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Abstract

In this paper, we present a denotational semantic for non-wellfounded proofs of μLL_∞, linear logic extended with least and greatest fixed points, by adapting the categorical semantics of μLL_∞ [EJ21]. Two instances of this categorical setting are **REL** (category of sets and relations), and **NUTS** (category of sets equipped with a notion of totality and relations preserving it) which is studied in [EJ21]. In particular, we relate validity condition for non-wellfounded proofs and totality of **NUTS**. More precisely, we show each μLL_∞ valid proof will have interpreted as a total element in **NUTS**.

1   Introduction

μLL is a version of propositional Linear Logic with least and greatest fixed points extending propositional $\mu MALL$ with exponentials [Bae12]. In [Dou17] and [BDS16], the $\mu MALL_\infty$ system, which is multiplicative and additive linear logic with two rules for unfolding fixed-points, is studied. And they have defined a syntactic notion of validity on proofs in order to distinguish sound and unsound proofs, as we can see a same notion in [Bro06] and [BS07].

One of our purpose is to develop a more Curry-Howard oriented point of view on μLL_∞ through the denotational semantic. And as there are different validity conditions on the μLL_∞ proofs (such as straight-thread, bouncing, etc. [BDKS20]), we hope that the denotational semantic helps us to understand which one is more appropriate.

There is a categorical semantics of μLL [EJ21], and two instances of this categorical setting are **REL** (category of sets and relations), and **NUTS** (category of sets equipped with a notion of totality and relations preserving it) which is studied in [EJ21].

We have considered μLL_∞, an extension of $\mu MALL_\infty$ with exponentials as the main system. And as the denotational semantic of μLL_∞, we have examined **REL** and **NUTS**.

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Organization of the paper   We first recap the language and the inference rules of $\mu\mathrm{LL}_\infty$ in the section 2. In the section 3 and 4, the interpretation of formula and proofs in **REL** and **NUTS** are provided. And finally, the main contribution is mentioned in the section 5 which relates validity condition and totality of **NUTS**. More precisely, we show that each $\mu\mathrm{LL}_\infty$ valid proof will have interpreted as a total element in **NUTS**.

2   Syntax of $\mu\mathrm{LL}_\infty$

The syntax of $\mu\mathrm{LL}_\infty$ is an extension of $\mu\mathrm{MALL}_\infty$ ([Dou17] and [BDS16]) with exponentials. We assumed to be given an infinite set of type variables names $X, Y, \cdots$.
    The formulas of $\mu\mathrm{LL}_\infty$ are defined as follows:

$$A, B, \cdots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \,\mathbin{\bindnasrepma}\, B \mid\; ?A \mid\; !B \mid X \mid \mu X.F \mid \nu X.F \tag{1}$$

The definition of negation for linear logic formula is as usual. However the orthogonality acts as the identity on variable, i.e $X^\perp = X$, and for the fixed-point: $(\mu X.A)^\perp = \nu X.A^\perp$ and $(\nu X.A)^\perp = \mu X.A^\perp$. One motivation for this definition is that it prevents the definition of recursive types with negative dependencies which leads to non-terminating programs.

Remark 1. *We can define the exponentials in $\mu\mathrm{MALL}_\infty$ by taking !A as the formula $\nu X.(1 \,\&\, A \,\&\, (X \otimes X))$ ([BDS16]). And indeed all the structural rules are derivable but we do not have the Seely isomorphism semantically.*

We consider the inference rules in a standard one-sided Linear Logic sequent calculus as in [Dou17] and [BDS16]. We just mention the fixed-point rules and refer to [Gir87] for the linear logic inference rules. The fixed-point rules are as follows:

$$\frac{\vdash \Gamma, A[\mu X.A/X]}{\vdash \Gamma, \mu X.A}\,\mu \qquad \frac{\vdash \Gamma, A[\nu X.A/X]}{\vdash \Gamma, \nu X.A}\,\nu$$

2.1 $\mu\mathrm{LL}_\infty$ proofs

A $\mu\mathrm{LL}_\infty$ pre-proof is a possibly infinite tree, generated by the given inference rules of previous section. The pre-proofs can be unsound. For instance the sequent $\vdash$ is provable:

$$\cfrac{\cfrac{\vdots}{\cfrac{\mu X.A}{\mu X.A}} \qquad \cfrac{\vdots}{\cfrac{\nu X.A}{\nu X.A}}}{\vdash}$$

To distinguish proper proofs from pre-proofs we consider a criterion, called *validity condition*, which is summed up in two following definitions here; we refer to [Dou17] for details.

