On the (Parameterized) Complexity of Recognizing Well-Covered (r, l)-graphs

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Abstract

An (r, ℓ)-*partition* of a graph G is a partition of its vertex set into r independent sets and ℓ cliques. A graph is (r, ℓ) if it admits an (r, ℓ)-partition. A graph is *well-covered* if every maximal independent set is also maximum. A graph is (r, ℓ)-*well-covered* if it is both (r, ℓ) and well-covered. In this paper we consider two different decision problems. In the (r, ℓ)-Well-Covered Graph problem ((r, ℓ)wc-g for short), we are given a graph G, and the question is whether G is an (r, ℓ)-well-covered graph. In the Well-Covered (r, ℓ)-Graph problem (wc-(r, ℓ)g for short), we are given an (r, ℓ)-graph G together with an (r, ℓ)-partition, and the question is whether G is well-covered. This generates two infinite families of problems, for any fixed non-negative integers r and ℓ, which we classify as being P, coNP-complete, NP-complete, NP-hard, or coNP-hard. Only the cases wc-$(r, 0)$g for $r \geq 3$ remain open. In addition, we consider the parameterized complexity of these problems for several choices of parameters, such as the size α of a maximum independent set of the input graph, its neighborhood diversity, its clique-width, or the number ℓ of cliques in an (r, ℓ)-partition. In particular, we show that the parameterized problem of determining whether every maximal independent set of an input graph G has cardinality equal to k can be reduced to the wc-$(0, \ell)$g problem parameterized by ℓ. In addition, we prove that both problems are coW[2]-hard but can be solved in XP-time.

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1. Introduction

One of the most important combinatorial problems is Maximum Independent Set (MIS), where the objective is to find a maximum sized subset $S \subseteq V$ of pairwise non-adjacent vertices in a graph $G = (V, E)$. Maximum independent sets appear naturally in a wide range of situations, and MIS also finds a number of "real world" relevant applications.

Unfortunately, the decision version of MIS is an NP-complete problem [24], and thus it cannot be solved in polynomial time unless P = NP. In spite of the fact that finding a *maximum* independent set is a computationally hard problem, a *maximal* independent set of a graph can easily be found in linear time. Indeed, a naive greedy algorithm for finding maximal independent sets consists simply of selecting an arbitrary vertex v to add to a set S, and updating the current graph by removing the closed neighborhood $N[v]$ of v. This algorithm always outputs a maximal independent set in linear time, but clearly not all choices lead to a maximum independent set.

Well-covered graphs were first introduced by Plummer [30] in 1970. Plummer defined that "a graph is said to be *well-covered* if every minimal point cover is also a minimum cover". This is equivalent to demanding that all maximal independent set have the same cardinality. Therefore, well-covered graphs can be equivalently defined as the class of graphs for which the naive greedy algorithm discussed above *always* outputs a maximum independent set.

The problem of recognizing a well-covered graph, which we denote by Well-Covered Graph, was proved to be coNP-complete by Chvátal and Slater [4] and independently by Sankaranarayana and Stewart [34]. On the other hand, the Well-Covered Graph problem is in P when the input is known to be a perfect graph of bounded clique size [13] or a claw-free graph [27, 36].

Let $r, \ell \geq 0$ be two fixed integers. An (r, ℓ)-*partition* of a graph $G = (V, E)$ is a partition of V into r independent sets $S^1, \ldots, S^r$ and ℓ cliques $K^1, \ldots, K^\ell$. For convenience, we allow these sets to be empty. A graph is (r, ℓ) if it admits an (r, ℓ)-partition. Note that the notion of (r, ℓ)-graphs is a generalization of that of r-colorable graphs.

A P versus NP-complete dichotomy for recognizing (r, ℓ)-graphs was proved by Brandstädt [2]: the problem is in P if $\max\{r, \ell\} \leq 2$, and NP-complete otherwise. The class of (r, ℓ)-graphs and its subclasses have been extensively studied in the literature. For instance, list partitions of (r, ℓ)-graphs were studied by Feder et al. [18]. In another paper, Feder et al. [19] proved that recognizing graphs that are both chordal and (r, ℓ) is in P.

A graph is (r, ℓ)-*well-covered* if it is both (r, ℓ) and well-covered. In this paper we analyze the complexity of the (r, ℓ)-Well-Covered Graph problem, which consists of deciding whether a graph is (r, ℓ)-well-covered. In particular, we give a complete classification of the complexity of this problem.

Additionally, we analyze the complexity of the WELL-COVERED-(r,ℓ)-GRAPH problem, which consists of deciding, given an (r,ℓ)-graph $G=(V,E)$ together with an (r,ℓ) partition, whether G is well-covered or not. We classify the complexity of this problem for every pair (r,ℓ), except for the cases when $\ell=0$ and $r\geq 3$, which we leave open.

We note that similar restrictions have been considered in the literature. For instance, Kolay et al. [25] recently considered the problem of removing a small number of vertices from a perfect graph so that it additionally becomes (r,ℓ).

To the best of our knowledge, this is the first time in the literature that a decision problem obtained by "intersecting" two recognition NP-complete and coNP-complete properties has been studied. From our results, the (r,ℓ)WC-G problem has a very peculiar property, namely that some cases of the problem are in NP, but other cases are in coNP. And if P $\neq$ NP, there are some cases where the decision problem is neither in NP nor in coNP.

In addition, according to the state of the art for the WELL-COVERED GRAPH problem, to the best of our knowledge this is the first work that associates the hardness of WELL-COVERED GRAPH with the number of independent sets and the number of cliques of an (r,ℓ)-partition of the input graph. This shows an important structural property for classifying the complexity of subclasses of well-covered graphs.

As a by-product of this paper, an infinite class of decision problems was classified as being both NP-hard and coNP-hard. Hence, unless P = NP these decision problems are neither in NP nor in coNP.

More formally, in this paper we focus on the following two decision problems.

(r,ℓ)-WELL-COVERED GRAPH ((r,ℓ)WC-G)
Input: A graph G.
Question: Is G (r,ℓ)-well-covered?

WELL-COVERED (r,ℓ)-GRAPH (WC-(r,ℓ)G)
Input: An (r,ℓ)-graph G, together with a partition of $V(G)$ into r independent sets and ℓ cliques.
Question: Is G well-covered?

We establish an almost complete characterization of the complexity of the (r,ℓ)WC-G and WC-(r,ℓ)G problems. Our results are shown in the following tables, where r (resp. ℓ) corresponds to the rows (resp. columns) of the tables, and where coNPc stands for coNP-complete, NPh stands for NP-hard, NPc stands for NP-complete, and (co)NPh stands for both NP-hard and coNP-hard. The symbol '?' denotes that the complexity of the corresponding problem is open.

(r,ℓ)WC-G	0	1	2	≥ 3
0	–	P	P	NPc
1	P	P	P	NPc
2	P	coNPc	coNPc	(co)NPh
≥ 3	NPh	(co)NPh	(co)NPh	(co)NPh

WC-(r,ℓ)G	0	1	2	≥ 3
0	–	P	P	P
1	P	P	P	P
2	P	coNPc	coNPc	coNPc
≥ 3	?	coNPc	coNPc	coNPc

We note the following simple facts, which we will use to fill the above tables:

Fact 1. *If (r,ℓ)WC-G is in P, then WC-(r,ℓ)G is in P.*

Fact 2. *If WC-(r,ℓ)G is coNP-hard, then (r,ℓ)WC-G is coNP-hard.*

Note that WC-(r,ℓ)G is in coNP, since a certificate for a NO-instance consists just of two maximal independent sets of different size. On the other hand, for (r,ℓ)WC-G we have the following facts, which are easy to verify:

Fact 3. *For any pair of integers (r,ℓ) such that the problem of recognizing an (r,ℓ)-graph is in P, the (r,ℓ)WC-G problem is in coNP.*

Fact 4. *For any pair of integers (r,ℓ) such that the WC-(r,ℓ)G problem is in P, the (r,ℓ)WC-G problem is in NP.*

In this paper we prove that (r,ℓ)WC-G with $(r,\ell) \in \{(0,1),(1,0),(0,2),(1,1),(2,0),(1,2)\}$ can be solved in polynomial time, which by Fact 1 yields that WC-(r,ℓ)G with $(r,\ell) \in \{(0,1),(1,0),(0,2),(1,1),(2,0),(1,2)\}$ can also be solved in polynomial time. On the other hand, we prove that WC-$(2,1)$G is coNP-complete, which by Fact 2 and Fact 3 yields that $(2,1)$WC-G is also coNP-complete. Furthermore, we also prove that WC-$(0,\ell)$G and WC-$(1,\ell)$G are in P, and that (r,ℓ)WC-G with $(r,\ell) \in \{(0,3),(3,0),(1,3)\}$ are NP-hard. Finally, we state and prove a "monotonicity" result, namely Theorem 1, stating how to extend the NP-hardness or coNP-hardness of WC-(r,ℓ)G (resp. (r,ℓ)WC-G) to WC-$(r+1,\ell)$G (resp. $(r+1,\ell)$WC-G), and WC-$(r,\ell+1)$G (resp. $(r,\ell+1)$WC-G). Together, these results correspond to those shown in the above tables.

