Totally linear proofs

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From the proceedings of Trends in Linear Logic and Applications 2021 (TLLA 2021, eds E. Pimentel, Ch. Retoré)
https://hal.archives-ouvertes.fr/TLLA2021

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Abstract

We show a proof system for propositional classical logic where all the inference rules are linear and there are no unit equations. There is no creation and no destruction inside derivations at all.

1 Outline of the result

Recent research has shown how to obtain standard proof theoretic results for non-linear logics, such as cut-elimination, out of linear proof systems [ATG17, BG21]. It can be done because all the common inference rules, no matter if linear or non-linear, can be recovered from a common linear shape via an interpretation function that takes care of the multiplicities of atoms and their negation. To achieve that result, we treat atoms as non-commutative self-dual connectives whose arguments are their truth values. In other words, atoms are superpositions of truth values. We need to allow inference rules to have access to formulae in the scope of atomic connectives, in deep-inference fashion. One strong reason to use deep inference is that Tiu has shown that it is the only way to obtain cut elimination for self-dual non-commutative connectives in linear proof systems [Tiu06].

For example, here is how the interpretation function for classical logic obtains a contraction and a cut from linear rules:

$$\frac{(0\,\mathbf{a}\,1)\vee(0\,\mathbf{a}\,1)}{(0\vee 0)\,\mathbf{a}\,(1\vee 1)} \quad \overset{\mathsf{C}}{\mapsto} \quad \frac{a\vee a}{a} \quad ,$$

$$\frac{(1\,\mathbf{a}\,0)\wedge(0\,\mathbf{a}\,1)}{(1\wedge 0)\,\mathbf{a}\,(0\wedge 1)} \quad \overset{\mathsf{C}}{\mapsto} \quad \frac{\bar{a}\wedge a}{0} \quad .$$

On the left of the arrows, we have derivations in subatomic proof systems, where atoms are represented by a denumerable supply of self-dual non-commutative connectives (in boldface). On the right, we have standard derivations in deep inference. In the cases above, we have single inference steps but the interpretation can be lifted to whole derivations in the natural way. The interpretation function computes the values of unit expressions for any given logic and then interprets the

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atoms accordingly. The same derivation can have different interpretations. For example, contraction in linear logic is interpreted as

$$\frac{(0\,\mathbf{a}\,1)\vee(0\,\mathbf{a}\,1)}{(0\vee 0)\,\mathbf{a}\,(1\vee 1)} \quad \overset{\mathsf{L}}{\mapsto} \quad \frac{a \parr a}{0\,\mathbf{a}\,(1\parr 1)} \quad .$$

In other words, the interpretation of contraction in linear logic falls outside of the standard language for linear logic because $1 \parr 1$ is not equivalent to 1 in linear logic. Perhaps surprisingly, one can use the subatomic system to prove cut-elimination for linear logic because non-interpretable formulae are not created in proofs of interpretable formulae. For this reason, it is possible to prove very general normalisation results in subatomic systems that can be automatically specialised to standard systems for several logics (including classical and linear logic).

Remarkably, in subatomic proof systems, the inference rules are obtained from what we call the 'subatomic shape'

$$\frac{(\mathbf{A}\,\alpha\,\mathbf{B})\,\beta\,(\mathbf{C}\,\alpha\,\mathbf{D})}{(\mathbf{A}\,\beta\,\mathbf{C})\,\alpha\,(\mathbf{B}\,\beta\,\mathbf{D})}$$

by applying a 'saturation function' to one of the instances of the connectives shown in red. Saturation is so defined: $\breve{\vee} = \breve{\wedge} = \vee$; $\hat{\vee} = \hat{\wedge} = \wedge$ and $\breve{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$; if the language has more connectives, saturation is defined for those as well. For example, two of the four possible instances of the shape are

$$\hat{\alpha}\beta\,\frac{(\mathbf{A}\,\hat{\alpha}\,\mathbf{B})\,\beta\,(\mathbf{C}\,\alpha\,\mathbf{D})}{(\mathbf{A}\,\beta\,\mathbf{C})\,\alpha\,(\mathbf{B}\,\beta\,\mathbf{D})} \qquad \text{and} \qquad \alpha\breve{\beta}\,\frac{(\mathbf{A}\,\alpha\,\mathbf{B})\,\beta\,(\mathbf{C}\,\alpha\,\mathbf{D})}{(\mathbf{A}\,\beta\,\mathbf{C})\,\alpha\,(\mathbf{B}\,\breve{\beta}\,\mathbf{D})} \quad ,$$

where we name the rules after the connectives shown in red. If, for example, we fix $\alpha = \vee$ and $\beta = \wedge$, and we pick the top right corner for saturation, we obtain the rule

$$\wedge\hat{\vee}\,\frac{(\mathbf{A}\vee\mathbf{B})\wedge(\mathbf{C}\wedge\mathbf{D})}{(\mathbf{A}\wedge\mathbf{C})\vee(\mathbf{B}\wedge\mathbf{D})} \quad .$$