Definition 2. *We define the relation* $\rightarrow_{FL}$ *on formulas as follows:*

- $A * B \rightarrow_{FL} A$ *and* $A * B \rightarrow_{FL} B$ *where* $*$ *is a linear logic connective.*

- $@A \rightarrow_{FL} A$ *where* $@$ *is either* $?$ *or* $!$.

- $\sigma X.F \rightarrow_{FL} F[\sigma X.F/X]$ *where* σ *is either* ν *or* μ.

The Fischer-Ladner sub-formula of a formula F is the formula G such that $F \rightarrow_{FL}^* G$ where $\rightarrow_{FL}^*$ is the reflexive transitive closure of $\rightarrow_{FL}$.

Definition 3. *A thread is a sequence $t = (A_i)_{i \in \omega}$ such that for all i either A_{i+1} is Fischer-Ladner sub-formula of A_i or $A_i = A_{i+1}$.*

Definition 4. *A valid thread is a thread t such that $min(Inf(t))$ is a ν-formula where $Inf(t)$ is set of formulas that happens infinitely often in t and minimum is respect to the usual sub-formula ordering (not Fischer-Ladner).*

Definition 5. *A valid μLL_∞ proof is a pre-proof π such that for any infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$, there is a non stationary valid thread $t = (A_i)_{i>j}$ where $j \in \omega$ and $\forall i > j(A_i \in \Gamma_i)$ and A_{i+1} is a suboccurrence of A_i.*

3 Interpreting fixed-point μLL_∞ formulas in REL

REL is the well-known model of linear logic interpreting formulas and proofs in the category of sets and relations. In this section, we provide an interpretation of fixed-point formula in **REL**.

To do that, we first define the notion of variable set as [EJ21]:

Definition 6. *A (n-ary) variable set (VS) is a endofunctor $\mathscr{E} : \mathbf{REL} \rightarrow \mathbf{REL}$ such that $\mathscr{E}$ is monotonic and continuous on objects and on morphisms, meaning that*

- *If when $\overrightarrow{f}, \overrightarrow{g} \in \mathbf{REL}^n(\overrightarrow{X}, \overrightarrow{Y})$ satisfy $\overrightarrow{f} \subseteq \overrightarrow{g}$, then $\mathscr{E}(\overrightarrow{f}) \subseteq \mathscr{E}(\overrightarrow{g})$, and*

- *if D is a directed subset of $\mathbf{REL}^n(X,Y)$, then $\mathscr{E}(\bigcup D) = \bigcup_{\overrightarrow{f} \in D} \mathscr{E}(\overrightarrow{f})$.*

As we can see in [EJ21], the linear logic operations $\otimes, \oplus, \parr, \&, ?, !$ are VS and VS are closed under De Morgan duality and composition.

Given a VS, $\mathscr{E} : \mathbf{REL} \rightarrow \mathbf{REL}$, we define $\sigma \mathscr{E}$ as $\bigcup_{n=0}^{\infty} \mathscr{E}^n(\emptyset)$. Then one can check that $\mathscr{E}(\sigma \mathscr{E}) = \sigma \mathscr{E}$.

Note that we only dealt with unary VS's to avoid some further technicalities.

With any formula A and any repetition-free sequence $\overrightarrow{X} = (X_1, \cdots, X_n)$ of variables containing all variables free in A, one can associates an n-ary VS $[\![A]\!]_{\overrightarrow{X}}$ by induction on A. So, we can take $\sigma [\![A]\!]_{\overrightarrow{X}}$ as the interpretation of $\mu X.A$ and $\nu X.A$, i.e $[\![\mu X.A]\!]_{\overrightarrow{X}} = [\![\nu X.A]\!]_{\overrightarrow{X}} = \sigma [\![A]\!]_{X,\overrightarrow{X}}$.

