Relating broadcast independence and independence

Stéphane Bessy, Dieter Rautenbach

Relating broadcast independence and independence

S. Bessy[1]   D. Rautenbach[2]

[1] Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier,
Montpellier, France, `stephane.bessy@lirmm.fr`

[2] Institute of Optimization and Operations Research, Ulm University,
Ulm, Germany, `dieter.rautenbach@uni-ulm.de`

Abstract

An independent broadcast on a connected graph G is a function $f : V(G) \to \mathbb{N}_0$ such that, for every vertex x of G, the value $f(x)$ is at most the eccentricity of x in G, and $f(x) > 0$ implies that $f(y) = 0$ for every vertex y of G within distance at most $f(x)$ from x. The broadcast independence number $\alpha_b(G)$ of G is the largest weight $\sum_{x \in V(G)} f(x)$ of an independent broadcast f on G. Clearly, $\alpha_b(G)$ is at least the independence number $\alpha(G)$ for every connected graph G. Our main result implies $\alpha_b(G) \leq 4\alpha(G)$. We prove a tight inequality and characterize all extremal graphs.

Keywords: broadcast independence; independence

1 Introduction

In his PhD thesis [6] Erwin introduced the notions of broadcast domination and broadcast independence in graphs, cf. also [5]. While broadcast domination was studied in detail [3, 7–11], only little research exists on broadcast independence [1, 2]. In the present paper we relate broadcast independence to ordinary independence in graphs; one of the most fundamental and well studied notions in graph theory.

We consider finite, simple, and undirected graphs, and use standard terminology and notation. Let $\mathbb{N}_0$ be the set of nonnegative integers. For a connected graph G, a function $f : V(G) \to \mathbb{N}_0$ is an *independent broadcast on G* if

(B1) $f(x) \leq \mathrm{ecc}_G(x)$ for every vertex x of G, where $\mathrm{ecc}_G(x)$ is the eccentricity of x in G, and

(B2) $\mathrm{dist}_G(x, y) > \max\{f(x), f(y)\}$ for every two distinct vertices x and y of G with $f(x), f(y) > 0$, where $\mathrm{dist}_G(x, y)$ is the distance of x and y in G.

The *weight* of f is $\sum_{x \in V(G)} f(x)$. The *broadcast independence number* $\alpha_b(G)$ of G is the maximum weight of an independent broadcast on G, and an independent broadcast on G of weight $\alpha_b(G)$ is *optimal*.[1] Let $\alpha(G)$ be the usual independence number of G, that is, $\alpha(G)$ is the maximum cardinality of an independent set in G, which is a set of pairwise nonadjacent vertices of G. For an integer k, let $[k]$ be the set of all positive integers at most k, and let $[k]_0 = \{0\} \cup [k]$.

Clearly, assigning the value 1 to every vertex in an independent set in some connected graph G, and 0 to all remaining vertices of G, yields an independent broadcast on G, which implies

$$\alpha_b(G) \geq \alpha(G) \text{ for every connected graph } G.$$

A consequence of our main result is that

$$\alpha_b(G) \leq 4\alpha(G) \text{ for every connected graph } G.$$

The fact that the broadcast independence number and the independence number are within a constant factor from each other immediately implies the computational hardness of the broadcast independence number, and also yields efficient constant factor approximation algorithms for the broadcast independence number on every class of graphs for which the independence number can efficiently be approximated within a constant factor.

In order to phrase our main result, we introduce some special graphs. For a positive integer k, a graph H is a *k-strip with partition* $(B_0, \ldots, B_k)$ if $V(H)$ can be partitioned into k nonempty cliques $B_0, \ldots, B_k$ such that

- B_0 contains a unique vertex x,

- all vertices in B_i have distance i in H from x, and

---

[1]Note that, for a disconnected graph G, (B1) and (B2) allow to assign an arbitrarily large value to one vertex in each component of G, which means that the weight of independent broadcasts on G would be unbounded. To avoid this issue, $\mathrm{ecc}_G(x)$ in (B1) could be replaced by the eccentricity of x in the connected component of G that contains x.

- B_i is completely joined to B_{i+1} for every even index i in $[k-1]_0$.

