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Abstract

The input of the Maximum Colored Cut problem consists of a graph $G = (V, E)$ with an edge-coloring $c : E \to \{1, 2, 3, \dots, p\}$ and a positive integer k, and the question is whether G has a nontrivial edge cut using at least k colors. The Colorful Cut problem has the same input but asks for a nontrivial edge cut using all p colors. Unlike what happens for the classical Maximum Cut problem, we prove that both problems are NP-complete even on complete, planar, or bounded treewidth graphs. Furthermore, we prove that Colorful Cut is NP-complete even when each color class induces a clique of size at most three, but is trivially solvable when each color induces an edge. On the positive side, we prove that Maximum Colored Cut is fixed-parameter tractable when parameterized by either k or p, by constructing a cubic kernel in both cases.

Keywords: colored cut; edge cut; maximum cut; planar graph; parameterized complexity; polynomial kernel.

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1. Introduction

Given an edge-colored graph G and an edge-set property Π, in maximum (minimum) colored/labeled Π problems we are asked to find a subset of edges satisfying property Π with respect to G that uses the maximum (minimum) number of colors/labels. These problems have a lot of applications and have been widely studied in recent years, for instance when Π is the property of being a spanning tree[4], a path between two designated vertices [5], a perfect matching [14], a Hamiltonian cycle [11], or an edge dominating set [11].

In this work, we focus on colored problems where Π is the property of being an *edge cut* of the input graph G. More precisely, let $G = (V, E)$ be a simple graph with an edge coloring $c : E \to \{1, 2, \ldots, p\}$, not necessarily proper. Given a proper subset $S \subset V$, we define the *edge cut* ∂S as the subset of E where the edges have one endpoint in S and the other in $V \setminus S$. We represent by $c(\partial S)$ the set of colors that appear in ∂S, i.e., $c(\partial S) = \{c(e) \mid e \in \partial S\}$. The problem of finding a subset $S \subset V$ such that $|c(\partial S)| \leq |c(\partial T)|$ for every $T \subset V$ is called Minimum Colored Cut, and its decision version is stated as follows.

Minimum Colored Cut
Instance: A graph $G = (V, E)$ with an edge coloring $c : E \to \{1, 2, \ldots, p\}$ and an integer $k > 0$.
Question: Is there a proper subset $S \subset V$ such that $|c(\partial S)| \leq k$?

Associated with Minimum Colored Cut, we have the Minimum Colored (s, t)-Cut problem, in which we are asked to find an edge cut that separates a given pair s, t of vertices using as few colors as possible.

Minimum Colored (s, t)-Cut
Instance: A graph $G = (V, E)$ with an edge coloring $c : E \to \{1, 2, \ldots, p\}$, a pair s, t of vertices of G, and an integer $k > 0$.
Question: Is there a proper subset $S \subset V$ such that $s \in S$, $t \notin S$ and $|c(\partial S)| \leq k$?

Analogously, the problem of finding a subset $S \subset V$ such that $|c(\partial S)| \geq |c(\partial T)|$ for every $T \subset V$ is called Maximum Colored Cut, and its decision version is stated as follows.

MAXIMUM COLORED CUT
Instance: A graph $G = (V, E)$ with an edge coloring $c : E \to \{1, 2, \ldots, p\}$ and an integer $k > 0$.
Question: Is there a proper subset $S \subset V$ such that $|c(\partial S)| \geq k$?

Note that the classical (simple) MAXIMUM CUT problem [12] is the particular case of MAXIMUM COLORED CUT when $c : E \to \mathbb{N}$ is an injective function. Therefore, for the MAXIMUM COLORED CUT problem we are interested in analyzing its complexity on graph classes $\mathcal{C}$ for which MAXIMUM CUT is solvable in polynomial time.

In addition, we are also interested in the complexity of determining if the input graph has a subset $S \subset V$ such that $|c(\partial S)| = p$, i.e., if there is an edge cut ∂S using *all* the colors; we call this problem COLORFUL CUT.

