A Parametrised Functional Interpretation of Affine Logic

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From the proceedings of Trends in Linear Logic and Applications 2021 (TLLA 2021, eds E. Pimentel, Ch. Retoré)
https://hal.archives-ouvertes.fr/TLLA2021

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Abstract

This paper presents an abstract parametrised functional interpretation of Affine Logic. It is based on families of parameters allowing for different degrees of freedom on the design of the interpretation. In this way we are able to generalise previous work on unifying functional interpretations, by including in the unification the more recent bounded and Herbrandized functional interpretations.

1 Introduction

Since Gödel [G58̈] published his functional ("Dialectica") interpretation in 1958, various other functional interpretations have been proposed[1]. These include Kreisel's modified realizability [Kre59], the Diller-Nahm variant of the Dialectica interpretation [DN74], Stein's family of interpretations [Ste79], and more recently, the bounded functional interpretation [FO05], the bounded modified realizability [FN06], and "Herbrandized" versions of modified realizability and the Dialectica [BBS12]. In view of this picture, several natural questions arise: How are these different interpretations related to each other? What is the common structure behind all of them? Are there any other interpretations out there waiting to be discovered?

These questions were addressed by the second author (and various co-authors) in a series of papers on *unifying functional interpretations*. Starting with a unification of interpretations of intuitionistic logic [Oli06], which was followed by various analysis of functional interpretations within the finer setting of affine and linear logic [FO11, Oli07, Oli08, Oli10], a proposal on how functional interpretations could actually be combined in so-called *hybrid functional interpretations* [HO08, Oli12], and the inclusion of truth variants in the unification [GO10].

Functional interpretations associate with each formula A a new formula $|A|^x_y$ where x and y are fresh tuples of variables. Intuitively, x captures the "positive"

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[1] See [AF98] for a survey on the "Dialectica" interpretation.

quantifications in A, while y captures the "negative" quantifications. This is done in such a way that, in a suitable system, the truth of A is equivalent to that of $\exists x \forall y |A|^x_y$. The key insight which arises from the programme of "unifying functional interpretations" is that we have some degree of freedom when choosing the interpretation of the exponentials of linear logic $!A$ and $?A$. For instance, we can take

$\lvert !A\rvert^x_y :\equiv !\lvert A\rvert^x_y$	(giving rise to the Dialectica interpretation)
$\lvert !A\rvert^x_a :\equiv !\forall y \in a \lvert A\rvert^x_y$	(giving rise to the Diller-Nahm interpretation)
$\lvert !A\rvert^x :\equiv !\forall y \lvert A\rvert^x_y$	(giving rise to modified realizability)
$\lvert !A\rvert^x :\equiv !\forall y \lvert A\rvert^x_y \otimes !A$	(giving rise to modified realizability with truth)
and so on...	

showing that each of these interpretations only differ in the way they treat the *contraction axiom*. In particular, in the *pure fragment* of linear logic (i.e. linear logic without the exponentials) all these interpretations coincide!

So, it makes sense to introduce an *abstract* bounded quantification $\forall y \sqsubset_\tau a A$, capturing this degree of freedom on the design of a functional interpretation, and to try to isolate the properties of this parameter which ensure the soundness of the interpretation. With this one is able to define a "unifying functional interpretation" which when instantiated gave rise to several of the existing functional interpretations, including the Dialectica interpretation, modified realizability (its q- and truth variants), Stein's family of interpretations, and the Diller-Nahm interpretation [Oli06, Oli10]. This process led to the design of a "Diller-Nahm with truth" interpretation [GO10], which at the time was not thought to be possible.

But the *unifying functional interpretation* programme has so far been unable to capture the two more recent families of functional interpretations, namely the bounded functional interpretations [DG18, FG15, FN06, FO05], and the Herbrandized functional interpretations [BBS12, FF17].

We propose a framework for a more general unification, introducing other families of parameters which allow for different interpretations of *typed quantifications*. When devising a functional interpretation, we in fact have *two crucial degrees of freedom*: we can choose how to interpret the contraction axiom, as discussed above, but also, we can choose how to interpret typed quantifications, which ultimately boils down to the choice of how predicate symbols are interpreted. Due to the page limitation we do not give proofs here and skip some details. The interested reader may consult [DO20].