In addition, we consider the parameterized complexity of these problems for several choices of the parameters, such as the size α of a maximum independent set of the input graph, its neighborhood diversity, its clique-width or the number ℓ of cliques in an (r,ℓ)-partition. We obtain several positive and negative results. In particular, we show that the parameterized problem of determining whether every maximal independent set of an input graph G has cardinality equal to k can be reduced to the WC-$(0,\ell)$G problem parameterized by ℓ. In addition, we prove that both problems are coW[2]-hard, but can be solved in XP-time.

The rest of this paper is organized as follows. We start in Section 2 with some basic preliminaries about graphs, parameterized complexity, and width parameters. In Section 3 we prove our results concerning the classical complexity of both problems, and in Section 4 we focus on their parameterized complexity. We conclude the paper with Section 5.

2. Preliminaries

Graphs. We use standard graph-theoretic notation, and we refer the reader to [14] for any undefined notation. A *graph* $G = (V, E)$ consists of a finite non-empty set V of vertices and a set E of unordered pairs (edges) of distinct elements of V. If $uv \in E(G)$, then u, v are said to be *adjacent*, and u is said to be a *neighbor* of v. A *clique* (resp. *independent set*) is a set of pairwise adjacent (resp. non-adjacent) vertices. A *vertex cover* is a set of vertices containing at least one endpoint of every edge in the graph. The *open neighborhood* $N(v)$ or *neighborhood*, for short, of a vertex $v \in V$ is the set of vertices adjacent to v. The *closed neighborhood* of a vertex v is defined as $N[v] = N(v) \cup \{v\}$. A *dominating set* is a set of vertices $S \subseteq V$ such that $\bigcup_{v \in S} N[v] = V$. Given $S \subseteq V$ and $v \in V$, the *neighborhood* $N_S(v)$ *of* v *in* S is the set $N_S(v) = N(v) \cap S$.

Throughout the paper, we let n denote the number of vertices in the input graph for the problem under consideration.

Parameterized complexity. We refer the reader to [11, 15, 20, 28] for basic background on parameterized complexity, and we recall here only some basic definitions. A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. For an instance $I = (x, k) \in \Sigma^* \times \mathbb{N}$, k is called the *parameter*. A parameterized problem is *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathcal{A}$, a computable function f, and a constant c such that given an instance $I = (x, k)$, $\mathcal{A}$ (called an FPT-*algorithm*) correctly decides whether $I \in L$ in time bounded by $f(k)|I|^c$.

Within parameterized problems, the class $\mathsf{W}[1]$ may be seen as the parameterized equivalent to the class NP of classical optimization problems. Without entering into details (see [11, 15, 20, 28] for the formal definitions), a parameterized problem being $\mathsf{W}[1]$-*hard* can be seen as a strong evidence that this problem is *not* FPT. The canonical example of a $\mathsf{W}[1]$-hard problem is Independent Set parameterized by the size of the solution[2].

The class $\mathsf{W}[2]$ of parameterized problems is a class that contains $\mathsf{W}[1]$, and so the problems that are $\mathsf{W}[2]$-*hard* are even more unlikely to be FPT than those that are $\mathsf{W}[1]$-hard (again, see [11, 15, 20, 28] for the formal definitions). The canonical example of a $\mathsf{W}[2]$-hard problem is Dominating Set parameterized by the size of the solution[3].

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[2] Given a graph G and a parameter k, the problem is to decide whether there exists an independent set $S \subseteq V(G)$ such that $|S| \geq k$.

[3] Given a graph G and a parameter k, the problem is to decide whether there exists a dominating set $S \subseteq V(G)$ such that $|S| \leq k$.

For $i \in [1, 2]$, to transfer $\mathsf{W}[i]$-hardness from one problem to another, one uses an *fpt-reduction*, which given an input $I = (x, k)$ of the source problem, computes in time $f(k)|I|^c$, for some computable function f and a constant c, an equivalent instance $I' = (x', k')$ of the target problem, such that k' is bounded by a function depending only on k.

Hence, an equivalent definition of $\mathsf{W}[1]$-hard (resp. $\mathsf{W}[2]$-hard) problem is any problem that admits an fpt-reduction from Independent Set (resp. Dominating Set) parameterized by the size of the solution.

Even if a parameterized problem is $\mathsf{W}[1]$-hard or $\mathsf{W}[2]$-hard, it may still be solvable in polynomial time for *fixed* values of the parameter; such problems are said to belong to the complexity class XP. Formally, a parameterized problem whose instances consist of a pair (x, k) is in XP if it can be solved by an algorithm with running time $f(k)|x|^{g(k)}$, where f, g are computable functions depending only on the parameter and $|x|$ represents the input size. For example, Independent Set and Dominating Set parameterized by the solution size are easily seen to belong to XP.

Width parameters. A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{X})$, where $T = (I, F)$ is a tree, and $\mathcal{X} = \{B_i\},\ i \in I$ is a family of subsets of $V(G)$, called *bags* and indexed by the nodes of T, such that

1. each vertex $v \in V$ appears in at least one bag, i.e., $\bigcup_{i \in I} B_i = V$;
2. for each edge $e = \{x, y\} \in E$, there is an $i \in I$ such that $x, y \in B_i$; and
3. for each $v \in V$ the set of nodes indexed by $\{i \mid i \in I,\ v \in B_i\}$ forms a subtree of T.

The *width* of a tree-decomposition is defined as $\max_{i \in I}\{|B_i| - 1\}$. The *treewidth* of G, denoted by $\mathsf{tw}(G)$, is the minimum width of a tree-decomposition of G.

The *clique-width* of a graph G, denoted by $\mathsf{cw}(G)$, is defined as the minimum number of labels needed to construct G, using the following four operations:

1. Create a single vertex v with an integer label ℓ (denoted by $\ell(v)$);
2. Take the disjoint union (i.e., co-join) of two graphs (denoted by $\oplus$);
3. Join by an edge every vertex labeled i to every vertex labeled j for $i \neq j$ (denoted by $\eta(i, j)$);
4. Relabel all vertices with label i by label j (denoted by $\rho(i, j)$).

An algebraic term that represents such a construction of G and uses at most k labels is said to be a *k-expression* of G (i.e., the clique-width of G is the minimum k for which G has a k-expression).

Graph classes with bounded clique-width include cographs [3], distance-hereditary graphs [21], graphs of bounded treewidth [10], graphs of bounded branchwidth [33], and graphs of bounded rank-width [23].

3. Classical Complexity of the Problems

We start with a monotonicity theorem that will be very helpful to fill the tables presented in Section 1. The remainder of this section is divided into four subsections according to whether (r,ℓ)WC-G and WC-(r,ℓ)G are polynomial or "hard" problems.

Theorem 1. *Let $r,\ell \geq 0$ be two fixed integers. Then it holds that:*

(i) if WC-(r,ℓ)G *is* coNP*-complete then* WC-$(r+1,\ell)$G *and* WC-$(r,\ell+1)$G *are* coNP*-complete;*

*(ii) if (r,ℓ)*WC-G *is* NP*-hard (resp.* coNP*-hard) then $(r,\ell+1)$*WC-G *is* NP*-hard (resp.* coNP*-hard);*

*(iii) supposing that $r \geq 1$, if (r,ℓ)*WC-G *is* NP*-hard (resp.* coNP*-hard) then $(r+1,\ell)$*WC-G *is* NP*-hard (resp.* coNP*-hard).*

Proof. (i) This follows immediately from the fact that every (r,ℓ)-graph is also an $(r+1,\ell)$-graph and an $(r,\ell+1)$-graph.