Complexity issues related to MINIMUM COLORED (s, t)-CUT and MINIMUM COLORED CUT have been widely investigated in recent years (cf. [1, 6, 7, 11, 17, 18, 19, 20, 21]). The goal of this work is to present a complexity analysis of MAXIMUM COLORED CUT, which, to the best of our knowledge, was missing in the literature. As COLORFUL CUT is a particular case of MAXIMUM COLORED CUT, our hardness results deal with COLORFUL CUT, while the tractable cases will be presented for MAXIMUM COLORED CUT.

The remainder of the article is organized as follows. In Section 2 we provide several NP-completeness results for restricted versions of COLORFUL CUT, and in Section 3 we present cubic kernels for MAXIMUM COLORED CUT parameterized either by p or by k. We use standard graph-theoretic notation; see [9] for any undefined notation. For the basic definitions of parameterized complexity, such as fixed-parameter tractability, W[2]-hardness, para-NP-hardness, or (polynomial) kernelization, we refer the reader to [8].

2. NP-completeness results for Colorful Cut

Hadlock [13] proved that (simple) MAXIMUM CUT is polynomial-time solvable on planar graphs. In this section we prove, among other results, the NP-completeness of COLORFUL CUT on a particular subclass of planar graphs. We start with general planar graphs, and then we discuss how the construction can be modified to get stronger hardness results.

Theorem 1. COLORFUL CUT *is* NP*-complete on planar graphs.*

Proof. Let $I = (U, C)$ be an instance of 3-SAT. We construct in polynomial time a planar instance $G = (V, E)$ with an edge coloring c such that $I = (U, C)$ is satisfiable if and only if (G, c) has an edge cut using all colors of c.

With each clause $c_j = (x \vee y \vee z) \in C$ we associate a K_3 where each edge is labeled by a literal of c_j together with its occurrence in C, obtaining a graph G' with m connected components, each of them associated with a clause of C. Starting with G', we construct a multigraph G equipped with a coloring c, for short G^c, where each literal in a clause of C is associated with a colored (multi)edge as follows:

- For each pair $\{x_i^j, \overline{x_i}^k\}$, where the integers j and k represent occurrences of the literals x_i and $\overline{x_i}$ in the clauses of C, respectively, create a color denoted by $S_i^{j,k}$.

- The edge labeled with x_i^j in G' is replaced with parallel edges colored with $S_i^{j,k}$ in G^c for all k. Analogously, the edge labeled by $\overline{x_i}^k$ in G' is replaced with parallel edges in G^c colored with $S_i^{j,k}$ for all j.

Without loss of generality, we may assume that all variables have both positive and negative literals in I (if not, the clauses containing such variables are trivially satisfiable and can be removed). From a truth assignment A_I of I, we can construct a colorful cut of G^c as follows. For each clause c_j of C, arbitrarily pick one edge $\{v, w\}$ corresponding to a true literal of C. Then, put v and w in the same part of the partition, leaving the remaining vertex of the clause in the other part. This procedure gives a cut using all colors. Indeed, for each true literal x_i^j, there is at least one false literal $\overline{x_i}^k$ that places the color $S_i^{j,k}$ in the cut.

Conversely, suppose that G^c has a colorful cut. Without loss of generality, we may assume that each K_3 has a cut edge. Indeed, as the cut has all the colors of the edge coloring, if there is some K_3 in a part of the partition, we can choose any vertex of this clique and place this vertex in the other part, without prejudice, because all the colors are still in the cut. Beginning with this cut, we construct a truth assignment A_I that satisfies I, putting $x_i = 1$ if at least one of the edges associated with some x_i^j is inside a part of the partition, and $x_i = 0$ otherwise. Note that this assignment is well-defined: there is no pair of literals $\{x_i^j, \overline{x_i}^k\}$ such that the edges corresponding to both literals are inside a part of the partition, otherwise the color $S_i^{j,k}$ is missing and the cut does not contain all colors. Besides that, each K_3 has the edges corresponding to some literal inside a part of the partition, which defines a truth assignment for I.