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Remark 7. *$\mu X.A$ and $\nu X.A$ have the same interpretation in* **REL**. *But we can distinguish them in non-uniform totality space [EJ21]. We summarize this construction in the next section and refer to [EJ21] for more details.*

4 Non-uniform totality space

Consider $\mathscr{U} \subseteq \mathscr{P}(X)$ for a given set X, we then define:

$$\mathscr{U}^{\perp} = \{u' \subseteq X \mid \forall u \in \mathscr{U} \, (u \cap u' \neq \emptyset)\}$$

A non-uniform totality space X is a pair $(|X|, \mathscr{T}X)$ such that $|X|$ is a set (at most countable) and $(\mathscr{T}X)^{\perp\perp} = \mathscr{T}X$. One of the crucial property of the non-uniform totality spaces is that we have a description of bi-orthogonality by the following lemma [EJ21]:

Lemma 8. *For any $\mathscr{U} \subseteq \mathscr{P}(X)$, we have $(\mathscr{U})^{\perp\perp} = \mathscr{U}$ if and only if $\mathscr{U}$ is upward closed.*

We define the category **NUTS** whose objects are non-uniform totality spaces and as the morphism $t \in \mathbf{NUTS}(X,Y)$ iff $\forall u \in \mathscr{T}X(t.u \in \mathscr{T}Y)$ where $t.u = \{y \in |Y| \mid \exists x \in u \, (x,y) \in t\}$

We now mention a lemma that will be useful for the construction of interpretation of fixed-point in **NUTS**. This lemma is proved in [EJ21].

Lemma 9. *Let $\mathsf{Tot}(X)$ be the set of all totality candidates, i.e. $\mathsf{Tot}(X) = \{\mathscr{U} \subseteq \mathscr{P}(X) \mid (\mathscr{U})^{\perp\perp} = \mathscr{U}\}$. Then $\mathsf{Tot}(X)$ is a complete lattice ordered by $\subseteq$.*

4.1 Interpretation of the μLL_{∞} formulas in NUTS

For any formula A of μLL_{∞}, we provide an interpretation $[\![A]\!] = (|[\![A]\!]|, \mathscr{T}[\![A]\!])$ as a non-uniform totality space. We first deal with linear logic formula:

- $[\![1]\!] = [\![\perp]\!] = (\{*\}, \{\{*\}\})$, $[\![A^{\perp}]\!] = (|[\![A]\!]|, (\mathscr{T}[\![A]\!])^{\perp})$

- $[\![A \otimes B]\!] = (|[\![A]\!]| \times |[\![B]\!]|, \{w \mid \exists u \in \mathscr{T}[\![A]\!] \text{ and } \exists v \in \mathscr{T}[\![B]\!] \text{ such that } u \times v \subseteq w\})$

- $[\![A \,\&\, B]\!] = (\{1\} \times |[\![A]\!]| \cup \{2\} \times |[\![B]\!]|, \{u \subseteq |[\![A \,\&\, B]\!]| \mid \pi_1(u) \in \mathscr{T}[\![A]\!] \wedge \pi_2(u) \in \mathscr{T}[\![B]\!]\}$

- $[\![!A]\!] = (\mathscr{M}_{fin}(|[\![A]\!]|), \{y \mid \exists x \in \mathscr{T}[\![A]\!] \wedge x^! \subseteq y\})$ where $x^! = \mathscr{P}_{fin}(x)$.

Note that the case of $\oplus, \mathscr{V}, ?$ are done by duality.

Lemma 10 (From [EJ21]). *$t \in NUTS([\![A]\!], [\![B]\!])$ iff $t \in [\![A \multimap B]\!] = [\![(A \otimes B^{\perp})^{\perp}]\!]$.*

For fixed-point formula, we define the notion variable of non-uniform totality space as follows [EJ21]:

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Definition 11. *An n-ary variable of non-uniform totality space (*VNUTS*)*

$\mathbb{E}$ *is a pair* $(|\mathbb{E}|, \mathscr{T}\mathbb{E})$ *such that* $|\mathbb{E}| : \mathbf{REL} \to \mathbf{REL}$ *is a (n-ary)* VS*, and* $\mathscr{T}\mathbb{E}$ *is an operation such that*