1.1 Intuitionistic logic and theories

We assume a sequent calculus for *intuitionistic affine logic* **AL**, with negation $A^\perp$ defined as $A \multimap \perp$. An extension of **AL** with new predicate and function symbols, and non-logical axioms, will be called an *intuitionistic affine theory*, or **AL**-theory, for short. Given an intuitionistic affine theory $\mathscr{A}$ we will denote its set of predicate symbols by $\mathbf{Pred}_{\mathscr{A}}$, its set of formulas by $\mathbf{Form}_{\mathscr{A}}$, and its set of non-logical axioms by $\mathbf{Ax}_{\mathscr{A}}$. The extension of **AL** with finite types will be denoted by $\mathbf{AL}^{\omega}$. The use of affine logic here derives from the fact that these functional interpretations trivially interpret the weakening axiom, whereas the contraction axiom requires particular care, as we will see.

2 Parametrised Interpretation of AL

We present now a parametrised interpretation of a "source" **AL**-theory $\mathscr{A}_{\mathbf{s}}$ into a "target" **AL**-theory $\mathscr{A}_{\mathbf{t}}$. In order to ensure that the parametrised interpretation is sound, we will need to stipulate a few assumption about $\mathscr{A}_{\mathbf{s}}$ and $\mathscr{A}_{\mathbf{t}}$:

(A1) The target theory $\mathscr{A}_{\mathbf{t}}$ is an extension of $\mathbf{AL}^{\omega}$ so that we can work with typed λ-terms as witnesses.

(A2) In the source theory $\mathscr{A}_{\mathbf{s}}$, the predicate symbols are divided into two groups: the *computational predicates*, denoted by $\mathbf{Pred}^{c}_{\mathscr{A}_{\mathrm{s}}}$, and the *non-computational predicates*, denoted by $\mathbf{Pred}^{nc}_{\mathscr{A}_{\mathrm{s}}}$. The predicate symbols of $\mathscr{A}_{\mathbf{s}}$ are also assumed to be predicate symbols of $\mathscr{A}_{\mathbf{t}}$.

(A3) For each computational predicate symbol $P(x) \in \mathbf{Pred}^{c}_{\mathscr{A}_{\mathrm{s}}}$ of $\mathscr{A}_{\mathbf{s}}$, of arity n, we have associated in $\mathscr{A}_{\mathbf{t}}$ a $(n+1)$-ary formula $x \prec^{P} a$, and a finite type $\mathrm{wt}(P)$ in which the witnesses a of $P(x)$ will live in. We will call $\mathrm{wt}(P)$ the *witnessing type* of P. We write $\forall x \prec^{P} aA$ and $\exists x \prec^{P} aA$ as abbreviations for $\forall x(x \prec^{P} a \multimap A)$ and $\exists x(x \prec^{P} a \otimes A)$, respectively. We assume that, over $\mathscr{A}_{\mathbf{t}}$, $x \prec^{P} a$ is stronger than $P(x)$, i.e.

$$x \prec^{P} a \vdash_{\mathscr{A}_{\mathrm{t}}} P(x).$$

(A4) For each finite type τ we associate in $\mathscr{A}_{\mathbf{t}}$ a formula $\mathrm{W}_{\tau}(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses. We also assume that $x \prec^{P} a$ implies that a is in W, i.e.

$$x \prec^{P} a \vdash_{\mathscr{A}_{\mathrm{t}}} \mathrm{W}_{\mathrm{wt}(P)}(a).$$

When τ is a tuple of finite types $\tau_1, \ldots, \tau_n$, we write $\mathrm{W}_{\tau}(x_1, \ldots, x_n)$ as an abbreviation for $\mathrm{W}_{\tau_1}(x_1), \ldots, \mathrm{W}_{\tau_n}(x_n)$, when this appears in the context of a sequent, or for $\mathrm{W}_{\tau_1}(x_1) \otimes \ldots \otimes \mathrm{W}_{\tau_n}(x_n)$, when this appears in the conclusion of a sequent. We assume that, provably in $\mathscr{A}_{\mathbf{t}}$, the combinators $\mathrm{S}_{\rho,\tau,\sigma}$ and $\mathrm{K}_{\rho,\tau}$ are in W, and that the application of a function in W to an argument in W will also be in W, i.e.