Given a set of connectives, which we take to be $\mathscr{C} = \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$, we call $\star\breve{\star}$ the proof system $\{\alpha\breve{\beta} \mid \alpha, \beta \in \mathscr{C}\}$, and we define analogously the proof systems $\star\hat{\star}$, $\breve{\star}\star$ and $\hat{\star}\star$. We define $\star\star = \breve{\star}\star \cup \hat{\star}\star \cup \star\breve{\star} \cup \star\hat{\star}$, *i.e*, $\star\star$ contains all the possible instances of the subatomic shape. By closing the system this way, the amount of structure *decreases* because everything obeys a scheme without exceptions.

All this sets the stage for the result in this paper. Research so far has shown that inside $\star\star$ there are several proof systems available for several logics, but all of them have been defined with rules for unit equations in addition to the rules derived from the subatomic scheme. For example, System SAKS, shown in Figure 1 has been shown in [ATG17] to be complete for propositional classical logic. In [BG21], it is shown that $\star\star$ is complete for propositional classical logic added with decision trees (which are the interpretation of nested atoms). Again, all the equation rules

$$\vee\check{\mathbf{a}}\,\frac{(\mathbf{A}\vee\mathbf{B})\,\mathbf{a}\,(\mathbf{C}\vee\mathbf{D})}{(\mathbf{A}\,\mathbf{a}\,\mathbf{C})\vee(\mathbf{B}\,\mathbf{a}\,\mathbf{D})} \qquad \wedge\hat{\mathbf{a}}\,\frac{(\mathbf{A}\,\mathbf{a}\,\mathbf{B})\wedge(\mathbf{C}\,\mathbf{a}\,\mathbf{D})}{(\mathbf{A}\wedge\mathbf{C})\,\mathbf{a}\,(\mathbf{B}\wedge\mathbf{D})}$$

$$\vee\check{\wedge}\,\frac{(\mathbf{A}\vee\mathbf{B})\wedge(\mathbf{C}\vee\mathbf{D})}{(\mathbf{A}\wedge\mathbf{C})\vee(\mathbf{B}\vee\mathbf{D})} \qquad \wedge\hat{\vee}\,\frac{(\mathbf{A}\vee\mathbf{B})\wedge(\mathbf{C}\wedge\mathbf{D})}{(\mathbf{A}\wedge\mathbf{C})\vee(\mathbf{B}\wedge\mathbf{D})}$$

$$\vee\hat{\wedge}\,\frac{(\mathbf{A}\wedge\mathbf{B})\vee(\mathbf{C}\wedge\mathbf{D})}{(\mathbf{A}\vee\mathbf{C})\wedge(\mathbf{B}\vee\mathbf{D})}$$

$$=\frac{\mathbf{A}}{\mathbf{A}\vee 0} \qquad =\frac{\mathbf{A}\wedge 1}{\mathbf{A}} \qquad =\frac{\mathbf{A}\vee 0}{\mathbf{A}} \qquad =\frac{\mathbf{A}}{\mathbf{A}\wedge 1}$$

$$=\frac{1\vee 1}{1} \qquad =\frac{0}{0\wedge 0} \qquad =\frac{1}{1\,\mathbf{a}\,1} \qquad =\frac{0\,\mathbf{a}\,0}{0} \qquad =\frac{0}{0\,\mathbf{a}\,0} \qquad =\frac{1\,\mathbf{a}\,1}{1}$$

Figure 1: System SAKS.

named $=$ in Figure 1 are used in that work. A natural questions arises: given that interpretation functions already contain the structure captured by unit equations, are unit equations necessary? The answer to the question is 'no', and that is the content of the present contribution.

Theorem System $\star\star$ for language $\mathscr{C}$ is complete for propositional classical logic under the interpretation $\overset{\mathsf{C}}{\mapsto}$. In other words, the $=$ rules in Figure 1 are admissible.

In a certain sense, this work is the most extreme form of deep inference. One of the main motivations behind deep inference is locality, which can be understood as the ability to check the correctness of an inference step with constant-bound complexity. Contractions and cuts are not local, but they can give rise to many local instances if we can reduce them to atomic form. That is one of the successes of deep inference, made possible by more extensive use of linearity than is possible in other formalisms. Since, as we show here, derivations in most logics among those amenable to a subatomic treatment can be completely captured by totally linear proof systems, this is the end of the journey in this particular direction.

We find the result in this work surprising. It also was unexpected. Of course, the question arises of what can be the use of such a finding. While deep inference has reduced the complexity of proofs and promises to reduce the complexity of proof search (via nonelementary speed-ups in the first order), the subatomic version of deep inference moves in the opposite direction: it increases complexity, and even more so in the totally linear version studied here. Therefore, we cannot foresee a use in complexity.