For a positive integer k, let $\mathcal{G}_2(k)$ be the class of all connected graphs that arise from the disjoint union of two $(2k+1)$-strips H_1 with partition $(B_0^1, \ldots, B_{2k+1}^1)$ and H_2 with partition $(B_0^2, \ldots, B_{2k+1}^2)$ by adding some edges between B_{2k+1}^1 and B_{2k+1}^2. An example of such a graph is depicted in Figure 1.
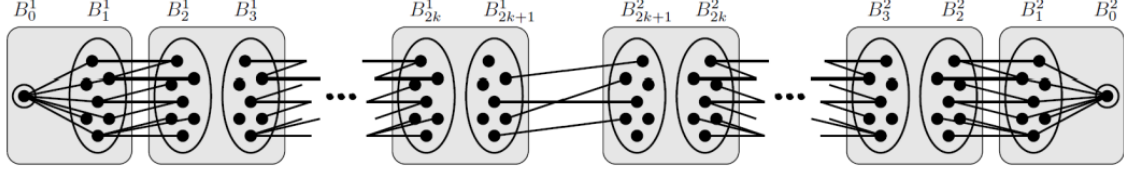

Figure 1: A graph from the family $\mathcal{G}_2(k)$. The vertices in each gray box form a clique.

For positive integers k and ℓ with $\ell \geq 2$, let $\mathcal{G}_0(k, \ell)$ be the class of all graphs that arise from the disjoint union of ℓ $2k$-strips $H_1, \ldots, H_\ell$, where H_i has partition $(B_0^i, \ldots, B_{2k}^i)$ for i in $[\ell]$, and a possibly empty set R of vertices by adding all possible edges within $R \cup \bigcup_{i=1}^{\ell} B_{2k}^i$. A graph from the family $\mathcal{G}_0(k, \ell)$ is depicted in Figure 2.
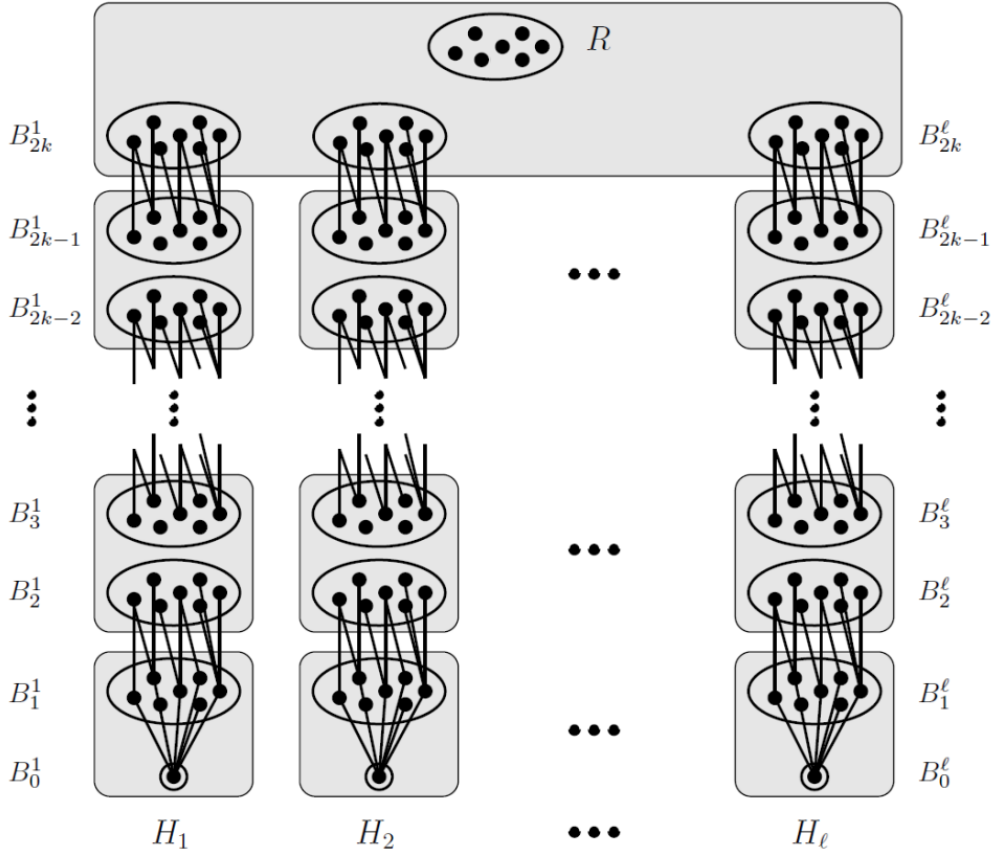

Figure 2: A graph from the family $\mathcal{G}_0(k, \ell)$. Also here, the vertices in each gray box form a clique.