$(\mathbf{W}_{\mathrm{K}})$ $\vdash_{\mathscr{A}_{\mathrm{t}}} \mathrm{W}_{\rho \to \tau \to \rho}(\mathrm{K}_{\rho,\tau})$

$(\mathbf{W}_{\mathrm{S}})$ $\vdash_{\mathscr{A}_{\mathrm{t}}} \mathrm{W}_{(\rho \to \tau \to \sigma) \to (\rho \to \tau) \to \rho \to \sigma}(\mathrm{S}_{\rho,\tau,\sigma})$

$(\mathbf{W}_{\mathrm{Ap}})$ $\mathrm{W}_{\tau}(x), \mathrm{W}_{\tau \to \rho}(f) \vdash_{\mathscr{A}_{\mathrm{t}}} \mathrm{W}_{\rho}(fx)$

(A5) For each formula A of $\mathscr{A}_\mathbf{t}$, tuple of variables $x = x_1, \ldots, x_n$, and finite types $\tau = \tau_1, \ldots, \tau_n$ we associate a tuple of *bounding types* $\mathrm{bt}(\tau)$ and a formula $\forall x \sqsubset_\tau a A$, in which the variables x are no longer free. We do not assume that the tuple of finite types $\mathrm{bt}(\tau)$ has the same length as τ. The intuition is that x ranges over elements of type τ, whereas the bounds a range over possibly different types $\mathrm{bt}(\tau)$. We use this parameter to interpret $!A$. This parameter is assumed to satisfy:

(Q$_1$**)** If $A \vdash_{\mathscr{A}_\mathbf{t}} B$ then $\forall x \sqsubset_\tau a A \vdash_{\mathscr{A}_\mathbf{t}} \forall x \sqsubset_\tau a B$

(Q$_2$**)** $\vdash_{\mathscr{A}_\mathbf{t}} \forall x \sqsubset_\tau a \, !\mathrm{W}_\tau(x)$

and, for each formula A of $\mathscr{A}_\mathbf{t}$, tuple of variables x, and types τ and ρ we assume that there exist terms $\eta(\cdot), (\cdot) \sqcup (\cdot)$ and $(\cdot) \circ (\cdot)$ of $\mathscr{A}_\mathbf{t}$ such that

(C$_\eta$**)** $!\mathrm{W}_\tau(z), !\forall x \sqsubset_\tau \eta(z) A \vdash_{\mathscr{A}_\mathbf{t}} A[z/x]$
$\vdash_{\mathscr{A}_\mathbf{t}} \mathrm{W}_{\tau \to \mathrm{bt}(\tau)}(\eta)$

(C$_\sqcup$**)** $!\mathrm{W}_{\tau,\tau}(x_1, x_2), !\forall x \sqsubset_\tau (x_1 \sqcup x_2) A \vdash_{\mathscr{A}_\mathbf{t}} \forall x \sqsubset_\tau x_1 A \otimes \forall x \sqsubset_\tau x_2 A$
$\vdash_{\mathscr{A}_\mathbf{t}} \mathrm{W}_{\mathrm{bt}(\tau) \to \mathrm{bt}(\tau) \to \mathrm{bt}(\tau)}(\lambda x_1, x_2 . x_1 \sqcup x_2)$