Regarding semantics of proofs, things are not necessarily better because the subatomic constructions are highly non-confluent, and this paper provides an example of that. On the other hand, the simplicity of totally linear proof systems might eventually help in semantics. We do not know.

That said, there is one thing that totally linear proofs are good at, and that is their behaviour under substitutions. Contrary to standard deep-inference proofs and their atomic flows, totally linear proofs know no bifurcations and U-turns. They are just a fabric of threads that can accept unlimited substitutions, with no restrictions or constraints. It is impossible to be extensive about that in this limited space, but that ability allows us to settle convincingly a dilemma about how to define proof substitution in a form of second-order deep inference. That problem has resisted all attempts for several years (and it is still a work in progress). More immediate evidence of the usefulness of substitutions in subatomic systems is in the cited recent work with Chris Barrett [BG21]. In that paper, cut-elimination and some other core proof-theoretic results for a new logic extending classical logic have been proved only using substitutions.

In conclusion, we believe that the techniques shown here will be useful in the design of proof systems, also as a way to check the robustness of the constructions that will be used at the standard level.

2 Sketch of the proof of the theorem

In this section, we sketch the proof of the admissibility of all the equational rules. The proof has two phases, which we call 'local' and 'global'. The local phase entails a standard permutation argument: some inference steps are permuted up or down until all the steps of their kind disappear from a given derivation. The global phase acts on the whole derivation and eliminates, one by one, all the remaining equality inference steps. The global phase is where most of the innovation of this work lies.

Before all that, we need to generalise some equality rules. We stipulate that each of the following rules

$$=\frac{\mathbf{A}}{\mathbf{A}\vee 0}\quad,\qquad =\frac{\mathbf{A}\wedge 1}{\mathbf{A}}\quad,\qquad =\frac{\mathbf{A}\vee 0}{\mathbf{A}}\qquad\text{and}\qquad =\frac{\mathbf{A}}{\mathbf{A}\wedge 1}$$

is replaced, respectively, by one of the following:

$$='\frac{\mathbf{A}}{\mathbf{A}\vee\mathbf{B}}\quad,\qquad ='\frac{\mathbf{A}\wedge\mathbf{C}}{\mathbf{A}}\quad,\qquad ='\frac{\mathbf{A}\vee\mathbf{B}}{\mathbf{A}}\qquad\text{and}\qquad ='\frac{\mathbf{A}}{\mathbf{A}\wedge\mathbf{C}}\quad,$$

where $\mathbf{B}=0$ and $\mathbf{C}=1$. We require the same for the mirror images of those rules, where left and right are exchanged.

So, we are given a derivation, and we replace all the $=$ inference steps as above. Initially, all the steps will be instances of the original $=$ rules. However, both the local and the global phase grow formulae around the units in $='$ steps, without

changing their truth value, of course. Once this is done, we collect all the units in substitutions that we keep until the end of the procedure, and we operate on a derivation where no unit appears, *i.e*, only variables and connectives are allowed.

The local phase consists in

- moving up all the instances of the following rules:

$$=\frac{0}{0\,\mathbf{a}\,0} \quad , \qquad =\frac{1}{1\,\mathbf{a}\,1} \qquad \text{and} \qquad =\frac{0}{0\wedge 0} \quad ;$$

- moving down all the instances of the following rules:

$$=\frac{0\,\mathbf{a}\,0}{0} \quad , \qquad =\frac{1\,\mathbf{a}\,1}{1} \qquad \text{and} \qquad =\frac{1\vee 1}{1} \quad .$$

Let us concentrate on moving up instances of $=\frac{0}{0\wedge 0}$. Unless the step goes upwards through another step, one of the following can happen:

- the step reaches the top of the derivation and it is replaced by $0\wedge 0$;
- the step reaches an instance of $='$ inside a $\mathbf{B}$ or a $\mathbf{C}$ and it is replaced by $0\wedge 0$;
- we get $\dfrac{=\dfrac{0\,\mathbf{a}\,0}{0}}{=\dfrac{0}{0\wedge 0}}$, which is replaced by $\wedge\breve{\mathbf{a}}\dfrac{\boxed{=\dfrac{0}{0\wedge 0}}\,\mathbf{a}\,\boxed{=\dfrac{0}{0\wedge 0}}}{\boxed{=\dfrac{0\,\mathbf{a}\,0}{0}}\wedge\boxed{=\dfrac{0\,\mathbf{a}\,0}{0}}}$.