(ii) Let G be an instance of (r,ℓ)WC-G. Let H be an $(r,\ell+1)$WC-G INSTANCE defined as the disjoint union of G and a clique Z with $V(Z) = \{z_1,\dots,z_{r+1}\}$. Clearly G is well-covered if and only if H is well-covered. If G is an (r,ℓ)-well-covered graph then H is an $(r,\ell+1)$-well-covered graph. Suppose H is an $(r,\ell+1)$-well-covered graph, with a partition into r independent sets $S^1,\dots,S^r$ and $\ell+1$ cliques $K^1,\dots,K^{\ell+1}$. Each independent set S^i can contain at most one vertex of the clique Z. Therefore, there must be a vertex z_i in some clique K^j. Assume without loss of generality that there is a vertex of Z in $K^{\ell+1}$. Then $K^{\ell+1}$ cannot contain any vertex outside of $V(Z)$, so we may assume that $K^{\ell+1}$ contains all vertices of Z. Now $S^1,\dots,S^r,K^1,\dots,K^\ell$ is an (r,ℓ)-partition of G, so G is an (r,ℓ)-well-covered graph. Hence, H is a YES-instance of $(r,\ell+1)$WC-G if and only if G is a YES-instance of (r,ℓ)WC-G.

(iii) Let G be an instance of (r,ℓ)WC-G. Let G' be an $(r+1,\ell)$WC-G instance obtained from G by adding $\ell+1$ isolated vertices. (This guarantees that every maximal independent set in G' contains at least $\ell+1$ vertices.) Since $r \geq 1$, it follows that G' is an (r,ℓ)-graph if and only if G is. Clearly G' is well-covered if and only if G is.

Next, find an arbitrary maximal independent set in G' and let p be the number of vertices in this set. Note that $p \geq \ell+1$. Let H be the join of G' and a set of p independent vertices $Z = \{z_1,\dots,z_p\}$, i.e., $N_H(z_i) = V(G')$ for all i. Every maximal independent set of H is either Z or a maximal independent set of G' and every maximal independent set of G' is a maximal independent set of H. Therefore, H is well-covered if and only if G' is well-covered. Clearly, if G' is an (r,ℓ)-graph then H is an $(r+1,\ell)$-graph. Suppose H is an $(r+1,\ell)$-graph, with a partition into $r+1$ independent sets $S^1,\dots,S^{r+1}$ and ℓ cliques $K^1,\dots,K^\ell$. Each clique set K^i can contain at most one vertex of Z. Therefore there must be a vertex z_i in some independent set S^j. Suppose that there is a vertex of Z in S^{r+1}. Then S^{r+1} cannot contain any vertex outside of Z.

Without loss of generality, we may assume that S^{r+1} contains all vertices of Z. Now $S^1, \ldots, S^r, K^1, \ldots, K^\ell$ is an (r, ℓ)-partition of G, so G is an (r, ℓ)-graph. Thus H is a YES-instance of $(r+1, \ell)$WC-G if and only if G is a YES-instance of (r, ℓ)WC-G. $\square$

3.1. Polynomial Cases for WC-(r, ℓ)G

Theorem 2. WC-$(0, \ell)$G *and* WC-$(1, \ell)$G *are in* P *for every integer* $\ell \geq 0$.

Proof. It is enough to prove that WC-$(1, \ell)$G is in P. Let $V = (S, K^1, K^2, K^3, \ldots, K^\ell)$ be a $(1, \ell)$-partition for G. Then each maximal independent set I of G admits a partition $I = (I_K, S \setminus N_S(I_K))$, where I_K is an independent set of $K^1 \cup K^2 \cup K^3 \cup \cdots \cup K^\ell$.

Observe that there are at most $O(n^\ell)$ choices for an independent set I_K of $K^1 \cup K^2 \cup K^3 \cup \cdots \cup K^\ell$, which can be listed in time $O(n^\ell)$, since ℓ is constant and $(K^1, K^2, K^3, \ldots, K^\ell)$ is given. For each of them, we consider the independent set $I = I_K \cup (S \setminus N_S(I_K))$. If I is not maximal (which may happen if a vertex in $(K^1 \cup K^2 \cup K^3 \cup \cdots \cup K^\ell) \setminus I_K$ has no neighbors in I), we discard this choice of I_K. Hence, we have a polynomial number $O(n^\ell)$ of maximal independent sets to check in order to decide whether G is a well-covered graph. $\square$

3.2. Polynomial Cases for (r, ℓ)WC-G

Fact 5. *The graph induced by a clique or by an independent set is well-covered.*

The following corollary is a simple application of Fact 5.

Corollary 3. *G is a $(0, 1)$-well-covered graph if and only if G is a $(0, 1)$-graph. Similarly, G is a $(1, 0)$-well-covered graph if and only if G is a $(1, 0)$-graph.*

The following is an easy observation.

Lemma 4. $(0, 2)$WC-G *can be solved in polynomial time.*

Proof. By definition, a graph $G = (V, E)$ is a $(0, 2)$-graph if and only if its vertex set can be partitioned into two cliques, and this can be tested in polynomial time. It follows that every $(0, 2)$-graph has maximum independent sets of size at most 2. Let G be a $(0, 2)$-graph with $(0, 2)$-partition (K^1, K^2). If V is a clique, then G is a $(0, 1)$-well-covered graph, and hence a $(0, 2)$-well-covered graph. If V is not a clique, then G is a $(0, 2)$-well-covered graph if and only if G has no universal vertex. $\square$

In the next three lemmas we give a characterization of $(1, 1)$-well-covered graphs in terms of their graph degree sequence. Note that $(1, 1)$-graphs are better known in the literature as *split graphs*.

Lemma 5. *Let $G = (V, E)$ be a $(1, 1)$-well-covered graph with $(1, 1)$-partition $V = (S, K)$, where S is a independent set and K is a clique. If $x \in K$, then $|N_S(x)| \leq 1$.*

Proof. Suppose that G is a $(1,1)$-well-covered graph with $(1,1)$-partition $V = (S, K)$, where S is a independent set and K is a clique. Let I be a maximal independent set of G such that $x \in I \cap K$. Suppose for contradiction that $|N_S(x)| \geq 2$, and let $y, z \in N_S(x)$. Since $y, z \in S$, $N_G(y), N_G(z) \subseteq K$. Since K is a clique, vertex x is the only vertex of I in K. Hence, we have that $N_G(y) \cap (I \setminus \{x\}) = N_G(z) \cap (I \setminus \{x\}) = \emptyset$. Therefore $I' = (I \setminus \{x\}) \cup \{y, z\}$ is an independent set of G such that $|I'| = |I| + 1$. Thus, I is a maximal independent set that is not maximum, so G is not well-covered. Thus, $|N_S(x)| \leq 1$. □

Lemma 6. *A graph G is a $(1,1)$-well-covered graph if and only if it admits a $(1,1)$-partition $V = (S, K)$ such that either for every $x \in K$, $|N_S(x)| = 0$, or for every $x \in K$, $|N_S(x)| = 1$.*

Proof. Let G be a $(1,1)$-well-covered graph. By Lemma 5 we have that, given a vertex $x \in K$, either $|N_S(x)| = 0$ or $|N_S(x)| = 1$. Suppose for contradiction that there are two vertices $x, y \in K$ such that $|N_S(x)| = 0$ and $|N_S(y)| = 1$. Let z be the vertex of S adjacent to y. Let I be a maximal independent set containing vertex y. Note that the vertex x is non-adjacent to every vertex of $I \setminus \{y\}$ since there is at most one vertex of I in K. The same applies to the vertex z. Hence, a larger independent set I', with size $|I'| = |I| + 1$, can be obtained from I by replacing vertex y with the non-adjacent vertices x, z, i.e., I is a maximal independent set of G that is not maximum, a contradiction. Thus, either for every $x \in K$, $|N_S(x)| = 0$, or for every $x \in K$, $|N_S(x)| = 1$.