Finally, let

$$\mathcal{G}_2 = \bigcup_{k \geq 1} \mathcal{G}_2(k) \qquad \text{and} \qquad \mathcal{G}_0 = \bigcup_{k \geq 1} \bigcup_{\ell \geq 2} \mathcal{G}_0(k, \ell).$$

3

The following is our main result; proofs are given in the following section.

Theorem 1.1. *If G is a connected graph such that G has diameter at least 3 or $\alpha(G) \geq 3$, and f is an optimal broadcast on G, then*

$$\alpha_b(G) \leq 4\alpha(G) - 4\min\left\{1, \frac{2\alpha(G)}{f_{\max} + 2}\right\}, \tag{1}$$

where $f_{\max} = \max\{f(x) : x \in V(G)\}$. Equality holds in (1) if and only if $G \in \mathcal{G}_0 \cup \mathcal{G}_2$.

The assumption that G has diameter at least 3 or $\alpha(G) \geq 3$ excludes some trivial cases; suppose that a nonempty connected graph G has diameter at most 2 and $\alpha(G) \leq 2$. If $\alpha(G) = 1$, then G is a clique, which implies $\alpha_b(G) = \alpha(G)$, and, if $\alpha(G) = 2$, then (B1) and (B2) imply $\alpha_b(G) = 2$, that is, both parameters are equal in these cases.

2 Proofs

For the proof of Theorem 1.1, we need some properties of the graphs in $\mathcal{G}_0 \cup \mathcal{G}_2$.

Lemma 2.1. *Let k and ℓ be positive integers with $\ell \geq 2$.*

(i) *If $G \in \mathcal{G}_2(k)$, then $\alpha(G) = 2k + 2$, $\alpha_b(G) = 8k + 4$, and $\max\{f(x) : x \in V(G)\} = 4k + 2$ for every optimal independent broadcast f on G.*

(ii) *If $G \in \mathcal{G}_0(k, \ell)$, then $\alpha(G) = k\ell + 1$, $\alpha_b(G) = 4k\ell$, and $\max\{f(x) : x \in V(G)\} = 4k$ for every optimal independent broadcast f on G.*

Proof. We only give details for the proof of (ii); the simpler proof of (i) can be obtained in a similar way. Let $G \in \mathcal{G}_0(k, \ell)$. Let $H_1, \ldots, H_\ell$ be as in the definition of $\mathcal{G}_0(k, \ell)$.

Since $B_{2i}^j \cup B_{2i+1}^j$ is a clique for every i in $[k-1]_0$ and every $j \in [\ell]$, and since $R \cup \bigcup_{i=1}^{\ell} B_{2k}^i$ is a clique, we obtain $\alpha(G) \leq k\ell + 1$. Since a set containing one vertex from B_{2i}^j for every i in $[k-1]_0$ and every $j \in [\ell]$, and one vertex from B_{2k}^1 is independent, we obtain $\alpha(G) = k\ell + 1$.

Let f be an optimal independent broadcast on G. Let j be an arbitrary index in $[\ell]$. Let $i_1, \ldots, i_r$ be all indices such that $0 \leq i_1 < \ldots < i_p \leq 2k - 1$, and f has a positive value on some vertex x_q in $B_{i_q}^j$ for every q in $[p]$. Since each B_i^j is a clique, the vertices $x_1, \ldots, x_p$ are unique. By the structure of G, the distance between a vertex in B_r^j and a vertex in B_s^j for r and s with $r, s \in [2k]_0$ and $r < s$ is at most $s - r + 1$, and at most $s - r$ if $r = 0$. Therefore, (B2) implies that $i_{q+1} \geq i_q + f(x_q)$ for every q in $[p-1]$, and that $i_2 \geq i_1 + f(x_1) + 1$ if $p \geq 2$ and $i_1 = 0$. If $p \geq 2$ and $i_1 > 0$, then

$$\sum_{q=1}^{p-1} f(x_q) \leq \sum_{q=1}^{p-1} (i_{q+1} - i_q) = i_p - i_1 \leq i_p - 1,$$

and, if $p \geq 2$ and $i_1 = 0$, then

$$\sum_{q=1}^{p-1} f(x_q) \leq (i_2 - i_1 - 1) + \sum_{q=2}^{p-1} (i_{q+1} - i_q) = i_p - i_1 - 1 \leq i_p - 1,$$

that is, the same bound holds in both cases.