A simple measure based on the height of the derivation shows termination. This and its dual are the more challenging cases. The other rules entail simpler arguments. Each reduction application removes one instance of $='$

The global phase consists of the application of the two global reductions in Figure 2, plus their symmetric variants via duality and left-right symmetry. At the left of the figure we see the derivations to be reduced: the inference step to be eliminated appears inside the formula context $\mathbf{K}\{\ \}$. The dotted line represents the vertical (categorical, or 'synchronal' in the deep-inference terminology) composition of that derivation with ψ above and χ below.

On the right, the two reductions have a very different nature. Intuitively, the top one works 'from the inside' and the bottom one 'from the outside'. Both are based on substitutions, for which we have developed a special notation. With $[\mathbf{B}\Rightarrow x_i]_{\underline{A}}\psi$ we denote the derivation obtained from ψ as follows; we consider the ordered sequence $\underline{A}$ of all the variables appearing in A, and we denote them by x_i; then each x_i in ψ is replaced with $\mathbf{B}$. The notation $\begin{matrix}\mathbf{A}\\ \|\mathscr{S}\\ \mathbf{B}\end{matrix}$ represents a derivation from $\mathbf{A}$ to $\mathbf{B}$ in system $\mathscr{S}$.

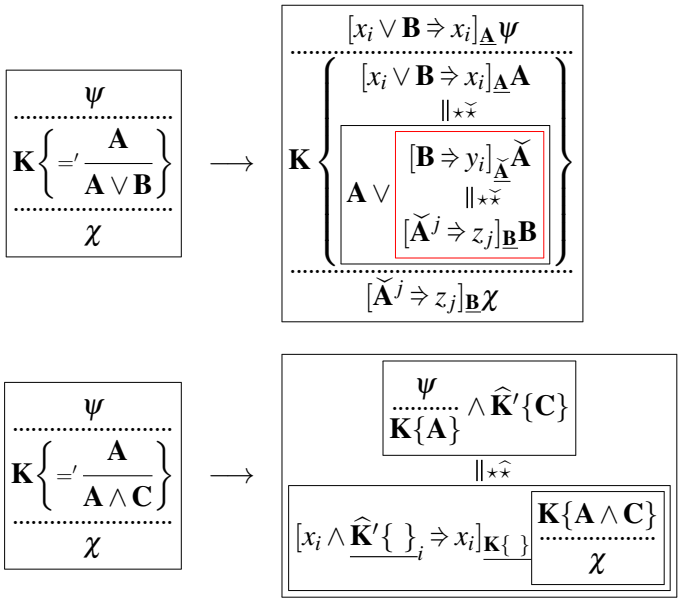

Figure 2: The two reductions of the global phase. In the top reduction, $\breve{\mathbf{A}}^j$s are renamings of $\breve{\mathbf{A}}$ such that $\underline{\breve{\mathbf{A}}^j} = \{z_j\}$. The derivation in the red box is guaranteed by a (non-trivial) 'eversion lemma'; the other unspecified derivations are standard fare in deep inference.

We try now to give an intuitive understanding of what goes on in the two constructions because we think that the technique involved has a wide applicability. Let us consider the topmost reduction and let us read the derivation from the top. The problem that we intend to solve is to introduce appropriate logical material so that $\mathbf{A}$ can generate a disjunction with $\mathbf{B}$, and, of course, we need to do so in such a way that the interpretation of the whole derivation remains the same. The idea is to substitute each variable of $\mathbf{A}$ with a copy of itself in a disjunction with a copy of $\mathbf{B}$. Since the value of $\mathbf{B}$ is 0, this does not change the interpretation. This way, we obtain the saturation-down of $\mathbf{A}$, *i.e.*, $\breve{\mathbf{A}}$, where all its leaves contain a duplicate of $\mathbf{B}$. Remarkably, we can then perform an eversion of this formula via a certain construction, so obtaining $\mathbf{B}$, albeit in a form where each variable is substituted by several copies of itself in the form of $\breve{\mathbf{A}}$. All that can be propagated down the proof, where either it reaches the conclusion or it increments some $='$ step premiss.

The bottom reduction in Figure 2 is simpler. In this case, we need $\mathbf{A}$ to produce a conjunction with $\mathbf{C}$. $\mathbf{C}$ can be brought in from the outside, and we need it to fit into a context that mimics $\mathbf{K}\{\ \}$. This can be done by using a renaming of $\mathbf{K}\{\ \}$, $\mathbf{K}'\{\ \}$, which we saturate up in order to be merged with $\mathbf{K}\{\ \}$ via, essentially, the quintessential deep-inference rule called 'switch'. In order to maintain the interpretation, all the new variables of $\mathbf{K}'\{\ \}$ will be, eventually, substituted by 1. Again, the substitutions are carried down through the derivation, until either they

reach the conclusion or they increment some $='$ step premiss.

Acknowledgments

This research has been supported by Royal Society International Exchange *Sharing and Sequentiality in Proof Systems with Locality* and by EPSRC Project EP/K018868/1 *Efficient and Natural Proof Systems.*

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