Conversely, suppose that there is a $(1,1)$-partition $V = (S, K)$ of G such that either for every $x \in K$, $|N_S(x)| = 0$, or for every $x \in K$, $|N_S(x)| = 1$. If $K = \emptyset$, then G is $(1,0)$ and then G is well-covered. Hence we assume $K \neq \emptyset$. If for every $x \in K$, $|N_S(x)| = 0$, then every maximal independent set consists of all the vertices of S and exactly one vertex $v \in K$. If for every $x \in K$, $|N_S(x)| = 1$, then every maximal independent set is either $I = S$, or $I = \{x\} \cup (S \setminus N_S(x))$ for some $x \in K$. Since $|N_S(x)| = 1$ we have $|I| = 1 + |S| - 1 = |S|$, and hence G is a $(1,1)$-well-covered graph. □

Corollary 7. $(1,1)$WC-G *can be solved in polynomial time.*

Proof. Since we can check in polynomial time whether G is a $(1,1)$ graph [2], and one can enumerate all $(1,1)$-partitions of a split graph in polynomial time, we can solve the $(1,1)$WG-G problem in polynomial time.

□

The next lemma shows that $(1,1)$-well-covered graphs can be recognized from their degree sequences.

Lemma 8. *G is a $(1,1)$-well-covered graph if and only if there is a positive integer k such that G is a graph with a $(1,1)$-partition $V = (S, K)$ where $|K| = k$, such that the degree sequence of V is either $(k, k, k, \ldots, k,\ i_1, i_2, \ldots, i_s,\ 0, 0, 0, \ldots, 0)$ with $\sum_{j=1}^{s}(i_j) = k$, or $(k-1, k-1, k-1, \ldots, k-1,\ 0, 0, 0, \ldots, 0)$, where the subsequences $k, \ldots, k$ (resp. $k-1, \ldots, k-1$) have length k.*

Proof. Let G be a $(1,1)$-well-covered graph. Then G admits a $(1,1)$-partition $V=(S,K)$ where $k:=|K|, k\geq 0$. If $k=0$, then the degree sequence is $(0,0,0,\ldots,0)$. If $k\geq 1$, then by Lemma 6 either for every $x\in K$, $|N_S(x)|=0$, or for every $x\in K$, $|N_S(x)|=1$. If for every $x\in K$, $|N_S(x)|=0$, then the degree sequence of G is $(k-1,k-1,k-1,\ldots,k-1,\ 0,0,0,\ldots,0)$. If for every $x\in K$, $|N_S(x)|=1$, then the degree sequence of G is $(k,k,k,\ldots,k,\ i_1,i_2,\ldots,i_s,$ $0,0,0,\ldots,0)$, with $\sum_{j=1}^{s}(i_j)=k$.

Suppose that there is a positive integer k such that G is a graph with $(1,1)$-partition $V=(S,K)$ where $|K|=k$, with degree sequence either $(k,k,k,\ldots,k,$ $i_1,i_2,\ldots,i_s,\ 0,0,0,\ldots,0)$, or $(k-1,k-1,k-1,\ldots,k-1,\ 0,0,0,\ldots,0)$, such that $\sum_{j=1}^{s}(i_j)=k$. If the degree sequence of G is $(k,k,k,\ldots,k,\ i_1,i_2,\ldots,i_s,$ $0,0,0,\ldots,0)$, then the vertices of K are adjacent to $k-1$ vertices of K and exactly one of S, since the vertices with degree $i_1,i_2,\ldots,i_s$, have degree at most k and the vertices with degree 0 are isolated. If the degree sequence of G is $(k-1,k-1,k-1,\ldots,k-1,\ 0,0,0,\ldots,0)$, then the vertices of K are adjacent to $k-1$ vertices of K and none of S and the vertices with degree 0 are isolated. By Lemma 6 we have that G is a well-covered graph. $\square$

Ravindra [32] gave the following characterization of $(2,0)$-well-covered graphs.

Proposition 9 (Ravindra [32])**.** *Let G be a connected graph. G is a $(2,0)$-well-covered graph if and only if G contains a perfect matching F such that for every edge $e=uv$ in F, $G[N(u)\cup N(v)]$ is a complete bipartite graph.*

We now prove that Proposition 9 leads to a polynomial-time algorithm.

Lemma 10. $(2,0)$WC-G *can be solved in polynomial time.*

Proof. Assume that G is connected and consider the weighted graph (G,ω) with $\omega: E(G)\to\{0,1\}$ satisfying $\omega(uv)=1$, if $G[N(u)\cup N(v)]$ is a complete bipartite graph, and 0 otherwise. By Proposition 9, G is well-covered if and only if (G,ω) has a weighted perfect matching with weight at least $n/2$, and this can be decided in polynomial time [16]. $\square$

Lemma 11. $(1,2)$WC-G *can be solved in polynomial time.*

Proof. We can find a $(1,2)$-partition of a graph G (if such a partition exists) in polynomial time [2]. After that, we use the algorithm for WC-$(1,\ell)$G given by Theorem 2. $\square$

Below we summarize the cases for which we have shown that WC-(r,ℓ)G or (r,ℓ)WC-G can be solved in polynomial time.

Theorem 12. (r,ℓ)WC-G *with* $(r,\ell)\in\{(0,1),(0,2),(1,0),(1,1),(1,2),(2,0)\}$ *and* WC-(r,ℓ)G *with* $r\in\{0,1\}$ *or* $(r,\ell)=(2,0)$ *can be solved in polynomial time.*

Proof. The first part follows from Corollary 3, Lemma 4, Corollary 7, Lemma 11, and Lemma 10, respectively. The second part follows from Theorem 2, and Lemma 10 together with Fact 1. $\square$

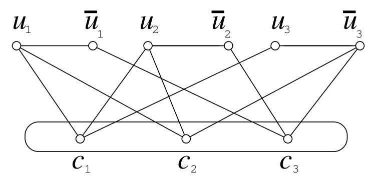

Figure 1: Chvátal and Slater's [4] WELL-COVERED GRAPH instance $G = (V, E)$ obtained from the satisfiable 3-SAT instance $I = (U, C) = (\{u_1, u_2, u_3\}, \{(u_1, u_2, u_3), (u_1, u_2, \overline{u}_3), (\overline{u}_1, \overline{u}_2, \overline{u}_3)\})$, where $\{c_1, c_2, \ldots, c_m\}$ is a clique of G. Observe that I is satisfiable if and only if G is not well-covered, since there is a maximal independent set with size $n + 1$ (e.g. $\{c_1, \overline{u}_1, \overline{u}_2, \overline{u}_3\}$) and there is a maximal independent set of size n (e.g. $\{u_1, u_2, \overline{u}_3\}$). Note also that G is a $(2, 1)$-graph with $(2, 1)$-partition $V = (\{u_1, u_2, \ldots, u_n\}, \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n\}, \{c_1, c_2, \ldots, c_m\})$.

3.3. coNP*-complete Cases for* WC-(r, ℓ)G

We note that the WELL-COVERED GRAPH instance G constructed in the reduction of Chvátal and Slater [4] is a $(2, 1)$-graph, directly implying that WC-$(2, 1)$G is coNP-complete.

Indeed, Chvátal and Slater [4] take a 3-SAT instance $I = (U, C) = (\{u_1, u_2, u_3, \ldots, u_n\}, \{c_1, c_2, c_3, \ldots, c_m\})$, and construct a WELL-COVERED GRAPH instance $G = (V, E) = (\ \{u_1, u_2, u_3,\ \ldots, u_n, \overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_n,\ c_1, c_2, c_3, \ldots, c_m\}, \{xc_j : x \text{ occurs in } c_j\} \cup \{u_i\overline{u}_i : 1 \leq i \leq n\} \cup \{c_ic_j : 1 \leq i < j \leq m\}\ )$. Note that $\{c_ic_j : 1 \leq i < j \leq m\}$ is a clique, and that $\{u_1, u_2, u_3,\ \ldots, u_n\}$, and $\{\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_n\}$ are independent sets. Hence, G is a $(2, 1)$-graph. An illustration of this construction can be found in Figure 1. This discussion can be summarized as follows.

Proposition 13 (Chvátal and Slater [4])**.** WC-$(2, 1)$G *is* coNP*-complete.*

As $(2, 1)$-graphs can be recognized in polynomial time [2], we obtain the following.

