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Towards Unifying (Co)induction and Structural Control

Siva Somayyajula [*]

Abstract

In this work-in-progress paper, we extend hybrid multiplicative-additive linear logic with modalities for recursion so that infinite formulas and proofs correspond to (co)inductive types and programs. Working towards a unified logical framework, we consider various approaches to incorporating limited rules of weakening and contraction that reuse the hybrid structure.

1 Introduction

(Co)induction [BDS16] and limited rules of weakening and contraction [GSS92] (which we call *structural control*) have separately been introduced into linear logic to reason about various computational phenomena. In this work-in-progress paper, we suggest hybrid logic as a unifying formalism for both features. First, we develop guarded HyMALL, which extends hybrid multiplicative-additive linear logic (HyMALL) [COPD19] with Nakano's later modality (●) [Nak00] and its dual (○). Then, we show how infinite formulas and proofs correspond to (co)inductive types and programs. Lastly, we show how the hybrid judgment can provide structural control in the sense of bounded linear logic. While both systems can be combined naively, their unified treatment is of theoretical interest and could lead to a concise and expressive logical framework for resource-sensitive systems with (in)finite behaviors.

2 Guarded HyMALL

In this section, we introduce guarded HyMALL and work through some examples, ending with sketches of its metatheory. In short, we transport Vezzosi's [Vez15] insight, which was to encode ● and ○ into *sized types* by appealing to their Kripke semantics as necessity resp. possibility modalities, to the hybrid setting. However, Nakano and Vezzosi both encode recursion via a fixed point combinator. Since recursive calls can be made zero, one, or many times, such a combinator must

[*]Carnegie Mellon University

be typed with !. On the other hand, we encode recursion via infinite proofs, decoupling it from ! [SP21]. This affords us much more control on the inclusion of structural rules in the next section. Moreover, our use of modalities to control recursion obviates the need for extrinsic validity conditions on proofs as is common in infinitary proof theory [BDS16].

Definition 1 (Worlds)**.** *Let* $(W, 0, +)$ *be a monoid and* $u, v, w \in W$ *be* worlds. *As usual, let* $u \leq w \triangleq \exists v. u + v = w$ *and* $<$ *be the corresponding strict order.*

Definition 2 (Formulas)**.** *The formulas of guarded HyMALL, given below, are stratified into two syntactic categories:* modal formulas M *and* quantified/hybrid formulas Q*. We will see later that limiting the scope of quantifiers and "at" in this fashion is central to our sketch of cut admissibility. The ellipses stand for copies of the grammar for A and B replacing the subterms A and B with M resp. Q, indicating that M and Q each include the connectives of MALL and* $\downarrow$*. The double pipes in the definition of M indicate the portion of its grammar that is coinductive. That is, viewing some M as a tree, all of its infinite paths must pass through* $\bullet$ *or* $\bigcirc$ *infinitely often [DA09]. Lastly, negation* $M^{\perp}$ *and* $Q^{\perp}$ *is defined as usual with* $\bullet$ *and* $\bigcirc$ *as duals but "*at*" and* $\downarrow$ *are self-dual.*

$$
\begin{aligned}
A, B &:= \mathbf{1} \mid \bot \mid A \otimes B \mid A \parr B \mid \top \mid \mathbf{0} \mid A \& B \mid A \oplus B \mid \downarrow u. A(u) \\
M &:= \ldots \parallel \bullet M \parallel \bigcirc M \\
Q &:= \ldots \mid M \mid Q \,\mathrm{at}\, w \mid \forall u. Q(u) \mid \exists u. Q(u)
\end{aligned}
$$

The contexts Γ and Δ are multisets of judgments $A \mathbin{@} w$ indicating that A holds at world w. Proof rules are given in table 1 with the usual freshness condition on eigenvariables, where A and B draw from the MALL subgrammars of M and Q. Indicated by the double line, the rule for $\bullet$ is coinductive, delimiting a potentially infinite branch as explained in the case of M. Moreover, the eigenvariable u is paired with the assumption that $u < w$, agreeing with the usual Kripke semantics for necessity. Dually, the rule for $\bigcirc$ is inductive and has the corresponding side condition. Since our presentation is quite abstract, let us work through some examples of (co)induction, viewing formulas as types and proofs as programs. For the remainder of this section, let $W = (\mathbb{N}, 0, +)$, noting that $<$ is well-founded.