Finally, we can transform G^c into a simple graph replacing each edge $\{v, w\}$ colored with $S_i^{j,k}$ by a path $\{v, x, y, w\}$ such that $c(\{x, y\}) = S_i^{j,k}$ and the remaining edges of this path receive new different colors. It is not difficult to see that the multigraph G^c has a colorful cut if and only if the associated graph has a colorful cut. $\square$

Several NP-hard problems, such as Maximum Cut, are polynomial-time solvable on bounded treewidth graphs [3]. An important class of graphs that belongs to the intersection of planar graphs and bounded treewidth graphs is that of K_4-minor-free graphs. It is well-known that a graph G is K_4-minor free if and only if each 2-connected component of G is a series-parallel graph [2]. We can modify the graph obtained in the construction presented in the proof of Theorem 1 in order to obtain the following corollary.

Corollary 2. Colorful Cut *remains* NP-*complete even when the input graph G satisfies simultaneously that it is K_4-minor-free, connected, has maximum degree three, and each color class contains at most two edges.*

Proof. Let H be an instance of Colorful Cut constructed as described in the proof of Theorem 1. First observe that each connected component of H (clause gadget) can be obtained from a K_2 by either duplicating an edge or subdividing an edge. Therefore, each connected component of H is series-parallel.

In order to make the graph H connected and of bounded degree, just create a binary tree T with m leaves and add edges by connecting a vertex with maximum degree of each gadget clause of H to a distinct leaf of T. Assigning a new distinct color for each edge previously created, it holds that H is K_4-minor-free and each color class of H contains at most two edges (as in proof of Theorem 1). Finally, each vertex v of degree greater than three (i.e., four or five) can be replaced by a P_3 where each pendant vertex will be neighbor of vertices (at most two) that were adjacent to v in H and came from the same edge in G' (the edges of these P_3's also get new colors), and the middle vertex of the P_3 will be adjacent to the vertex of T that was neighbor of v, if any.

As the set of edges that we add in the graph induces a tree having a new color by each edge, it is easy to see that the modified graph has a colorful cut if and only if the original graph has a colorful cut. $\square$

Clearly, every bipartite graph has a colorful cut. Thus, it is natural to ask about the complexity of the problem on graphs with a small odd cycle transversal. By picking a vertex of each gadget of (G, c) constructed as described in the proof of Theorem 1, and identifying them into a single vertex, we get the following corollary.

Corollary 3. Colorful Cut *remains* NP*-complete even when the input graph* G *has odd cycle transversal number one.*

Note that Maximum Cut is trivial on complete graphs, and that it is polynomial-time solvable on cographs [3]. By adding a new vertex and edges colored with a new color, we can construct a hard instance in order to show the NP-completeness of Colorful Cut on complete graphs.

Theorem 4. Colorful Cut *is* NP*-complete on complete graphs.*

Proof. Given an instance (G, c) of Colorful Cut, we create another instance (G', c') such that G' is a clique as follows. Start from (G, c), add all the missing edges to G, add a new vertex v adjacent to all the vertices of G, and give to the edges in $E(G') \setminus E(G)$ the same color, different from the colors appearing in $E(G)$. Clearly, this new color appears in all the maximum colored cuts of G', and therefore (G', c') has a colorful cut if and only if (G, c) has one. $\square$

Note that if each color class of a graph G induces a K_2, then G has a colorful cut if and only if G is bipartite, which can be decided in polynomial time. The next result shows that this is best possible, in the sense that Colorful Cut is NP-complete when each color class induces either a K_2 or a K_3.

Theorem 5. Colorful Cut *is* NP*-complete when each color class induces a clique of size at most three.*

Proof. We present a reduction from Not All Equal 3-sat (nae 3-sat), which is NP-complete [15]. Let $I = (U, C)$ be an instance of nae 3-sat such that $U = \{u_1, u_2, \ldots, u_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$.