(C$_\circ$**)** $!\mathrm{W}_{\rho \to \mathrm{bt}(\tau)}(f), !\mathrm{W}_{\mathrm{bt}(\rho)}(z), !\forall x \sqsubset_\tau (f \circ z) A \vdash_{\mathscr{A}_\mathbf{t}} \forall y \sqsubset_\rho z \, !\forall x \sqsubset_\tau f y A$
$\vdash_{\mathscr{A}_\mathbf{t}} \mathrm{W}_{(\rho \to \mathrm{bt}(\tau)) \to \mathrm{bt}(\rho) \to \mathrm{bt}(\tau)}(\lambda f, z . f \circ z)$

A term $t[x]$, with free variables x, is called *typable* in $\mathscr{A}_\mathbf{t}$ if $\rho(x) \vdash_{\mathscr{A}_\mathbf{t}} \tau(t[x])$ for some ρ and τ. We say that a typable term $t[x]$ *is in* W if $\mathrm{W}_\rho(x) \vdash_{\mathscr{A}_\mathbf{t}} \mathrm{W}_\tau(t[x])$.

In each instantiation we will consider different choices for the parameters $\{x \prec^P a\}_{P \in \mathbf{Pred}^c_{\mathscr{A}_\mathbf{s}}}$, $\{\mathrm{wt}(P)\}_{P \in \mathbf{Pred}^c_{\mathscr{A}_\mathbf{s}}}$, $\{\mathrm{W}_\tau(x)\}_{\tau \in \mathscr{T}}$, and $\{\forall x \sqsubset_\tau a A\}_{A \in \mathbf{Form}_{\mathscr{A}_\mathbf{t}}, \tau \in \mathscr{T}}$ for each choice of variables x.

Definition 1 (Adequate parameters in **AL**)**.** *Given theories $\mathscr{A}_\mathbf{s}$ and $\mathscr{A}_\mathbf{t}$, a choice of parameters will be called* adequate *for $(\mathscr{A}_\mathbf{s}, \mathscr{A}_\mathbf{t})$ if assumptions (***A1***) – (***A5***) hold. Given a class of formulas $\mathscr{C} \subseteq \mathbf{Form}_{\mathscr{A}_\mathbf{t}}$, we say that the choice of parameters in $\mathscr{A}_\mathbf{t}$ is* $\mathscr{C}$-adequate *for $(\mathscr{A}_\mathbf{s}, \mathscr{A}_\mathbf{t})$ if it is adequate for $(\mathscr{A}_\mathbf{s}, \mathscr{A}_\mathbf{t})$ when assumption (***A5***) is only required to hold for formulas in $\mathscr{C}$.*

2.1 Parametrised interpretation of AL theories

Assume now a given choice of **AL**-theories $\mathscr{A}_\mathbf{s}$ (source theory) and $\mathscr{A}_\mathbf{t}$ (target theory) satisfying the assumptions stated above. Recall that we write ε for the empty tuple of terms. Let us use the same notation, and write ε for an empty tuple of types as well.

Definition 2. *We generalise the notion of* witnessing type *to all formulas by defining for each formula A tuples of types τ_A^+ and τ_A^- inductively as*

$$
\begin{array}{lcllcl}
\tau_P^+ & :\equiv & \mathrm{wt}(P), \quad \textit{for } P \in \mathbf{Pred}^c_{\mathscr{A}_\mathbf{s}} & \tau_P^- & :\equiv & \varepsilon, \quad \textit{for } P \in \mathbf{Pred}^c_{\mathscr{A}_\mathbf{s}} \\
\tau_P^+ & :\equiv & \varepsilon, \quad \textit{for } P \in \mathbf{Pred}^{nc}_{\mathscr{A}_\mathbf{s}} & \tau_P^- & :\equiv & \varepsilon, \quad \textit{for } P \in \mathbf{Pred}^{nc}_{\mathscr{A}_\mathbf{s}} \\
\tau^+_{A\multimap B} & :\equiv & \tau_A^+ \to \tau_B^+, \tau_A^+ \to \tau_B^- \to \tau_A^- & \tau^-_{A\multimap B} & :\equiv & \tau_A^+, \tau_B^- \\
\tau^+_{A\otimes B} & :\equiv & \tau_A^+, \tau_B^+ & \tau^-_{A\otimes B} & :\equiv & \tau_A^-, \tau_B^- \\
\tau^+_{\exists zA} & :\equiv & \tau_A^+ & \tau^-_{\exists zA} & :\equiv & \tau_A^- \\
\tau^+_{\forall zA} & :\equiv & \tau_A^+ & \tau^-_{\forall zA} & :\equiv & \tau_A^- \\
\tau^+_{!A} & :\equiv & \tau_A^+ & \tau^-_{!A} & :\equiv & \mathrm{bt}(\tau)^-_A
\end{array}
$$