Corollary 14. $(2, 1)$WC-G *is* coNP*-complete.*

3.4. NP*-hard Cases for* (r, ℓ)WC-G

Now we prove that $(0, 3)$WC-G is NP-complete. For this purpose, we slightly modify an NP-completeness proof of Stockmeyer [35].

Stockmeyer's [35] NP-completeness proof of 3-coloring considers a 3-SAT instance $I = (U, C) = (\ \{u_1, u_2, u_3, \ldots, u_n\}, \{c_1, c_2, c_3, \ldots, c_m\}\ )$, and constructs a 3-COLORING instance $G = (V, E) = (\ \{u_1, u_2, u_3, \ldots, u_n, \overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_n\} \cup \{v_1[j], v_2[j], v_3[j], v_4[j], v_5[j], v_6[j] : j \in \{1, 2, 3, \ldots, m\}\} \cup \{t_1, t_2\}, \{u_i\overline{u}_i : i \in \{1, 2, 3, \ldots, n\}\} \cup \{v_1[j]v_2[j], v_2[j]v_4[j], v_4[j]v_1[j], v_4[j]v_5[j], v_5[j]v_6[j], v_6[j]v_3[j], v_3[j]v_5[j] : j \in \{1, 2, 3, \ldots, m\}\} \cup \{v_1[j]x, v_2[j]y, v_3[j]z : c_j = (x, y, z)\} \cup \{t_1u_i, t_1\overline{u}_i : i \in \{1, 2, 3, \ldots, n\}\} \cup \{t_2v_6[j] : j \in \{1, 2, 3, \ldots, m\}\}\ )$; see Figure 2(a).

Lemma 15. $(0, 3)$WC-G *is* NP*-complete.*

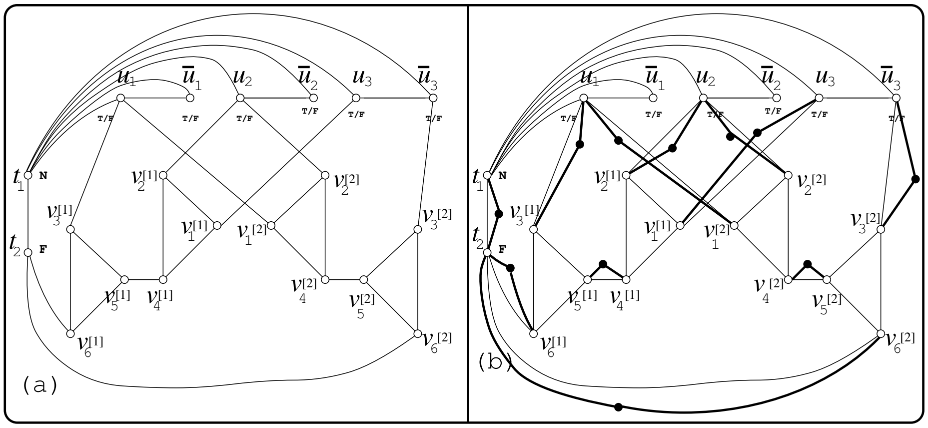

Figure 2: (a) Stockmeyer's [35] 3-COLORING instance G obtained from the 3-SAT instance $I = (U, C) = (\{u_1, u_2, u_3\}, \{(u_3, u_2, u_1), (u_1, u_2, \overline{u}_3)\})$. (b) The graph G' obtained from G by adding a vertex x_{uv} with $N_{G'}(x_{uv}) = \{u, v\}$ for every edge uv of G not belonging to a triangle.

Proof. As by Theorem 2 the WELL-COVERED GRAPH problem can be solved in polynomial time on $(0,3)$-graphs, by Fact 4 $(0,3)$WC-G is in NP.

Let $I = (U, C)$ be a 3-SAT instance. We produce, in polynomial time in the size of I, a $(0,3)$WC-G instance H, such that I is satisfiable if and only if H is $(0,3)$-well-covered. Let $G = (V, E)$ be the graph of [35] obtained from I, and let G' be the graph obtained from G by adding to V a vertex x_{uv} for every edge uv of G not belonging to a triangle, and by adding to E edges ux_{uv} and vx_{uv}; see Figure 2(b). Finally, we define $H = \overline{G'}$ as the complement of G'. Note that, by [35], I is satisfiable if and only if G is 3-colorable. Since x_{uv} is adjacent to only two different colors of G, clearly G is 3-colorable if and only if G' is 3-colorable. Hence, I is satisfiable if and only if H is a $(0,3)$-graph. We prove next that I is satisfiable if and only if H is a $(0,3)$-well-covered graph.

Suppose that I is satisfiable. Then, since H is a $(0,3)$-graph, every maximal independent set of H has size 3, 2, or 1. If there is a maximal independent set I in H with size 1 or 2, then I is a maximal clique of G' of size 1 or 2. This contradicts the construction of G', since every maximal clique of G' is a triangle. Therefore, G is well-covered.

Suppose that H is $(0,3)$-well-covered. Then G' is 3-colorable, so G is also 3-colorable. Thus, by [35], I is satisfiable. □

We next prove that $(3,0)$WC-G is NP-hard. For this, we again use the proof of Stockmeyer [35], together with the following theorem.

Proposition 16 (Topp and Volkmann [37]). *Let $G = (V, E)$ be an n-vertex graph, $V = \{v_1, v_2, v_3, \ldots, v_n\}$, and let H be obtained from G such that $V(H) = V \cup \{u_1, u_2, u_3, \ldots, u_n\}$ and $E(H) = E \cup \{v_iu_i : i \in \{1, 2, 3, \ldots, n\}\}$. Then H is a well-covered graph where every maximal independent set has size n.*

Proof. Observe that every maximal independent set I of H has a subset $I_G = I \cap V$. Let $\mathcal{U} \subseteq \{1, 2, 3, \ldots, n\}$ be the set of indices i such that $v_i \in I$. Since I is maximal, the set $\{u_i : i \in \{1, 2, 3, \ldots, n\} \setminus \mathcal{U}\}$ must be contained in I, so $|I| = n$. $\square$

Lemma 17. $(3, 0)$WC-G *is* NP*-hard.*

Proof. Let $I = (U, C)$ be a 3-SAT instance; let $G = (V, E)$ be the graph obtained from I in Stockmeyer's [35] NP-completeness proof for 3-COLORING; and let H be the graph obtained from G by the transformation described in Proposition 16. We prove that I is satisfiable if and only if H is a $(3, 0)$-well-covered graph. Suppose that I is satisfiable. Then by [35] we have that G is 3-colorable. Since a vertex $v \in V(H) \setminus V(G)$ has just one neighbor, there are 2 colors left for v to extend a 3-coloring of G, and so H is a $(3, 0)$-graph. Hence, by Proposition 16, H is a $(3, 0)$-well-covered graph. Suppose that H is a $(3, 0)$-well-covered graph. Then we have that G is a $(3, 0)$-graph. By [35], I is satisfiable. $\square$

Note that Theorem 1 combined with Lemma 15 does not imply that $(1, 3)$WC-G is NP-complete.

Lemma 18. $(1, 3)$WC-G *is* NP*-complete.*

Proof. As by Theorem 2 the WELL-COVERED GRAPH problem can be solved in polynomial time on $(1, 3)$-graphs, by Fact 4 $(1, 3)$WC-G is in NP.

Let $I = (U, C)$ be a 3-SAT instance. Without loss of generality, I has more than two clauses. We produce a $(1, 3)$WC-G instance H polynomial in the size of I, such that I is satisfiable if and only if H is $(1, 3)$-well-covered.

Let $G = (V, E)$ be the graph of Stockmeyer [35] obtained from I (see Figure 2(a)), and let H be the graph obtained from $\overline{G}$ (the complement of the graph G) by adding one pendant vertex p_v for each vertex v of $\overline{G}$. Note that $V(H) = V(G) \cup \{p_v : v \in V(G)\}$, $E(H) = E(\overline{G}) \cup \{p_v v : v \in V(G)\}$, and $N_H(p_v) = \{v\}$.

First suppose that I is satisfiable. Then by [35], G is a $(3, 0)$-graph, and $\overline{G}$ is a $(0, 3)$-graph with partition into cliques $V(\overline{G}) = (K^1_{\overline{G}}, K^2_{\overline{G}}, K^3_{\overline{G}})$. Thus it follows that $(S = \{p_v : v \in V(G)\}, K^1_{\overline{G}}, K^2_{\overline{G}}, K^3_{\overline{G}})$ is a $(1, 3)$-partition of $V(H)$. In addition, from Proposition 16 and by the construction of H, H is a well-covered graph. Hence H is $(1, 3)$-well-covered.