Example 3 (Induction)**.** *Let* $\mathrm{gnat} \triangleq \bigcirc(\mathbf{1} \oplus \mathrm{gnat})$ *be the type of* guarded natural numbers*; intuitively, a proof of* $\vdash \mathrm{gnat} \mathbin{@} w$ *is a natural number* $< w$*. As a result,* $\mathrm{nat} \triangleq \exists u.\, \mathrm{gnat}\,\mathrm{at}\, u$ *must be the type of all natural numbers. Then, the infinite proof at the top in table 2 represents a trivial program* $\mathrm{gnat} \multimap \mathbf{1}$ *using the notation* $D \vdash J$ *to indicate a derivation D of the judgment(s) J and* $[w/u]D$ *to substitute w for u in D.*

In general, letting $\Diamond Q \triangleq \exists u. Q \mathbin{@} u$, we call $M = \bigcirc f(M)$ a *guarded inductive type* and $\Diamond M$ the corresponding full *inductive type*. Let us now look at an example of coinduction.

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$$\dfrac{}{\vdash \mathbf{1} \mathbin{@} w}\ \mathbf{1} \qquad\qquad \dfrac{\vdash \Gamma, \Delta}{\vdash \Gamma, \bot \mathbin{@} w, \Delta}\ \bot$$

$$\dfrac{\vdash \Gamma, A \mathbin{@} w \quad \vdash B \mathbin{@} w, \Delta}{\vdash \Gamma, A \otimes B \mathbin{@} w, \Delta}\ \otimes \qquad\qquad \dfrac{\vdash \Gamma, A \mathbin{@} w, B \mathbin{@} w, \Delta}{\vdash \Gamma, A \parr B \mathbin{@} w, \Delta}\ \parr$$

$$\dfrac{}{\vdash \Gamma, \top \mathbin{@} w, \Delta}\ \top \qquad\qquad \text{(no rule for } \mathbf{0}\text{)}$$

$$\dfrac{\vdash \Gamma, A \mathbin{@} w, \Delta \quad \vdash \Gamma, B \mathbin{@} w, \Delta}{\vdash \Gamma, A \,\&\, B \mathbin{@} w, \Delta}\ \& \qquad \dfrac{\vdash \Gamma, A_i \mathbin{@} w, \Delta}{\vdash \Gamma, A_1 \oplus A_2 \mathbin{@} w, \Delta}\ \oplus_i, i \in \{1,2\}$$

$$\dfrac{\vdash \Gamma, Q \mathbin{@} u, \Delta}{\vdash \Gamma, Q \,\mathrm{at}\, u \mathbin{@} w, \Delta}\ \mathrm{at} \qquad\qquad \dfrac{\vdash \Gamma, A(w) \mathbin{@} w, \Delta}{\vdash \Gamma, {\downarrow} u.\, A(u) \mathbin{@} w, \Delta}\ \downarrow$$

$$\dfrac{\vdash \Gamma, M \mathbin{@} u, \Delta}{\vdash \Gamma, \bullet M \mathbin{@} w, \Delta}\ \bullet \qquad\qquad \dfrac{\vdash \Gamma, M \mathbin{@} u, \Delta}{\vdash \Gamma, \bigcirc M \mathbin{@} w, \Delta}\ \bigcirc, u < w$$

$$\dfrac{\vdash \Gamma, Q(u) \mathbin{@} w, \Delta}{\vdash \Gamma, \forall u.\, Q(u) \mathbin{@} w, \Delta}\ \forall \qquad\qquad \dfrac{\vdash \Gamma, Q(v) \mathbin{@} w, \Delta}{\vdash \Gamma, \exists u.\, Q(u) \mathbin{@} w, \Delta}\ \exists$$