Given a tuple of formulas $\Gamma = A_1, \ldots, A_n$ we write τ_Γ^+ (resp., τ_Γ^-) for the tuple $\tau^+_{A_1}, \ldots, \tau^+_{A_n}$ (resp. $\tau^-_{A_1}, \ldots, \tau^-_{A_n}$).

We can now present the parametrised interpretation of $\mathscr{A}_\mathbf{s}$ into $\mathscr{A}_\mathbf{t}$:

Definition 3 (Parametrised **AL**-interpretation)**.** *For each formula A of $\mathscr{A}_\mathbf{s}$, let us associate a formula $|A|^x_y$ of $\mathscr{A}_\mathbf{t}$, with two fresh lists of free-variables x and y, inductively as follows: for computational predicate symbols $P \in \mathbf{Pred}^c_{\mathscr{A}_\mathbf{s}}$ we let*

$$|P(x)|^a_\varepsilon \quad :\equiv \quad x \prec^P a,$$

whereas for non-computational predicate symbols $P \in \mathbf{Pred}^{nc}_{\mathscr{A}_\mathbf{s}}$ we let

$$|P(x)|^\varepsilon_\varepsilon \quad :\equiv \quad P(x).$$

Assuming A and B have interpretations $|A|^x_y$ and $|B|^v_w$, then we define

$$
\begin{array}{rcl@{\qquad}rcl}
|A \multimap B|^{f,g}_{x,w} & :\equiv & |A|^x_{gxw} \multimap |B|^{fx}_w & |\exists z A|^x_y & :\equiv & \exists z |A|^x_y \\
|A \otimes B|^{x,v}_{y,w} & :\equiv & |A|^x_y \otimes |B|^v_w & |\forall z A|^x_y & :\equiv & \forall z |A|^x_y \\
|!A|^x_a & :\equiv & !\forall y \sqsubset_{\tau_A^-} a\, |A|^x_y & & &
\end{array}
$$

Given a tuple of formulas $\Gamma = A_1, \ldots, A_n$, we write $|\Gamma|^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$ is an abbreviation for $|A_1|^{x_1}_{y_1}, \ldots, |A_n|^{x_n}_{y_n}$, assuming A_i has interpretation $|A|^{x_i}_{y_i}$.

If A has interpretation $|A|^x_y$ we call x the *witnesses* of A, and y the *counter-witnesses*. For example, suppose that both P and Q are computational predicates and $A = \,!(\forall x P(x) \multimap \exists y Q(y)$, then $|A|^f_b = \,!\forall a \sqsubset b\, (\forall x (x \prec^P a) \multimap \exists y (y \prec^Q fa))$. In this case, the function f serves as a witnessing function for A, mapping "bounds" for x into "bounds" for y, whereas b is a counter-witness, providing a "bound" on a which itself "bounds" x.

We say that a formula A has *no computational content* if its interpretation is $|A|^\varepsilon_\varepsilon$, i.e. if the tuples of witnesses and counter-witnesses are both empty. Note that all of the computational content of a formula comes from the interpretation of the computational predicate symbols. The logical connectives ($\multimap$ and $\otimes$), the

$$\begin{tikzcd}
\mathscr{I}_{\mathbf{s}} \arrow[rr, "\{\{\cdot\}\}^x_y;\, ((\cdot))^x_y"] \arrow[d, "(\cdot)^{\bullet};(\cdot)^{\circ}"] & & \mathscr{I}_{\mathbf{t}} \\
\mathscr{I}^{\bullet}_{\mathbf{s}} \simeq \mathscr{I}^{\circ}_{\mathbf{s}} \arrow[rr, "|\cdot|^x_y"] & & \mathscr{I}^{\bullet}_{\mathbf{t}} \simeq \mathscr{I}^{\circ}_{\mathbf{t}} \arrow[u, "(\cdot)^{\mathscr{F}}"']
\end{tikzcd}$$