- $\mathscr{T}\mathbb{E}((|X|, \mathscr{T}X)) \in \mathsf{Tot}(|\mathbb{E}|(|X|))$ *for any non-uniform totality space* $(|X|, \mathscr{T}X)$, *and*

- *for any* $\overrightarrow{\tau} \in \mathbf{NUTS}(\overrightarrow{X}, \overrightarrow{Y})$, *the morphism* $|\mathbb{E}|(\overrightarrow{\tau})$ *belongs actually to* $\mathbf{NUTS}(\mathbb{E}(\overrightarrow{X}, \mathbb{E}(\overrightarrow{Y}))$, *so that* $\mathbb{E}$ *defines a functor* $\mathbf{NUTS}^n \to \mathbf{NUTS}$ *(denoted simply as* $\mathbb{E}$*) .*

4.1.1 Fixed Points of VNUTS:

Let $\mathbb{E}$ be a VNUTS, we define a non-uniform totality space $\mu.\mathbb{E} = (|\mu.\mathbb{E}|, \mathscr{T}(\mu.E))$. First, we set $|\mu.\mathbb{E}| = \sigma|\mathbb{E}|$. Now we define a map $\theta_{\mathbb{E}} : \mathsf{Tot}(|\mu.\mathbb{E}|) \to \mathsf{Tot}(|\mu.\mathbb{E}|)$ such that it maps $T \in \mathscr{T}|\mu.\mathbb{E}|$ to $\mathscr{T}\mathbb{E}(|\mu.\mathbb{E}|, T)$. Note that by definition of VNUTS, $\mathscr{T}\mathbb{E}(|\mu.\mathbb{E}|, T) \in \mathsf{Tot}(|\mathbb{E}|(|\mu.\mathbb{E}|)) = \mathsf{Tot}(|\mu.\mathbb{E}|)$. The map $(\theta_{\mathbb{E}})$ is a monotonic map on the complete lattice $\mathsf{Tot}(|\mu.\mathbb{E}|)$ (as shown in [EJ21]). By Knaster-Tarski's Theorem, there is a least fixed point $\mathscr{U}$ of $\theta_{\mathbb{E}}$, and we set $\mathscr{T}(\mu.\mathbb{E}) = \mathscr{U}$.

We actually can describe that $\mathscr{U}$ by sequence of candidates of totality for $|\mu.\mathbb{E}|$, indexed by ordinals: $\mathscr{U}_{\alpha+1} = \theta_{\mathbb{E}}(\mathscr{U}_\alpha)$ and $\mathscr{U}_\lambda = (\bigcup_{\alpha<\lambda} \mathscr{U}_\alpha)^{\perp\perp}$. One can also define $\nu.\mathbb{E}$ by De Morgan duality, i.e. $\nu.\mathbb{E} = (\sigma|\mathbb{E}|, (\mathscr{T}(\mu.(\mathbb{E})^\perp))^\perp)$. A more explicit construction for $\nu.\mathbb{E}$ is given in the section 5.

Lemma 12. *Let A be a formula and* $\overrightarrow{X} = (X_1, \cdots, X_n)$ *be a repetition-free list of variables containing all free variables of A. Let* $B_1, \cdots, B_n$ *be a list of formulas and let* $\overrightarrow{Y} = (Y_1, \cdots, Y_n)$ *be a repetition-free list of variables containing all free variables of* $B_1, \cdots, B_n$. *Then we have the:*

$$\llbracket A[C_1/X_1, \cdots, C_n/X_n] \rrbracket_{\overrightarrow{Y}} = \llbracket A \rrbracket_{\overrightarrow{X}} \circ (\llbracket C_1 \rrbracket_{\overrightarrow{Y}}, \cdots, \llbracket C_n \rrbracket_{\overrightarrow{Y}})$$

4.2 Interpretation of the μLL_∞ proofs in NUTS

The interpretation of μLL_∞ inference rules in **NUTS** is same as their interpretation in **REL**. So, we only mention the case for fixed-point rules, and this is fairly simple by taking the interpretation of the premise. More precisely, let us say π' is the proof of $\vdash \Gamma, \mu X.A$, and π is obtained by applying the μ rule on π'. Then we take $\llbracket \pi \rrbracket$ same as $\llbracket \pi' \rrbracket$. And we do the same for the case of ν rule.