Conversely, suppose that H is $(1, 3)$-well-covered, and let $V(H) = (S, K^1, K^2, K^3)$ be a $(1, 3)$-partition for H. Then we claim that no vertex $p_v \in V(H) \setminus V(G)$ belongs to $K^i, i \in \{1, 2, 3\}$. Indeed, suppose for contradiction that $p_v \in K^i$ for some $i \in \{1, 2, 3\}$. Then, $K^i \subseteq \{p_v, v\}$. Hence, $H \setminus K^i$ is a $(1, 2)$-graph and $G \setminus \{v\}$ is an induced subgraph of a $(2, 1)$-graph. But by construction of G, $G \setminus \{v\}$ (for any $v \in V(G)$) contains at least one $2K_3$ (that is, two vertex-disjoint copies of K_3) as an induced subgraph, which is a contradiction given that $2K_3$ is clearly a forbidden subgraph for $(2, 1)$-graphs. Therefore, $\{p_v : v \in V(G)\} \subseteq S$, and since $\{p_v : v \in V(G)\}$ is a dominating set of H, $S = \{p_v : v \in V(G)\}$. Thus, $\overline{G}$ is a $(0, 3)$-graph with partition $V(\overline{G}) = (K^1, K^2, K^3)$, and therefore G is a $(3, 0)$-graph, i.e., a 3-colorable graph. Therefore, by [35], I is satisfiable. $\square$

Corollary 19. *If $r \geq 3$ and $\ell = 0$, then (r, ℓ)*WC-G *is* NP*-hard. If $r \in \{0, 1\}$ and $\ell \geq 3$, then (r, ℓ)*WC-G *is* NP*-complete.*

Proof. (r, ℓ)WC-G is NP-hard in all of these cases by combining Theorem 1, and Lemmas 15, 17 and 18. For $r \in \{0, 1\}$ and $\ell \geq 3$, the Well-Covered Graph problem can be solved in polynomial time on (r, ℓ)-graphs, so by Fact 4 (r, ℓ)WC-G is in NP. □

Below we summarize the cases for which we have shown that WC-(r, ℓ)G or (r, ℓ)WC-G is computationally hard.

Theorem 20. *The following classification holds:*

1. WC-(r, ℓ)G *with $r \geq 2$ and $\ell \geq 1$ are* coNP*-complete;*
2. $(0, \ell)$WC-G *and* $(1, \ell)$WC-G *with $\ell \geq 3$ are* NP*-complete;*
3. $(2, 1)$WC-G *and* $(2, 2)$WC-G *are* coNP*-complete;*
4. (r, ℓ)WC-G *with $r \geq 0$ and $\ell \geq 3$ is* NP*-hard;*
5. (r, ℓ)WC-G *with $r \geq 3$ and $\ell \geq 0$ is* NP*-hard;*
6. (r, ℓ)WC-G *with $r \geq 2$ and $\ell \geq 1$ is* coNP*-hard.*

Proof. Statement 1 follows from Proposition 13 and Theorem 1(i). Statement 2 follows from Corollary 19. Statement 3 follows from Statement 1, Facts 2 and 3 and the fact that recognizing (r, ℓ)-graphs is in P if $\max\{r, \ell\} \leq 2$ [2]. Statement 4 follows from Statement 2 and Theorem 1(ii)-(iii). Statement 5 follows from Lemma 17 and Theorem 1(ii)-(iii). Finally, Statement 6 follows from Corollary 14 and Theorem 1(ii)-(iii). □

4. Parameterized Complexity of the Problems

In this section we focus on the parameterized complexity of the Well-Covered Graph problem, with special emphasis on the case where the input graph is an (r, ℓ)-graph. Recall that the results presented in Section 2 show that WC-(r, ℓ)G is para-coNP-complete when parameterized by r and ℓ. Thus, additional parameters should be considered. Henceforth we let α (resp. ω) denote the size of a maximum independent set (resp. maximum clique) in the input graph G for the problem under consideration. Note that WC-(r, ℓ)G parameterized by r, ℓ, and ω generalizes WC-$(r, 0)$G, whose complexity was left open in the previous sections. Therefore, we focus on the complexity of WC-(r, ℓ)G parameterized by r, ℓ, and α, and on the complexity of the natural parameterized version of Well-Covered Graph, defined as follows:

k-WELL-COVERED GRAPH	
Input:	A graph G and an integer k.
Parameter:	k.
Question:	Does every maximal independent set of G have size exactly k?

The next lemma provides further motivation to study of the WC-$(0,\ell)$G problem, as it shows that k-WELL-COVERED GRAPH (on general graphs) can be reduced to the WC-$(0,\ell)$G problem parameterized by ℓ.

Lemma 21. *The k-WELL-COVERED GRAPH problem can be fpt-reduced to the WC-$(0,\ell)$G problem parameterized by ℓ.*

Proof. Consider an arbitrary input graph G with vertices $u_1,\dots,u_n$. First, we find an arbitrary maximal (with respect to set-inclusion) independent set I in G. Without loss of generality we may assume that $|I| = k$ and $I = \{u_1,\dots,u_k\}$. Let $\ell = k+1$.

We construct a $(0,\ell)$-graph G' with vertex set $\{v_{i,j} : i \in \{1,\dots,\ell\}, j \in \{1,\dots,n\}\}$ as follows:

- For all $i \in \{1,\dots,\ell\}$ add edges to make $V_i := \{v_{i,j} : j \in \{1,\dots,n\}\}$ into a clique.
- For all $j \in \{1,\dots,n\}$ add edges to make $W_j := \{v_{i,j} : i \in \{1,\dots,\ell\}\}$ into a clique.
- For all pairs of adjacent vertices u_a, u_b in G, add edges between $v_{i,a}$ and $v_{j,b}$ for all $i,j \in \{1,\dots,\ell\}$ (so that V_a is complete to V_b).

Note that the sets V_i partition G' into ℓ cliques, so G' is indeed a $(0,\ell)$-graph, where $\ell = k+1$.

The graph G' has a maximal independent set of size k, namely $\{v_{1,1},\dots, v_{k,k}\}$, so G' is well-covered if and only if every maximal independent set in G' has size exactly k. Every maximal independent set in G' has at most one vertex in any set V_i and at most one vertex in any set W_j, since V_i and W_j are cliques. As there are $\ell = k+1$ sets V_i, it follows that every independent set in G' contains at most $k+1$ vertices. If G' contains an independent set $\{v_{i_1,j_1},\dots,v_{i_x,j_x}\}$ for some x then $\{u_{j_1},\dots,u_{j_x}\}$ is an independent set in G. If G contains an independent set $\{u_{j_1},\dots,u_{j_x}\}$ for some x then $\{v_{1,j_1},\dots,v_{\min(x,k+1),j_{\min(x,k+1)}}\}$ is an independent set in G'. Therefore G contains a maximal independent set smaller than k if and only if G' contains a maximal independent set smaller than k and G contains a (not necessarily maximal) independent set of size at least $k+1$ if and only G' contains a maximal independent set of size exactly $k+1$. It follows that G' is well-covered if and only if G is. As $\ell = k+1$, this completes the proof. □

Recall that the WELL-COVERED GRAPH problem is coNP-complete [4, 34]. In order to analyze the parameterized complexity of the problem, we will need the following definition.

Definition 1. *The class* coW[2] *is the class of all parameterized problems whose complement is in* W[2].

For an overview of parameterized complexity classes, see [12, 20].

We are now ready to show the next result.

Theorem 22. *The* WC-$(0, \ell)$G *problem parameterized by* ℓ *is* coW[2]*-hard.*

Proof. Red-Blue Dominating Set (RBDS) is a well-known W[2]-complete problem [15], which consists of determining whether a given bipartite graph $G = (R \cup B, E)$ admits a set $D \subseteq R$ of size k (the parameter) such that D dominates B (that is, every vertex in B has a neighbor in D). To show the coW[2]-hardness of our problem, we present an fpt-reduction from Red-Blue Dominating Set to the problem of determining whether a given $(0, \ell)$-graph is *not* well-covered, where $\ell = k + 1$.