Figure 1: Parametrised interpretations of $\mathscr{I}_{\mathbf{s}}$ into $\mathscr{I}_{\mathbf{t}}$

quantifiers ($\forall$ and $\exists$) and the exponential ($!$) simply translate witnesses and counter-witness for the subformulas into witnesses and counter-witnesses for the compound formula. If the subformulas have no computational content then the compound formula will not have any computational content either.

Given a tuple of types $\rho = \rho_1, \ldots, \rho_n$ and a type σ, let us write $\rho \to \sigma$ as an abbreviation for the type $\rho_1 \to \ldots \to \rho_n \to \sigma$. Given tuples of terms $t = t_1, \ldots, t_n$ and s we write ts for the tuple $t_1 s, \ldots, t_n s$.

Definition 4 (Witnessable **AL** sequents)**.** *A sequent* $\Gamma \vdash A$ *of* $\mathscr{A}_{\mathbf{s}}$ *is said to be* witnessable *in* $\mathscr{A}_{\mathbf{t}}$ *if there are tuples of closed terms* γ, a *of* $\mathscr{A}_{\mathbf{t}}$ *such that*

(i) $\vdash_{\mathscr{A}_{\mathbf{t}}} \mathrm{W}_{\tau^+_\Gamma \to \tau^-_A \to \tau^-_\Gamma}(\gamma)$ *and* $\vdash_{\mathscr{A}_{\mathbf{t}}} \mathrm{W}_{\tau^+_\Gamma \to \tau^+_A}(a)$

(ii) $!\mathrm{W}_{\tau^+_\Gamma, \tau^-_A}(x, w), |\Gamma|^x_{\gamma x w} \vdash_{\mathscr{A}_{\mathbf{t}}} |A|^{ax}_w$

Definition 5 (Sound **AL**-interpretation)**.** *An* **AL***-interpretation of* $\mathscr{A}_{\mathbf{s}}$ *into* $\mathscr{A}_{\mathbf{t}}$ *is said to be* sound *if the provable sequents of* $\mathscr{A}_{\mathbf{s}}$ *are witnessable in* $\mathscr{A}_{\mathbf{t}}$*.*

Theorem 6 (Soundness of **AL**-interpretation)**.** *Assume a fixed choice of the parameters in* $\mathscr{A}_{\mathbf{t}}$*. If*

(i) this choice is adequate for the formulas $|A|^x_y$*, for all* A *in* $\mathscr{A}_{\mathbf{s}}$*, and*

(ii) the non-logical axioms of $\mathscr{A}_{\mathbf{s}}$ *are witnessable in* $\mathscr{A}_{\mathbf{t}}$*,*

then this instance of the parametrised **AL***-interpretation of Definition 3 is sound.*

Proof. This can be shown by induction on the derivation of $\Gamma \vdash A$, where we use assumption (ii) to deal with the non-logical axioms of the theory $\mathscr{A}_{\mathbf{s}}$, and (i) to construct witnessing terms for each logical rule of **AL**. $\square$

2.2 Parametrised interpretations of intuitionistic theories

The parametrised interpretation of intuitionistic affine logic presented above (Definition 3) has been used as the basis for two parametrised interpretations $\{\{\cdot\}\}^x_y$ and $((\cdot))^x_y$ of intuitionistic theories (**IL**-theories) [DO21]. As described in the diagram above (Figure 1), these interpretations of **IL**-theories were obtained by composing the interpretation of intuitionistic affine logic with the Girard translations (denoted in the digram as $(\cdot)^\circ$ and $(\cdot)^\bullet$) and the forgetful translation $(\cdot)^{\mathscr{F}}$ (to take us back to **IL**). In [DO21] we have also shown that these parametrised interpretations of **IL**-theories (when appropriately instantiated) give all the interpretations mentioned in the introduction as well as a few others which we believe to be new.