We can not simply interpret an infinite proof by induction on the proof tree, since we have to deal with an infinite object. The idea is to consider all finite approximations of the proof, and then take the union of the interpretation of all finite approximation. However there is another idea to interpret an infinite proof in [KPP21] which is based on the notion of the computation and an well-founded relation on them.

The cut-elimination theorem on $\mu MALL_\infty$ is provided in [Dou17]. One can define a set $\mathbb{R}$ of reduction rules on μLL_∞ by taking $\mathbb{R}$ as the cut-elimination rules of $\mu MALL_\infty$ plus the corresponding reduction rules for the exponentials. Then

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we can show that the denotational semantic **NUTS** respects the reduction rules as follows:

Theorem 13. *The interpretation of* μLL$_\infty$ *proofs in* **NUTS** *(***REL***) is preserved by the reduction rules* $\mathbb{R}$*.*

5   Validity implies totality

In this section, we prove our main result which says that the interpretation of any valid proof is a total element, i.e. theorem 19. The proof method is similar to the proof of soundness of *LKID*$^\omega$ in [Bro06]. However the system of [Bro06] is classical logic with inductive definitions, and the proof is for a Tarskian semantic. We need to adapt the proof in two aspects: considering μLL$_\infty$ instead of *LKID*$^\omega$, and try to deal with the denotational semantic instead of Tarskian semantics. The adapation for μLL$_\infty$ is somehow done in [Dou17], since there is soundness theorem for μLL$_\infty$ with respect to the truncated truth semantics (a Tarskian semantic). So, basically, the main point of this section is adapting a Tarskian soundness theorem to a denotational semantic soundness.

As we saw in the previous section, given a formula $\nu X.A$, we can define its interpretation by a transfinite induction considering sequences of totality candidate as follows:

- $U_0 = \mathscr{P}_{fin}(|[\![\nu X.A]\!]|)$,

- $U_{\alpha+1} = \mathscr{T}[\![A]\!](|[\![\nu X.A]\!]|, U_\alpha)$

- $U_\delta = \bigcap_{\alpha<\delta} U_\alpha$

- and finally, there is an ordinal, denoted as λ_A, such that $U_{\lambda_A} = U_{\lambda_A+1}$.

We use the notation U_α freely without mentioning the formula. One can find what the corresponding formula is from the context.

The following definition is borrowed from [Dou17].

Definition 14. *The marked formulas of* μLL$_\infty$ *are defined as follows where* α *is an ordinal:*

$$A,B,\dots := 1 \mid 0 \mid \perp \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \,\mathcal{Y}\, B \mid ?A \mid !B \mid X \mid \mu X.A \mid \nu^\alpha X.A \tag{2}$$

The interpretation of $\nu^\alpha X.A$ in **NUTS** is $[\![\nu^\alpha X.A]\!] = (|[\![\nu X.A]\!]|, U_\alpha)$. And the interpretation of the other marked formulas are same as the case of μLL$_\infty$ formula.

Proposition 15. *Let A be a* μLL$_\infty$ *formula. Then we have* $[\![\bar{A}]\!] = [\![A]\!]$ *where* $\bar{A}$ *is the marked formula, obtained from A by marking every* ν *binder of A by the ordinal* λ_A*.*

Lemma 16. *Given a marked formula $\nu^\alpha X.A$. If $x \notin \mathscr{T}[\![\nu^\alpha X.A]\!]$, then there exists an ordinal $\gamma < \alpha$ such that $x \notin \mathscr{T}[\![A[\nu^\gamma X.A/X]]\!]$.*

Proof. If α is a successor ordinal $\delta + 1$ then $U_\alpha = \mathscr{T}[\![A]\!]((|[\![\nu X.A]\!]|, U_\delta))$ by definition, and obviously $x \notin \mathscr{T}[\![A]\!]((|[\![\nu X.A]\!]|, U_\delta))$. And so $x \notin \mathscr{T}[\![A[\nu^\gamma X.A/X]]\!]$ for $\gamma = \delta$ using lemma 12.