First, we assume that f has a positive value on some vertex x in $R \cup \bigcup_{i=1}^{\ell} B_{2k}^i$. By the structure of G, we have $f(x) \leq \mathrm{ecc}_G(x) \leq 2k+1$. (B2) implies $f(x_p) \leq \mathrm{dist}_G(x_p, x) - 1 \leq 2k - i_p$. Hence, $\sum_{q=1}^{p} f(x_q) \leq 2k - 1$ if $p \geq 2$, and $\sum_{q=1}^{p} f(x_q) \leq 2k$ if $p = 1$ and $i_1 = 0$. Since j was chosen arbitrarily, we obtain $\alpha_b(G) \leq 2k\ell + 2k + 1$.

Next, we assume that f is 0 on $R \cup \bigcup_{i=1}^{\ell} B_{2k}^i$. This implies $f(x_p) \leq \mathrm{ecc}_G(x_p) = 4k - i_p + 1 \leq 4k + 1$. If $f(x_p) = \mathrm{ecc}_G(x_p)$, then, by (B2), x_p is the only vertex of G with a positive value of f, and, hence, $\alpha_b(G) \leq 4k + 1$. If $f(x_p) \leq 4k - i_p$, then $\sum_{q=1}^{p} f(x_q) \leq 4k - 1$ if $p \geq 2$, and $\sum_{q=1}^{p} f(x_q) \leq 4k$ if $p = 1$ and $i_1 = 0$. Since j was chosen arbitrarily, we obtain $\alpha_b(G) \leq 4k\ell$. Altogether, we obtain

$$\alpha_b(G) \leq \max\{2k\ell + 2k + 1, 4k + 1, 4k\ell\} = 4k\ell.$$

Since the function f^* that has value $4k$ on every vertex in $\bigcup_{i=1}^{\ell} B_0^i$ and value 0 everywhere else is an independent broadcast on G of weight $4k\ell$, we conclude

$$\alpha_b(G) = 4k\ell.$$

Since $\max\{2k\ell + 2k + 1, 4k + 1\} < 4k\ell$, the above arguments actually imply that f^* is the unique optimal broadcast on G, which completes the proof. $\qquad\square$

We are now in a position to prove our main result.

Proof of Theorem 1.1. Let $X = \{x \in V(G) : f(x) > 0\}$. For every vertex x in X and every nonnegative integer i, let

$$
\begin{aligned}
B_i(x) &= \{y \in V(G) : \mathrm{dist}_G(x, y) = i\}, \\
B(x) &= \bigcup_{i=0}^{\left\lfloor \frac{f(x)}{2} \right\rfloor} B_i(x), \\
\partial B(x) &= B_{\left\lfloor \frac{f(x)}{2} \right\rfloor}(x), \text{ and} \\
R &= V(G) \setminus \bigcup_{x \in X} B(x).
\end{aligned}
$$

If there are two distinct vertices x and x' in X such that the sets $B(x)$ and $B(x')$ intersect, then

$$\mathrm{dist}_G(x, x') \leq \frac{f(x)}{2} + \frac{f(x')}{2} \leq \max\{f(x), f(x')\},$$

which contradicts (B2). Hence,

the sets $B(x)$ for x in X are disjoint.

Note that no vertex y in $B(x) \setminus \partial B(x)$ has a neighbor outside of $B(x)$. For every x in X, let $p(x)$ be an arbitrary vertex in $\partial B(x)$, and let $P(x)$ be a shortest path in G between x and $p(x)$. Note that

$P(x)$ has order $\left\lfloor \frac{f(x)+2}{2} \right\rfloor$, that x and $p(x)$ coincide if and only if $f(x) = 1$, and that $p(x)$ is the only vertex on $P(x)$ that may have neighbors outside of $B(x)$.

For $i \in \{0, 1, 2, 3\}$, let $X_i = \{x \in X : f(x) \mod 4 \equiv i\}$. For every x in $X_0 \cup X_1$, the path $P(x)$ contains a unique independent set $I(x)$ of order $\left\lfloor \frac{f(x)+4}{4} \right\rfloor$ that contains $p(x)$, and for every x in $X_2 \cup X_3$, the path $P(x)$ contains a unique independent set $I(x)$ of order $\left\lfloor \frac{f(x)+2}{4} \right\rfloor$ that does not contain $p(x)$. The next table summarizes the different cases.

