial Y_i$ satisfies the conditions in the definition of $\mathrm{dyn}_i(X_i, \prec_i, \rho_i)$, and, hence,

$$\begin{aligned}
\mathrm{dyn}_i(X_i, \prec_i, \rho_i) &\leq |Y_i| \\
&= |\partial Y_i| + |X''_{i-1}| + |Y_{i-1}| \\
&= |\partial Y_i| + |X''_{i-1}| + \mathrm{dyn}_{i-1}\Big(\Big(X'_{i-1} \cup X''_{i-1}\Big), \prec_{i-1}, \rho_{i-1}\Big) \\
&= f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big),
\end{aligned}$$

which shows (1).

Since $\mathcal{Y}$ has polynomially many elements, and each $f\Big(X''_{i-1}, \partial Y_i, \prec_{(i-1,i)}\Big)$ can be determined in polynomial time, the claim follows. □

Since $k \leq n$, and there are only polynomially many local cascades for each G_i, the Claims 3, 4, and 5 complete the proof. $\square$

The algorithm described in the proof of Theorem 1.2 can easily be modified in such a way that it also determines a minimum dynamic monopoly of (G, τ) within the same time bound. While many ideas used in this proof extend to chordal graphs, the number of choices for the linear orders $\prec$ seems to be a problem for the extension of Theorem 1.2 to chordal graphs.

We proceed to the proof of our second result.

Proof of Theorem 1.3. Since the hull of a set in (G, τ) can be determined in polynomial time, the considered problem is in NP. In order to show hardness, we describe a reduction from the NP-complete problem Vertex Cover restricted to cubic graphs. Therefore, let G be a cubic graph of order n. Let G' arise from the complete graph K with vertex set $V(G)$ by adding, for every edge uv of G, a clique $K(uv)$ of order n as well as all $2n$ possible edges between $K(uv)$ and $\{u, v\}$. Let

$$\tau : V(G') \to \mathbb{N}_0 : u \mapsto \begin{cases} 3n+3 & \text{, if } u \in V(G), \text{ and} \\ 1 & \text{, otherwise.} \end{cases}$$

In order to complete the proof, it suffices to show that the vertex cover number of G equals $\mathrm{dyn}(G', \tau)$.

First, suppose that X is a vertex cover of G. Let H be the hull of X in (G', τ). Since every vertex in $V(G') \setminus V(G)$ has a neighbor in X and threshold value 1, the set H contains $V(G') \setminus V(G)$. Therefore, for every vertex u of G' in $V(G) \setminus X$, the set H contains all three neighbors of u in $V(G)$ as well as all $3n$ neighbors of u in $V(G') \setminus V(G)$, which implies that X is a dynamic monopoly of (G', τ).

Next, suppose that D is a dynamic monopoly of (G', τ). Since replacing a vertex in $D \setminus V(G)$ by some neighbor in $V(G)$ yields a dynamic monopoly, we may assume that $D \subseteq V(G)$. Suppose, for a contradiction, that $u_r, u_s \not\in D$ for some edge $u_r u_s$ in G, where $u_1 \prec \ldots \prec u_{n'}$ is a cascade for D, and $r < s$. It follows that $\{u_j : j \in [r-1]\}$ contains no vertex of $K(u_r u_s)$, which implies the contradiction $|N_{G'}(u_r) \cap \{u_j : j \in [r-1]\}| \leq 2 + 2n$. Hence, D is a vertex cover of G, which completes the proof. $\square$

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