If α is a limit ordinal, then: $U_\alpha = \bigcap_{\gamma < \alpha} U_\gamma$, and $x \notin \bigcap_{\gamma < \alpha} U_\gamma = \bigcap_{\delta+1 < \alpha} U_{\delta+1}$. So, there exists an ordinal $\delta + 1 < \alpha$ such that $x \notin U_{\delta+1}$ and we continue as before. $\square$

Lemma 17. $\mathscr{T}[\![A[\mu X.A/X]]\!] = \mathscr{T}[\![\mu X.A]\!]$.

Proof. The interpretation of $\mu X.A$ is the least fixed-point of θ_A. So, we have:

$$
\begin{aligned}
\mathscr{T}[\![\mu X.A]\!] &= \theta_A(\mathscr{T}[\![\mu X.A]\!]) \\
&= \mathscr{T}[\![A]\!]((|[\![\mu X.A]\!]|, \mathscr{T}[\![\mu X.A]\!])) \qquad \text{by definition of } \theta_A \\
&= \mathscr{T}[\![A[\mu X.A/X]]\!] \qquad \qquad \qquad \text{by Lemma 12}
\end{aligned}
$$

$\square$

Lemma 18. *Given a proof π of $\vdash \Gamma$. If $[\![\pi]\!] \notin \mathscr{T}([\![\Gamma]\!])$, then*

1. *there exists an infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$ such that $[\![\pi_i]\!] \notin \mathscr{T}([\![\Gamma_i]\!])$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;*

2. *there exists a sequence of functions $(f_i)_{i \in \omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that if $\Gamma_i = \Gamma'_i, C$, then there exists $x \in \mathscr{T}[\![f_i(\Gamma'_i)^\perp]\!]$ such that $[\![\pi_i]\!].x \notin \mathscr{T}[\![f_i(C)]\!]$ where $\Gamma'_i = A_0^i, \cdots, A_n^i$ and $[\![f_i(\Gamma'_i)^\perp]\!] = [\![f_i(A_0^i)]\!]^\perp \otimes \cdots \otimes [\![f_i(A_n^i)]\!]^\perp$.*

Theorem 19. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathscr{T}[\![\Gamma]\!]$.*

Proof. Let us assume $[\![\pi]\!] \notin \mathscr{T}[\![\Gamma]\!]$. We can then apply Lemma 18 to obtain an infinite branch $(\vdash \Gamma_i)_{i \in \omega}$ and a sequence $(f_i)_{i \in \omega}$ satisfying properties 1 and 2 of the lemma. By the definition of valid proof, there exists a valid thread $t = (F_i)_{i \in \omega}$ for the infinite branch $(\vdash \Gamma_i)_{i \in \omega}$. Take the suffix t' of t such that $D \in t'$ iff $D \in Inf(t)$. Let $\nu X.F$ be the minimal formula formula of t'. So, there are infinitely many times that we use a ν rule to unfold $\nu X.F$, and all of those $\nu X.F$ are in t' so that those are suboccurrence of each other. Let $(i_k)_{k \in \omega}$ be the sequence of indices where $\nu X.F$ gets unfolded. By the property 2 of lemma 18, $f_{i_k}(F_{i_k}) = \nu^{\alpha_k} X.f_{i_k}(F)$. Therefore, by the property 2 of lemma 18 and by the construction of the f_i in the proof of lemma 18, the sequence $(\alpha_k)_{k \in \omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $[\![\pi]\!] \in \mathscr{T}[\![\Gamma]\!]$. $\square$

And at the end, we conclude with the following remark.

Remark 20. *The converse of the theorem 19 is not necessarily true, and there are many counterexamples indeed. For instance, simply take the following proof:*

$$\cfrac{\cfrac{\overline{\vdash \nu X.X, \mu X.X}\ id}{\vdash \nu X.X, \mu X.X}\ \nu \qquad \cfrac{\star}{\vdash \nu X.X}}{\star \vdash \nu X.X}\ cut$$

The interpretation of this proof is $\emptyset$ and $\emptyset \in \mathcal{T}[\![\nu X.X]\!]$ but that is not a valid proof regrading the definition 5.

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