$f(x) \mod 4$	$\left\lfloor \frac{f(x)}{2} \right\rfloor \mod 2$	$|P(x)| \mod 2$	$|P(x)|$	$|I(x)|$
0	0	1	$\frac{f(x)+2}{2}$	$\frac{f(x)+4}{4}$ (and $I(x)$ contains $p(x)$)
1	0	1	$\frac{f(x)+1}{2}$	$\frac{f(x)+3}{4}$ (and $I(x)$ contains $p(x)$)
2	1	0	$\frac{f(x)+2}{2}$	$\frac{f(x)+2}{4}$ (and $I(x)$ does not contain $p(x)$)
3	1	0	$\frac{f(x)+1}{2}$	$\frac{f(x)+1}{4}$ (and $I(x)$ does not contain $p(x)$)

Table 1: Values of different parameters according to $f(x) \mod 4$.

We consider three cases.

Case 1 $X_0 = X_3 = \emptyset$.

Let $I = \bigcup_{x \in X} I(x)$. Suppose, for a contradiction, that I is not independent. Since $I(x)$ contains $p(x)$ only if x belongs to X_1, it follows that there are two distinct vertices x and x' in X_1 such that $p(x)$ is adjacent to $p(x')$. Now,

$$\mathrm{dist}_G(x, x') \leq |P(x)| + |P(x')| - 1 \leq \frac{f(x)+1}{2} + \frac{f(x')+1}{2} - 1 \leq \max\{f(x), f(x')\},$$

which contradicts (B2). Hence, I is independent. Since $X = X_1 \cup X_2$ using Table 1 we obtain

$$|I(x)| \geq \frac{f(x)+2}{4}$$

for every x in X. Since $f_{\max} \cdot |X| \geq \alpha_b(G)$, we obtain

$$\alpha(G) \geq |I| = \sum_{x \in X} |I(x)| \geq \sum_{x \in X} \frac{f(x)+2}{4} = \frac{1}{4}\left(\alpha_b(G) + 2|X|\right) \geq \frac{1}{4}\alpha_b(G)\left(1 + \frac{2}{f_{\max}}\right), \qquad (2)$$

and, hence,

$$\alpha_b(G) \leq 4\left(1 - \frac{2}{f_{\max}+2}\right)\alpha(G). \qquad (3)$$

Case 2 $X_0 = \emptyset$ and $X_3 \neq \emptyset$.

Let x_3 be some vertex in X_3. By (B1), we may assume that $p(x_3)$ is chosen in such a way that it has a neighbor y_3 outside of $B(x_3)$. Suppose, for a contradiction, that y_3 belongs to $B(x)$ for some x in

6

X. If $f(x_3) \geq f(x)$, then

$$\text{dist}_G(x_3, x) \leq |P(x_3)| + |P(x)| - 1 \leq \left\lfloor \frac{f(x_3) + 2}{2} \right\rfloor + \left\lfloor \frac{f(x) + 2}{2} \right\rfloor - 1 \leq \frac{f(x_3) + 1}{2} + \frac{f(x_3) + 1}{2} - 1 = f(x_3),$$

which contradicts (B2), and, if $f(x_3) < f(x)$, then $X_0 = \emptyset$ implies $f(x_3) \leq f(x) - 2$, and, hence,

$$\text{dist}_G(x_3, x) \leq |P(x_3)| + |P(x)| - 1 \leq \left\lfloor \frac{f(x_3) + 2}{2} \right\rfloor + \left\lfloor \frac{f(x) + 2}{2} \right\rfloor - 1 \leq \left\lfloor \frac{f(x)}{2} \right\rfloor + \left\lfloor \frac{f(x) + 2}{2} \right\rfloor - 1 \leq f(x),$$

which contradicts (B2). Hence

$$y_3 \in R.$$

Let $I = \{y_3\} \cup \bigcup_{x \in X} I(x)$. Suppose, for a contradiction, that I is not independent. In view of the argument in Case 1, it follows that y_3 is adjacent to a vertex $p(x)$ for some x in X. As $p(x)$ has a neighbor outside of $B(x)$, we have $x \in X_0 \cup X_1 = X_1$ in this case. If $f(x_3) \geq f(x)$, then $f(x) \leq f(x_3) - 2$, and

$$\text{dist}_G(x_3, x) \leq |P(x_3)| + |P(x)| \leq \frac{f(x_3) + 1}{2} + \frac{f(x) + 1}{2} \leq \frac{f(x_3) + 1}{2} + \frac{f(x_3) - 1}{2} = f(x_3),$$