References

[AF98] Jeremy Avigad and Solomon Feferman. Gödel's functional ("Dialectica") interpretation. In *Handbook of proof theory*, volume 137 of *Stud. Logic Found. Math.*, pages 337–405. North-Holland, Amsterdam, 1998.

[BBS12] Benno van den Berg, Eyvind Briseid, and Pavol Safarik. A functional interpretation for nonstandard arithmetic. *Annals of Pure and Applied Logic*, 163(12):1962–1994, December 2012.

[DG18] Bruno Dinis and Jaime Gaspar. Intuitionistic nonstandard bounded modified realisability and functional interpretation. *Ann. Pure Appl. Logic*, 169(5):392–412, 2018.

[DN74] Justus Diller and Werner Nahm. Eine Variante zur Dialectica-Interpretation der Heyting-Arithmetik endlicher Typen. *Arch. Math. Logik Grundlagenforsch.*, 16:49–66, 1974.

[DO20] Bruno Dinis and Paulo Oliva. Parametrised functional interpretations. *ArXiv 2005.05794*, May 2020.

[DO21] Bruno Dinis and Paulo Oliva. A parametrised functional interpretation of Heyting arithmetic. *Ann. Pure Appl. Logic*, 172(4):102940, 24, 2021.

[FF17] Fernando Ferreira and Gilda Ferreira. A herbrandized functional interpretation of classical first-order logic. *Archive for Mathematical Logic*, 56(5):523–539, Aug 2017.

[FG15] Fernando Ferreira and Jaime Gaspar. Nonstandardness and the bounded functional interpretation. *Annals of Pure and Applied Logic*, 166(6):701–712, June 2015.

[FN06] Fernando Ferreira and Ana Nunes. Bounded modified realizability. *The Journal of Symbolic Logic*, 71(1):329–346, March 2006.

[FO05] Fernando Ferreira and Paulo Oliva. Bounded functional interpretation. *Annals of Pure and Applied Logic*, 135(1–3):73–112, September 2005.

[FO11] Gilda Ferreira and Paulo Oliva. Functional interpretations of intuitionistic linear logic. *Log. Methods Comput. Sci.*, 7(1):1:9, 22, 2011.

[G5̈8] Kurt Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958.

[GO10] Jaime Gaspar and Paulo Oliva. Proof interpretations with truth. *MLQ Math. Log. Q.*, 56(6):591–610, 2010.

[HO08] Mircea-Dan Hernest and Paulo Oliva. Hybrid functional interpretations. In *Logic and theory of algorithms*, volume 5028 of *Lecture Notes in Comput. Sci.*, pages 251–260. Springer, Berlin, 2008.

[Kre59] Georg Kreisel. Interpretation of analysis by means of constructive functionals of finite types. In *Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)*, Studies in Logic and the Foundations of Mathematics, pages 101–128. North-Holland Publishing Co., Amsterdam, 1959.

[Oli06] Paulo Oliva. Unifying functional interpretations. *Notre Dame Journal of Formal Logic*, 47(2):263–290, 2006.

[Oli07] Paulo Oliva. Computational interpretations of classical linear logic. In *Logic, language, information and computation*, volume 4576 of *Lecture Notes in Comput. Sci.*, pages 285–296. Springer, Berlin, 2007.

[Oli08] Paulo Oliva. An analysis of Gödel's Dialectica interpretation via linear logic. *Dialectica*, 62(2):269–290, 2008.

[Oli10] Paulo Oliva. Functional interpretations of linear and intuitionistic logic. *Inform. and Comput.*, 208(5):565–577, 2010.

[Oli12] Paulo Oliva. Hybrid functional interpretations of linear and intuitionistic logic. *J. Logic Comput.*, 22(2):305–328, 2012.

[Ste79] Martin Stein. Interpretationen der Heyting-Arithmetik endlicher Typen. *Arch. Math. Logik Grundlag*, 19:175–189, 1979.