The construction of an instance (G, c) is given by the following procedure:

- For each clause $c_j = (x, y, z) \in C$, construct a clique $\{(x)_j, (y)_j, (z)_j\}$ with all the edges colored with color j.
- For each variable $u_i \in U$, add two new vertices a_i and b_i to V, such that a_i is only adjacent to all positive occurrences of u_i, and b_i is only adjacent to all negative occurrences of the same variable.
- For each variable $u_i \in U$, add an edge joining the vertices (in the clause cliques) corresponding to the first positive occurrence and the first negative occurrence of u_i.
- Excluding the edges of the clause cliques, all other edges are colored with new different integers strictly greater than m.

Figure 1 illustrates the instance (G, c) of Colorful Cut associated with an instance $I = (U, C)$ of Not All Equal 3-sat.

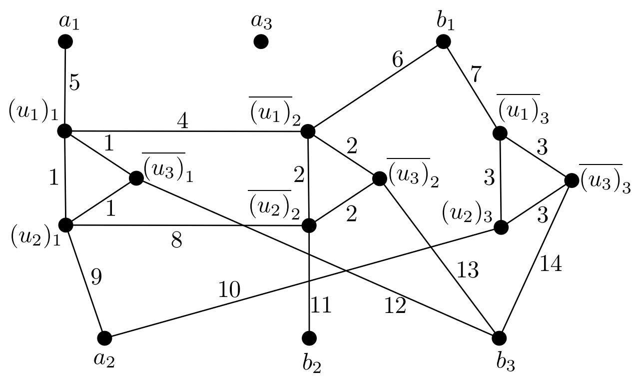

Figure 1: Graph associated with the formula $(u_1 \vee u_2 \vee \overline{u_3}) \wedge (\overline{u_1} \vee \overline{u_2} \vee \overline{u_3}) \wedge (\overline{u_1} \vee u_2 \vee \overline{u_3})$.

At this point, it is not difficult to see that $I = (U, C)$ is a satisfiable instance of nae 3-sat if and only if (G, c) has a colorful cut. Indeed, suppose first that $I = (U, C)$ is a satisfiable instance of nae 3-sat, and let η be a truth assignment of U that satisfies I. A colorful cut ∂S in G^c is obtained as follows: if $u_i = 1$, then put all its positive occurrences together with b_i in S, and put all its negative occurrences together with a_i in $V \setminus S$. By construction, all the colors greater than m are in the cut ∂S. Furthermore, each of the colors $j \leq m$ is in the cut because in each clause there is always a true and a false occurrence.

Conversely, suppose that (G, c) contains a colorful cut ∂S. All the clause cliques have vertices in different parts of the partition, because the colors j with $1 \leq j \leq m$ only appear in those clique edges. Thus we can produce a truth assignment η by setting to true to those literals corresponding to the clique vertices $\{x_j, y_j, z_j\}$ that belong to S, and by setting to false otherwise. This is a consistent truth assignment because the edge joining the first positive and negative occurrences of the variable u_i (if any) must be a cut edge, that is, its exclusive color must be in the cut, which means that those occurrences must be in different parts of the partition, thus having opposite truth assignments. As all positive occurrences of u_i are adjacent to the vertex a_i and those edges have pairwise different colors presented in the cut, it forces all positive occurrences of u_i to be in the same part of the partition, receiving the same truth assignment. Analogously, we can prove that all negative occurrences of u_i must be in the same part of the partition, getting the same truth assignment.