which contradicts (B2), and, if $f(x_3) \leq f(x)$, then $f(x_3) \leq f(x) - 2$, and

$$\text{dist}_G(x_3, x) \leq |P(x_3)| + |P(x)| \leq \frac{f(x_3) + 1}{2} + \frac{f(x) + 1}{2} \leq \frac{f(x) - 1}{2} + \frac{f(x) + 1}{2} = f(x),$$

which contradicts (B2). Hence, I is independent. Since $X_0 = \emptyset$, by Table 1 we obtain

$$|I(x)| \geq \frac{f(x) + 1}{4}$$

for every x in X. As before, $f_{\max} \cdot |X| \geq \alpha_b(G)$, and, hence,

$$\alpha(G) \geq 1 + \sum_{x \in X} |I(x)| \geq 1 + \sum_{x \in X} \frac{f(x) + 1}{4} = 1 + \frac{1}{4} \left(\alpha_b(G) + |X| \right) \geq 1 + \frac{1}{4} \alpha_b(G) \left(1 + \frac{1}{f_{\max}} \right),$$

which implies

$$\alpha_b(G) \leq 4 \left(1 - \frac{1}{f_{\max} + 1} \right) (\alpha(G) - 1). \tag{4}$$

Case 3 $X_0 \neq \emptyset$.

Let x_0 be some vertex in X_0, and let

$$I = I(x_0) \cup \bigcup_{x \in X_0 \setminus \{x_0\}} I(x) \setminus \{p(x)\} \bigcup_{x \in \cup X_1 \cup X_2 \cup X_3} I(x).$$

Exactly as in Case 1, it follows that $I \setminus \{p(x_0)\}$ is independent. Suppose, for a contradiction, that I itself is not independent. This implies that the vertex $p(x_0)$, which lies in $I(x_0)$, is adjacent to a vertex $p(x)$ for some x in X. As $p(x) \in I$ and $p(x)$ has a neighbor outside of $B(x)$, we have $x \in X_1$.

So if $f(x) \geq f(x_0)$, then $f(x_0) \leq f(x) - 1$ and, hence,

$$\text{dist}_G(x_0, x) \leq |P(x)| + |P(x_0)| - 1 \leq \frac{f(x) + 1}{2} + \frac{f(x_0) + 2}{2} - 1 \leq \frac{f(x) + 1}{2} + \frac{f(x) + 1}{2} - 1 = f(x),$$

which contradicts (B2), and, if $f(x) \leq f(x_0)$, then $f(x) \leq f(x_0) - 3$ and, hence,

$$\text{dist}_G(x_0, x) \leq |P(x)| + |P(x_0)| - 1 \leq \frac{f(x) + 1}{2} + \frac{f(x_0) + 2}{2} - 1 \leq \frac{f(x_0) - 2}{2} + \frac{f(x_0) + 2}{2} - 1 < f(x_0),$$

which again contradicts (B2). Hence, I is independent. Since $|I(x) \setminus \{p(x)\}| = \frac{f(x)}{4}$ for x in X_0, and $|I(x)| > \frac{f(x)}{4}$ for x in $X \setminus X_0$, we obtain

$$\alpha(G) \geq |I| = 1 + \sum_{x \in X_0} |I(x) \setminus \{p(x)\}| + \sum_{x \in X \setminus X_0} |I(x)| \geq 1 + \sum_{x \in X} \frac{f(x)}{4} = 1 + \frac{\alpha_b(G)}{4}, \tag{5}$$

and, hence,

$$\alpha_b(G) \leq 4\alpha(G) - 4. \tag{6}$$

Note that the inequality (6) is always strictly weaker than the inequality (4), and hence, the three inequalities (3), (4), and (6) together imply (1).

We proceed to the characterization of the extremal graphs. Lemma 2.1 implies that all graphs in $\mathcal{G}_0 \cup \mathcal{G}_2$ satisfy (1) with equality. Now, let G and f be such that (1) holds with equality. Since equality in (1) can not be achieved in Case 2, either Case 1 or Case 3 applies to G.

We consider two cases.