From an instance (G, k) of RBDS we construct a $(0, \ell)$-graph G' as follows. Replace the set $R = \{r_1, r_2, \ldots, r_m\}$ by k copies: $R_1 = \{r_1^1, r_2^1, \ldots, r_m^1\}$, $R_2 = \{r_1^2, r_2^2, \ldots, r_m^2\}, \ldots, R_k = \{r_1^k, r_2^k, \ldots, r_m^k\}$, where each new vertex has the same neighborhood as the corresponding vertex did in G. Add edges to make B, as well as each R_i for $1 \leq i \leq k$, induce a clique. For each $i \in \{1, \ldots, k\}$, create a vertex s_i, and add all possible edges between s_i and the vertices in R_i. Let G' be the resulting graph. Note that the vertex set of G' can be partitioned into $\ell = k+1$ cliques: $B, R_1 \cup \{s_1\}, R_2 \cup \{s_2\}, \ldots, R_k \cup \{s_k\}$.

Clearly, for every $b \in B$, the set $\{s_1, s_2, \ldots, s_k\} \cup \{b\}$ is an independent set of G' of size $k+1$. Note that such an independent set is maximum, as it contains one vertex from each of the $k+1$ cliques that partition $V(G')$. In addition, any maximal independent set of G' has size at least k, since every maximal independent set contains either s_i or a vertex of R_i. At this point, we claim that G has a set $D \subseteq R$ of size k which dominates B if and only if G' has a maximal independent set of size k (i.e., G' is not well-covered).

If $D = \{r_{i_1}, r_{i_2}, \ldots, r_{i_k}\}$ is a subset of R of size k which dominates B in G, then $D' = \{r_{i_1}^1, r_{i_2}^2, \ldots, r_{i_k}^k\}$ is a maximal independent set of G', implying that G' is not well-covered.

Conversely, if G' is not well-covered then there exists in G' a maximal independent set D' of size k. Note that $D' \cap B = \emptyset$ and each vertex in B has at least one neighbor in D', as otherwise D' would not be a maximal independent set of size k. Therefore, by letting D be the set of vertices in R that have copies in $D' \cap \{R_1 \cup R_2 \cup \ldots \cup R_k\}$, we find that D is a subset of R of size at most k which dominates B in G. □

From the previous theorem we immediately obtain the following corollaries.

Corollary 23. *The* k-Well-Covered Graph *problem is* coW[2]*-hard.*

Proof. This follows immediately from Lemma 21 and Theorem 22. □

Corollary 24. *Unless* FPT = coW[2]*, the* WC-(r, ℓ)G *problem cannot be solved in time* $f(\alpha + \ell)n^{g(r)}$ *for any computable function* f*.*

Proof. This follows from the fact that an algorithm running in time $f(\alpha+\ell)n^{g(r)}$, would be an FPT-algorithm for WC-$(0,\ell)$G parameterized by ℓ, and from the $\mathsf{coW[2]}$-hardness of the problem demonstrated in Theorem 22. $\square$

In contrast to Corollary 24, Lemma 25 shows that the WC-(r,ℓ)G problem can be solved in time $2^{r\alpha}n^{O(\ell)}$.

Lemma 25. *The* WC-(r,ℓ)G *problem can be solved in time $2^{r\alpha}n^{O(\ell)}$. In particular, it is* FPT *when ℓ is fixed and r,α are parameters.*

Proof. Note that each of the r independent sets $S^1,\ldots,S^r$ of the given partition of $V(G)$ must have size at most α. On the other hand, any maximal independent set of G contains at most one vertex in each of the ℓ cliques. The algorithm exhaustively constructs all maximal independent sets of G as follows: we start by guessing a subset of $\bigcup_{i=1}^{r} S^i$, and then choose at most one vertex in each clique. For each choice, we just have to verify whether the constructed set is a maximal independent set, and then check that all the constructed maximal independent sets have the same size. The claimed running time follows. In fact, in the statement of the lemma, one could replace $r\alpha$ with $\sum_{1\leq i\leq r}|S^i|$, which yields a stronger result. $\square$

Although WC-$(1,\ell)$G parameterized by ℓ is $\mathsf{coW[2]}$-hard (see Theorem 22), Theorem 2 shows that the problem is in XP.

Corollary 26. *The* WC-$(1,\ell)$G *problem can be solved in time $n^{O(\ell)}$. In other words, it is in* XP *when parameterized by ℓ.*

Proof. This follows from Theorem 2 by considering ℓ to not be a constant. $\square$

Table 1 summarizes the results presented so far. Note that, by Ramsey's Theorem [31], when both ω and α are parameters the input graph itself is a trivial kernel.

4.1. Taking the Neighborhood Diversity as the Parameter

Neighborhood diversity is a structural parameter based on a special way of partitioning a graph into independent sets and cliques. Therefore, it seems a natural parameter to consider for our problem, since an (r,ℓ)-partition of a graph G is also a partition of its vertex set into cliques and independent sets.

Definition 2 (Lampis [26]). *The* neighborhood diversity $\mathsf{nd}(G)$ *of a graph $G=(V,E)$ is the minimum integer t such that V can be partitioned into t sets $V_1,\ldots,V_t$ where for every $v\in V(G)$ and every $i\in\{1,\ldots,t\}$, either v is adjacent to every vertex in V_i or it is adjacent to none of them. Note that each part V_i of G is either a clique or an independent set.*

Another natural parameter to consider is the vertex cover number, because well-covered graphs can be equivalently defined as graphs in which every minimal vertex cover has the same size. However, neighborhood diversity is *stronger* than vertex cover, in the sense that every class of graphs with bounded vertex

Table 1: Parameterized complexity of WC-(r,ℓ)G.

Param.\Class	$(0,\ell)$	$(1,\ell)$	(r,ℓ)
r	–	–	para-coNP-h
ℓ	coW[2]-h XP	coW[2]-h XP	para-coNP-h
r,ℓ	coW[2]-h XP	coW[2]-h XP	para-coNP-h
r,ℓ,ω	FPT Trivial	FPT Trivial	Open (generalizes WC-(3,0)G)
r,ℓ,α	coW[2]-h XP	coW[2]-h XP	coW[2]-h, no $f(\alpha+\ell)n^{g(r)}$ algo. unless FPT=coW[2], algo. in time $2^{r\alpha}n^{O(\ell)}$
ω,α	FPT Ramsey's Thm.	FPT Ramsey's Thm.	FPT Ramsey's Thm.

cover number is also a class of graphs with bounded neighborhood diversity, but the reverse is not true [26]. Thus, for our analysis, it is enough to consider the neighborhood diversity as the parameter. In addition, neighborhood diversity is a graph parameter that captures more precisely than vertex cover number the property that two vertices with the same neighborhood are “equivalent”.

It is worth mentioning that an optimal neighborhood diversity decomposition of a graph G can be computed in time $O(n^3)$; see [26] for more details.

Lemma 27. *The* Well-Covered Graph *problem is* FPT *when parameterized by neighborhood diversity.*

Proof. Given a graph G, we first obtain a neighborhood partition of G with minimum width using the polynomial-time algorithm of Lampis [26]. Let $t := \mathsf{nd}(G)$ and let $V_1,\ldots,V_t$ be the partition of $V(G)$. As we can observe, for any pair u,v of non-adjacent vertices belonging to the same part V_i, if u is in a maximal independent set S then v also belongs to S, otherwise S cannot be maximum. On the other hand, if $N[u]=N[v]$ then for any maximal independent set S_u such that $u \in S_u$ there exists another maximal independent set S_v such that $S_v = S_u \setminus \{u\} \cup \{v\}$. Hence, we can contract each partition V_i that is an independent set into a single vertex v_i with weight $\tau(v_i)=|S_i|$, and contract each partition V_i that is a clique into a single vertex v_i with weight $\tau(v_i)=1$, in order to obtain a graph G_t with $|V(G_t)|=t$, where the weight of a vertex v_i of G_t means that any maximal independent set of G uses either none or exactly $\tau(v)$ vertices of V_i. At this point, we just need to analyze whether all maximal independent sets of G_t have the same weight (sum of the weights of its vertices), which can be done in time $2^t n^{O(1)}$. ☐

Corollary 28. *The* Well-Covered Graph *problem is* FPT *when parameterized by the vertex cover number $n-\alpha$.*

4.2. Taking the Clique-width as the Parameter

In the 90's, Courcelle proved that for every graph property Π that can be formulated in *monadic second order logic* (MSOL_1), there is an $f(k)n^{O(1)}$ algorithm that decides if a graph G of clique-width at most k satisfies Π (see [5, 6, 7, 9]), provided that a k-expression is given.