Table 1: Rules for Guarded HyMALL

Example 4 (Coinduction). $\mathrm{gstream}_A \triangleq \bullet(A \otimes \mathrm{gstream}_A)$ *is the type of* guarded streams *with elements in A; intuitively, a proof of* $\vdash \mathrm{gstream}_A \mathbin{@} w$ *is a stream with at most w observable elements. As a result,* $\mathrm{stream}_A \triangleq \forall u.\, \mathrm{gstream}_A \,\mathrm{at}\, u$ *is the type of all infinite streams. For example, consider the infinite proof at the bottom in table 2 representing a trivial stream of units.*

$$\mathrm{eat} = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1} \mathbin{@} v}\ \mathbf{1}}{\vdash \bot \mathbin{@} u, \mathbf{1} \mathbin{@} v}\ \bot \quad [u/w]\mathrm{eat} \vdash \mathrm{gnat}^\perp \mathbin{@} u, \mathbf{1} \mathbin{@} v}{\vdash \bot \,\&\, \mathrm{gnat}^\perp \mathbin{@} u, \mathbf{1} \mathbin{@} v}\ \&}{\vdash \bullet(\bot \,\&\, \mathrm{gnat}^\perp) \mathbin{@} w, \mathbf{1} \mathbin{@} v}\ \bullet$$

$$\mathrm{ones} = \dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1} \mathbin{@} u}\ \mathbf{1} \quad [u/w]\,\mathrm{ones} \vdash \mathrm{gstream}_{\mathbf{1}} \mathbin{@} u}{\vdash \mathbf{1} \otimes \mathrm{gstream}_{\mathbf{1}} \mathbin{@} u}\ \otimes}{\vdash \bullet(\mathbf{1} \otimes \mathrm{gstream}_{\mathbf{1}}) \mathbin{@} w}\ \bullet$$

Table 2: Examples of (Co)induction

In general, letting $\Box Q \triangleq \forall u.\, Q \,\mathrm{at}\, u$, we call $M = \bullet f(M)$ a *guarded coinductive type* and $\Box M$[1] the corresponding full *coinductive type*. We suspect that mixed

[1]Nakano introduced this additional modality; Clouston et al. [CBGB15] call it the *constant*

induction-coinduction can be encoded with world vectors ordered lexicographically [Abe12]. Now, let us comment on the metatheory of guarded HyMALL.

Definition 5 (Cut). *If $D \vdash \Gamma, Q @ w$ and $E \vdash Q^\perp @ w, \Delta$, then we define* $\mathrm{cut}(D,E) \vdash \Gamma, \Delta$ *in two parts: if $Q = M$, then we proceed by lexicographic induction on (w, M, D, E) otherwise proceed by the usual lexicographic induction on (Q, D, E) [Pfe95]. While we have not verified every case, consider the principal cut for $\bullet$.*

$$\mathrm{cut}\left(\dfrac{D' \vdash \Gamma, M @ u}{\vdash \Gamma, \bullet M @ w}\,\bullet\;, \dfrac{E' \vdash M^\perp @ v}{\vdash \bigcirc M^\perp @ w, \Delta}\,\bigcirc\right) = \mathrm{cut}\big([v/u]D', E'\big)$$

*w decreases to v, allowing M and D' to grow arbitrarily large. If we had not limited the scope of quantifiers and "*at*" to come before modalities, then we could not have guaranteed that w decreases or stays the same in every case.*