Case A *Either $2\alpha(G) > f_{\max} + 2$, or $2\alpha(G) \leq f_{\max} + 2$ and Case 3 applies to G.*

Since $2\alpha(G) > f_{\max} + 2$ implies $4\left(1 - \frac{2}{f_{\max}+2}\right)\alpha(G) < 4\alpha(G) - 4$, necessarily Case 3 applies to G, and we use the notation from that case. It follows that (6), and, hence, also (5) hold with equality. Since $|I(x)| > \frac{f(x)}{4}$ for x in $X \setminus X_0$, this implies

$$X = X_0.$$

We may assume that x_0 was chosen such that $f(x_0) = f_{\max}$.

If $f(x_1) < f_{\max}$ for some x_1 in X, then $f(x_1) \leq f(x_0) - 4$. Suppose, for a contradiction, that $p(x_0)$ and $p(x_1)$ are adjacent. In this case

$$\text{dist}_G(x_0, x_1) \leq \frac{f(x_0) + 4}{4} + \frac{f(x_1) + 4}{4} - 1 \leq \frac{f(x_0) + 4}{4} + \frac{f(x_0)}{4} - 1 = f(x_0),$$

which contradicts (B2). Hence, $I \cup \{p(x_1)\}$ is independent, which implies the contradiction $\alpha(G) > |I|$. Hence,

$$f(x) = f_{\max} \text{ for every } x \text{ in } X.$$

Let the integer k be such that $f_{\max} = 4k$.

If there is some x in X such that $\partial B(x)$ contains two nonadjacent vertices p and p', then $(I \setminus \{p(x_0)\}) \cup \{p, p'\}$ is independent, which implies the contradiction $\alpha(G) > |I|$. Hence, $\partial B(x)$ is a clique for every x in X. If there are two distinct vertices x and x' in X for which $p(x)$ and $p(x')$ are not adjacent, then $(I \setminus \{p(x_0)\}) \cup \{p(x), p(x')\}$ is independent, which implies the contradiction $\alpha(G) > |I|$. Since $p(x)$ was an arbitrary vertex in $\partial B(x)$, it follows that

$$\bigcup_{x \in X} \partial B(x) \text{ is a clique.}$$

Since G has diameter at least 3 or $\alpha(G) \geq 3$, and f is an optimal broadcast on G, it follows that

$$|X| \geq 2.$$

If R is not a clique, then adding two nonadjacent vertices from R to $I \setminus \{p(x_0)\}$ yields an independent set, which implies the contradiction $\alpha(G) > |I|$. Hence,

$$R \text{ is a clique.}$$

If some vertex p in $\bigcup_{x \in X} \partial B(x)$ is not adjacent to some vertex y in R, then we may assume that x_0 and $p(x_0)$ have been chosen such that $p(x_0) = p$, and $I \cup \{y\}$ is independent, which implies the contradiction $\alpha(G) > |I|$. Hence,

$$R \text{ is completely joined to } \bigcup_{x \in X} \partial B(x).$$

Let x be an arbitrary vertex in X, and let $H = G[B(x) \setminus \partial B(x)]$. Recall that

$$B(x) \setminus \partial B(x) = B_0(x) \cup \ldots \cup B_{2k-1}(x),$$

that $B_0(x)$ contains only x, and that there are no edges between $B_i(x)$ and $B_j(x)$ if $|j - i| \geq 2$.

If $\alpha(H) > k$, then we may assume that x_0 is distinct from x, and adding a maximum independent set in H to the set $I \setminus (I(x) \setminus \{p(x)\})$ yields an independent set in G, which implies the contradiction $\alpha(G) > |I|$. Hence,

$$\alpha(H) = k.$$

If $B_i(x)$ is not a clique for some i in $[2k - 1]$, then a set containing

- two nonadjacent vertices from $B_i(x)$, and

- one vertex from $B_j(x)$ for every j in $[2k - 1]_0$ such that j and i have the same parity modulo 2

is an independent set in H with more than k vertices, which is a contradiction. Hence,

$$B_i(x) \text{ is a clique for every } i \text{ in } [2k - 1]_0.$$

If there is an even integer i in $[2k - 1]$ such that some vertex x in $B_i(x)$ is not adjacent to some vertex x' in $B_{i+1}(x)$, then a set

- containing x and x',

9

- one vertex from $B_j(x)$ for every even j in $[2k-1]_0$ less than i, and

- one vertex from $B_j(x)$ for every odd j in $[2k-1]_0$ larger than $i+1$

is an independent set in H with more than k vertices, which is a contradiction. Hence,

$$B_{2i}(x) \text{ is completely joined to } B_{2i+1}(x) \text{ for every } i \text{ in } [k-1]_0.$$

Since x was an arbitrary vertex in X, at this point it follows that G contains a graph G_0 from $\mathcal{G}_0(k, \ell)$ with $\ell = |X|$ as a spanning subgraph. Since adding any further edge e to G_0 such that $G_0 + e \notin \mathcal{G}_0(k, \ell)$ results in a graph that has less than ℓ vertices of eccentricity $f_{\max} = 4k$, we obtain $G \in \mathcal{G}_0(k, \ell)$, which completes the proof in this case.