Figure 2 illustrates the colorful cut of (G, c) from the NAE 3-SAT instance $I = (U, C)$ with $U = \{u_1, u_2, u_3\}$ and $C = \{(u_1 \vee u_2 \vee \overline{u_3}), (\overline{u_1} \vee \overline{u_2} \vee \overline{u_3}), (\overline{u_1} \vee u_2 \vee \overline{u_3})\}$, satisfying the truth assignment $\overline{u_1} = u_2 = u_3 = 1$. $\square$

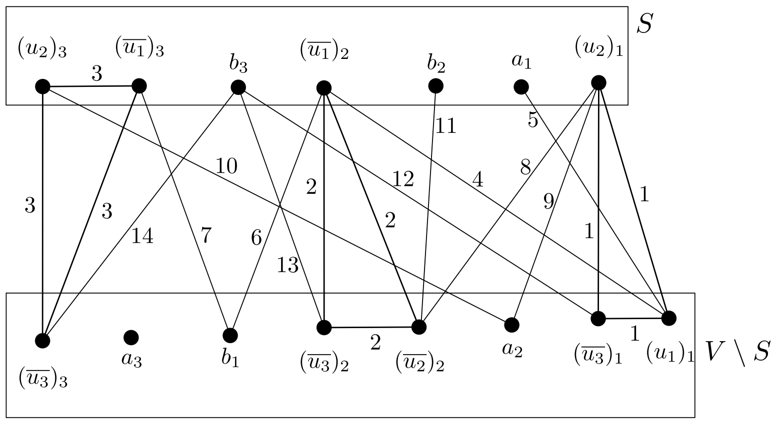

Figure 2: Colorful cut of $G^c = (V, E, f)$ from Figure 1 corresponding to the truth assignment $\overline{u_1} = u_2 = u_3 = 1$.

3. Polynomial kernelization of Maximum Colored Cut

From the results presented in Section 2, it follows that MAXIMUM COLORED CUT is *para*-NP-*hard* (see [8]) parameterized by any of these parameters: treewidth, neighborhood diversity, genus, degeneracy, odd cycle transversal number, $p - k$, and several combinations of such parameters. In contrast to these results, next we show the fixed-parameter tractability of MAXIMUM COLORED CUT when parameterized by either k or p, by means of the existence of a polynomial kernel.

Theorem 6. MAXIMUM COLORED CUT *admits a cubic kernel parameterized by the number of colors.*

Proof. First recall that a cut of a graph G is a bipartite subgraph of G. The following claim is an easy fact.

Claim 1. Let $H = (V_1, V_2, E)$ be a bipartite graph having β edges and no isolated vertices. The maximum number of edges having endpoints in the same part that can be added to H is $2\binom{\beta}{2}$, corresponding to the case where E induces a matching.

Now, suppose that λ is the maximum number of colors in a cut of $G = (V, E)$ and let $S \subset V$ be a set such that $|c(\partial S)| = \lambda$. Forming a bipartite graph H by selecting exactly one edge of each color class in $[S, V \setminus S]$, by Claim 1 it follows that any color class that is not in H has at most $2\binom{\lambda}{2}$ edges, otherwise λ would not be maximum. Let $E_i \subseteq E$ be the set of edges colored with color i. As $\lambda \leq p$, if $|E_i| > 2\binom{p}{2}$, then color i appears in any maximum colored cut. Such a property gives us the following reduction rule: If for some color i, $|E_i| > 2\binom{p}{2}$, decrease by one the number of colors and replace G by $G[E \setminus E_i]$. The exhaustive application of this rule yields a kernel of size $O(p^3)$. $\square$

Before our last result, we need the following lemma.

Lemma 7. *Any simple graph $G = (V, E)$ with an edge coloring $c : E \to \{1, 2, \ldots, p\}$ has an edge cut ∂S such that $|c(\partial S)| \geq \frac{p}{2}$.*

Proof. Let G' be an uncolored graph obtained from G by keeping one arbitrary edge from each color. Then the lemma follows by applying to G' the fact that any graph with at least m edges contains a bipartite subgraph with at least $\frac{m}{2}$ edges [10]. $\square$

Corollary 8. Maximum Colored Cut *admits a cubic kernel parameterized by the cost of the solution.*

Proof. If $k \geq p/2$, by Lemma 7 we conclude that we are dealing with a Yes-instance. Otherwise, $k < p/2$, and applying exhaustively the rule described in the proof of Claim 1 yields a kernel of size $O(k^3)$. $\square$

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