Definition 6 (Identity). *Define* $\mathrm{id}(F) \vdash Q @ w, Q^\perp @ w$ *in two parts as before: $Q = M$ goes by lexicographic induction on (w, M) otherwise by induction on Q. While we have not verified every case, consider $\bullet$ below: w decreases to u.*

$$\mathrm{id}(\bullet M) = \dfrac{\dfrac{\mathrm{id}(M) \vdash M @ u, M^\perp @ u}{\vdash M @ u, \bigcirc M^\perp @ w}\,\bigcirc}{\vdash \bullet M @ w, \bigcirc M^\perp @ w}\,\bullet$$

3 HyMALLSC

First, note that hybrid linear logic embeds into focused linear logic with quantified subexponentials [COPD19], suggesting a close relationship between worlds and the control of exponential modalities. In general, substructural logics can be embedded into focused first-order logic [RP10] where the sequent is modeled by the concatenation of worlds in an algebra such that its equations govern the structural rules. We aim to determine whether guarded HyMALL can be embedded similarly—the primary obstacle is reckoning with infinite proofs. However, consider the following development internal to HyMALL itself.

We begin with the intuition that $Q @ w$ means "w copies of the resource Q." Then, we could add to HyMALL rules of weakening and contraction where $Q @ 0$ is dropped resp. $Q @ u + w$ splits into $Q @ u$ and $Q @ w$. However, HyMALL rules for MALL connectives betray the new meaning of the judgment. Instead, let W be a semiring and let the rules for MALL connectives be defined at world 1, indicating that one copy of a formula is worked with at a time. Table 3 details representative rules of the resulting calculus, which we call HyMALL with structural control (HyMALLSC).

modality

$$\frac{\vdash \Gamma, A \mathbin{@} 1 \quad \vdash B \mathbin{@} 1, \Delta}{\vdash \Gamma, A \otimes B \mathbin{@} 1, \Delta}\otimes \qquad \frac{\vdash \Gamma, \Delta}{\vdash \Gamma, Q \mathbin{@} 0, \Delta}\,\mathrm{W} \qquad \frac{\vdash \Gamma, Q \mathbin{@} u, Q \mathbin{@} w, \Delta}{\vdash \Gamma, Q \mathbin{@} u+w, \Delta}\,\mathrm{C}$$

Table 3: Selected Rules for HyMALLSC

HyMALLSC is similar to Granule [OLEI19], albeit with a different judgmental structure. The latter extends intuitionistic linear logic with hypotheses $[A]_w$ standing for w copies of A such that $[A]_1$ collapses to A. Like HyMALLSC, its additive structure controls weakening and contraction. However, its multiplicative structure induces graded modalities ($\newmoon_w$ and $\bigcirc_w$) indicating an upper bound w on use in lieu of unlimited exponentials. Let us compare both systems through examples.

Example 7 (Reuse)**.** *Let $W = \mathbb{N}$. In Granule, contraction is internalized as $\newmoon_2 A \multimap A \otimes A$. HyMALLSC can ascribe a more specific type: $A \,\mathrm{at}\, 2 \multimap A \otimes A \,\mathrm{at}\, 1$.*

Example 8 (Security)**.** *Let $W = \{\mathrm{Private}, \mathrm{Public}\}$ be a Boolean semiring. In Granule, an information leak (not provable) looks like $\newmoon_{\mathrm{Private}} A \multimap \newmoon_{\mathrm{Public}} A$. Alternatively, consider $A \,\mathrm{at}\, \mathrm{Private} \multimap A \,\mathrm{at}\, \mathrm{Public}$ in HyMALLSC as a more specific ascription.*

4 Conclusion and Future Work

We have presented two logics—guarded HyMALL and HyMALLSC—that respectively extend HyMALL with (co)induction and structural control in the sense of bounded linearity. Both can be combined immediately by introducing multiple world sorts (one for (co)induction and one for structural control) with separate modalities for each. However, the Church encoding of natural numbers bounded above in bounded linear logic [GSS92] suggests a material connection between (co)induction and structural control, although the specifics are unclear to us. Moreover, we are interested in controlling the rule of exchange as well [BEI20]. Overall, the high-level notion that worlds restrict the structure of proofs seems promising.

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