Case B $2\alpha(G) \leq f_{\max} + 2$ and Case 1 applies to G.

We use the notation from Case 1. Since $4\left(1 - \frac{2}{f_{\max}+2}\right)\alpha(G) \geq 4\alpha(G) - 4$, it follows that (3), and, hence, also (2) hold with equality. This implies $f_{\max} \cdot |X| = \alpha_b(G)$, and, hence,

$$f(x) = f_{\max} \text{ for every } x \text{ in } X.$$

Furthermore, since $|I(x)| > \frac{f(x)+2}{4}$ for x in X_1, equality in (2) implies

$$X = X_2.$$

Let the integer k be such that $f_{\max} = 4k + 2$.

As in Case A, we have $|X| \geq 2$. If $|X| \geq 3$, then, by (2), $\alpha(G) \geq 3(\frac{f_{\max}+2}{4}) \geq 3(k+1)$, and, hence, $2\alpha(G) \geq 6k + 6 > 4k + 4 = f_{\max} + 2$. Hence,

$$|X| = 2.$$

If R is not empty, then adding a vertex from R to I yields an independent set, which implies the contradiction $\alpha(G) > |I|$. Hence,

$$R \text{ is empty.}$$

Let $X = \{x_1, x_2\}$, and let $B_i^j = B_i(x_j)$ for every i in $[2k+1]_0$ and j in $[2]$, cf. the definition of the graphs in $\mathcal{G}_2(k)$. Arguing similarly as in Case A, we obtain that

$$B_i^j \text{ is a clique for every } i \text{ in } [2k+1]_0 \text{ and } j \text{ in } [2],$$

and that

$$B_{2i}^j \text{ is completely joined to } B_{2i+1}^j \text{ for every } i \text{ in } [k]_0 \text{ and } j \text{ in } [2].$$

Since G is connected,

$$\text{there are some edges between } B_{2k+1}^1 \text{ and } B_{2k+1}^2.$$

Again, it follows that G contains a graph G_2 from $\mathcal{G}_2(k)$ as a spanning subgraph. Since adding any further edge e to G_2 such that $G_2 + e \notin \mathcal{G}_2(k)$ results in a graph of diameter less than $4k + 3$, we obtain $G \in \mathcal{G}_2(k)$, which completes the proof. $\quad\square$

References

[1] M. Ahmane, I. Bouchemakh, E. Sopena, On the broadcast independence number of caterpillars, Discrete Applied Mathematics 244 (2018) 20-35.

[2] I. Bouchemakh, M. Zemir, On the broadcast independence number of grid graph, Graphs and Combinatorics 30 (2014) 83-100.

[3] R.C. Brewster, C.M. Mynhardt, L.E. Teshima, New bounds for the broadcast domination number of a graph, Central European Journal of Mathematics 11 (2013) 1334-1343.

[4] R. Diestel, Graph theory. 2nd ed., Graduate Texts in Mathematics. 173. Berlin: Springer. xiv, 313 p. (2000).

[5] J.E. Dunbar, D.J. Erwin, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, Broadcasts in graphs, Discrete Applied Mathematics 154 (2006) 59-75.

[6] D.J. Erwin, Cost domination in graphs, (Ph.D. thesis), Western Michigan University, 2001.

[7] P. Heggernes, D. Lokshtanov, Optimal broadcast domination in polynomial time, Discrete Mathematics 36 (2006) 3267-3280.

[8] S. Herke, C.M. Mynhardt, Radial Trees, Discrete Mathematics 309 (2009) 5950-5962.

[9] S. Lunney, C.M. Mynhardt, More trees with equal broadcast and domination numbers, Australasian Journal of Combinatorics 61 (2015) 251-272.

[10] S.M. Seager, Dominating broadcasts of caterpillars, Ars Combinatorica 88 (2008) 307-319.

[11] K.W. Soh, K.M. Koh, Broadcast domination in graph products of paths, Australasian Journal of Combinatorics 59 (2014) 342-351.