LinEMSOL is an extension of MSOL_1 which allows searching for sets of vertices which are optimal with respect to some linear evaluation functions. Courcelle et al. [8] showed that every graph problem definable in LinEMSOL is linear-time solvable on graphs with clique-width at most k (i.e., FPT when parameterized by clique-width) if a k-expression is given as input. Using a result of Oum [29], the same result follows even if no k-expression is given.

Theorem 29. *The* Well-Covered Graph *problem is* FPT *when parameterized by clique-width.*

Proof. Given $S \subseteq V(G)$, first observe that the property "S is a maximal independent set" is MSOL_1-expressible. Indeed, we can construct a formula $\varphi(G,S)$ such that "S is a maximal independent set" $\Leftrightarrow \varphi(G,S)$ as follows:

$$[\nexists\ u,v \in S : edge(u,v)] \ \wedge\ [\nexists\ S' : (S \subseteq S')\ \wedge (\nexists\ x,y \in S' : edge(x,y))]$$

Since $\varphi(G,S)$ is an MSOL_1-expression, the problem of finding $goal(S)$: $\varphi(G,S)$ for $goal \in \{\max, \min\}$ is definable in LinEMSOL. Thus we can find $\max(S)$ and $\min(S)$ satisfying $\varphi(G,S)$ in time $f(\mathsf{cw}(G))n^{O(1)}$. Finally, G is well-covered if and only if $|\max(S)| = |\min(S)|$. $\square$

Corollary 30. *The* Well-Covered Graph *problem is* FPT *when parameterized by treewidth.*

Proof. This follows from the fact that graphs with treewidth bounded by k have clique-width bounded by a function of k [10]. $\square$

Corollary 31. *For any fixed r and ℓ, the* (r,ℓ)-Well-Covered Graph *problem is* FPT *when parameterized by clique-width.*

Proof. As r and ℓ are constants, the problem of determining whether G is an (r,ℓ)-graph is also MSOL_1-expressible. $\square$

Note that, since for every graph G we have $\mathsf{cw}(G) \leq \mathsf{nd}(G)+1$ [26], Lemma 27 is also a corollary of Theorem 29. Nevertheless, the algorithm derived from the proof of Lemma 27 is much simpler and faster than the one that follows from the meta-theorem of Courcelle et al. [8].

5. Further Research

Concerning the complexity of the (r,ℓ)wc-g and wc-(r,ℓ)g problems, note that the only remaining open cases are wc-$(r,0)$g for $r \geq 3$ (see the tables in Section 1). We do not even know if there exists some integer $r \geq 3$ such that WC-$(r,0)$G is coNP-complete, although we conjecture that this is indeed the case.

As another avenue for further research, it would be interesting to provide a complete characterization of well-covered tripartite graphs, as has been done for bipartite graphs [17, 32, 38]. So far, only partial characterizations exist [22, 39].

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References

[1] S. R. Alves, K. K. Dabrowski, L. Faria, S. Klein, I. Sau, and U. dos Santos Souza. On the (parameterized) complexity of recognizing well-covered (r,ℓ)-graphs. In *Proc. of the 10th International Conference on Combinatorial Optimization and Applications (COCOA)*, volume 10043 of *LNCS*, pages 423–437, 2016.

[2] A. Brandstädt. Partitions of graphs into one or two independent sets and cliques. *Discrete Mathematics*, 152(1-3):47–54, 1996.

[3] A. Brandstädt, F. F. Dragan, H.-O. Le, and R. Mosca. New graph classes of bounded clique-width. *Theory of Computing Systems*, 38(5):623–645, 2005.

[4] V. Chvátal and P. J. Slater. A note on well-covered graphs. *Annals of Discrete Mathematics*, 55:179–181, 1993.

[5] B. Courcelle. The monadic second-order logic of graphs. I. recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.

[6] B. Courcelle. The expression of graph properties and graph transformations in monadic second-order logic. *Handbook of Graph Grammars*, 1:313–400, 1997.

[7] B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach.* Cambridge University Press, 1st edition, 2012.

[8] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33(2):125–150, 2000.

[9] B. Courcelle and M. Mosbah. Monadic second-order evaluations on tree-decomposable graphs. *Theoretical Computer Science*, 109(1–2):49–82, 1993.

[10] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1–3):77–114, 2000.

[11] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms.* Springer, 2015.

[12] R. de Haan and S. Szeider. Machine Characterizations for Parameterized Complexity Classes Beyond Para-NP. In *Proc. of the 41st International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM)*, volume 8939 of *LNCS*, pages 217–229, 2015.

[13] N. Dean and J. S. Zito. Well-covered graphs and extendability. *Discrete Mathematics*, 126(1–3):67–80, 1994.

[14] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[15] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[16] J. Edmonds. Paths, trees and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.

[17] O. Favaron. Very well-covered graphs. *Discrete Mathematics*, 42:177–187, 1982.

[18] T. Feder, P. Hell, S. Klein, and R. Motwani. List partitions. *SIAM Journal on Discrete Mathematics*, 16(3):449–478, 2003.

[19] T. Feder, P. Hell, S. Klein, L. T. Nogueira, and F. Protti. List matrix partitions of chordal graphs. *Theoretical Computer Science*, 349(1):52–66, 2005.

[20] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006.

[21] M. C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. *International Journal of Foundations of Computer Science*, 11(03):423–443, 2000.

[22] H. Haghighi. A generalization of Villarreal's result for unmixed tripartite graphs. *Bulletin of the Iranian Mathematical Society*, 40(6):1505–1514, 2014.

[23] M. Kamiński, V. V. Lozin, and M. Milanič. Recent developments on graphs of bounded clique-width. *Discrete Applied Mathematics*, 157(12):2747–2761, 2009.

[24] R. M. Karp. Reducibility among combinatorial problems. *Complexity of Computer Computations*, pages 85–103, 1972.

[25] S. Kolay, F. Panolan, V. Raman, and S. Saurabh. Parameterized algorithms on perfect graphs for deletion to (r, ℓ)-graphs. In *Proc. of the 41st International Symposium on Mathematical Foundations of Computer Science (MFCS)*, volume 58 of *LIPIcs*, pages 75:1–75:13, 2016.

[26] M. Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64(1):19–37, 2012.

[27] M. Lesk, M. D. Plummer, and W. R. Pulleyblank. Equi-matchable graphs. *Graph Theory and Combinatorics. Academic Press*, pages 239–254, 1984.

[28] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*, volume 31. Oxford University Press, 2006.

[29] S.-I. Oum. Approximating rank-width and clique-width quickly. *ACM Transactions on Algorithms*, 5(1):10:1–10:20, 2008.

[30] M. D. Plummer. Some covering concepts in graphs. *Journal of Combinatorial Theory*, 8(1):91–98, 1970.

[31] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930.

[32] G. Ravindra. Well-covered graphs. *Journal of Combinatorics, Information & System Sciences*, 2(1):20–21, 1977.

[33] N. Robertson and P. D. Seymour. Graph minors. X. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.

[34] R. S. Sankaranarayana and L. K. Stewart. Complexity results for well-covered graphs. *Networks*, 22(3):247–262, 1992.

[35] L. Stockmeyer. Planar 3-colorability is polynomial complete. *ACM SIGACT News*, 5(3):19–25, 1973.

[36] D. Tankus and M. Tarsi. Well-covered claw-free graphs. *Journal of Combinatorial Theory, Series B*, 66(2):293–302, 1996.

[37] J. Topp and L. Volkmann. Well covered and well dominated block graphs and unicyclic graphs. *Mathematica Pannonica*, 1(2):55–66, 1990.

[38] R. H. Villarreal. Unmixed bipartite graphs. *Revista Colombiana de Matemáticas*, 41(2):393–395, 2007.

[39] R. Zaare-Nahandi and R. J. Golzari. Unmixed r-partite graphs. *Bulletin of the Iranian Mathematical Society*, in press, 2017.