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Probabilistic logic programming with multiplicative modules

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Abstract

This paper illustrates a simple idea on how to interpret undecompsable generalized multiplicative modules of linear logic as probabilistic methods of a logic prgramming language. These new modules/methods allow to express kinds of propagation of probability distribution inside Bayesian Networks that cannot be expressed by standard Prolog-like programming languages.

1 Logic programming with multiplicative modules

In the *logic programming* paradigm, a program execution is interpreted by the process of proof construction. A state of a computation is represented by a (cut-free) proof which may be partial, i.e. with open branches, or, equivalently, containing non-logical axioms. A program is defined by means of a set of *rules* (or *methods*) which can be applied to an open branch of a proof to close it (by an axiom) or to expand it with new branches. The standard methods of an abstract logic programming language (typically, Prolog-like) are represented as $H : -B_1, \ldots, B_{n\geq 0}$ where the *head* H is produced once the *body* is satisfied, i.e. for each $i \in \{1, \ldots, n\}$ the clause B_i is satisfied. Following Andreoli [2], we may use *bipoles* (a kind of "modules") of the pure multiplicative fragment of Linear Logic (MLL, [4, 3]) to represent a "multi-head" extension of methods defined as follows: the body is given by the MLL proof structure representing the formula tree of the clause set (a conjunctive normal form formula) connected to a bundle of axiom links – one for each conclusion in the head of the method – by a $\otimes$-link. The conclusion of this $\otimes$-link is called the *method name* or the *method handle*. An instance of bipole is given in Fig.1 where $X = \{a, b, c, d, h_1, h_2\}$ is called the *border* of the represented bipole β; any proper subset of the border is called a *restriction* of the border; in particular, $\{a, b, c, d\}$ (resp., $\{h_1, h_2\}$) is the special restriction of X called the *body/input border* (resp., the *head/output border*) of the bipole. Methods with empty body are called *facts* (see e.g. γ in Fig.1).

Given a set of methods/bipoles $\mathscr{U}$, and a goal G (i.e. a set of atoms, $a_1, \ldots, a_n$) a logic program $\langle \mathscr{U}, G\rangle$ is interpreted by the MLL bipolar proof structure built

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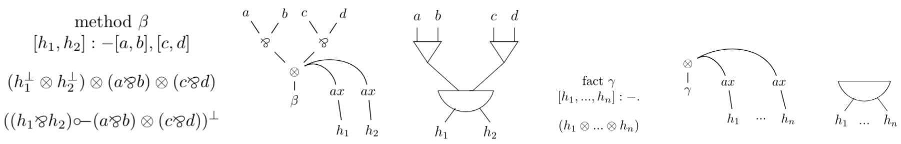
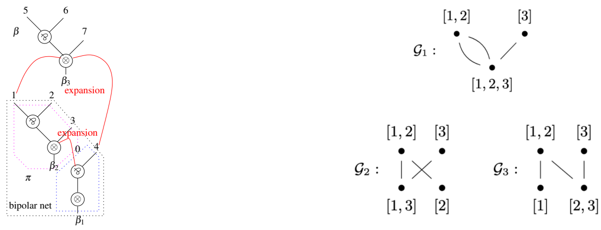


Figure 1: (from left to right) a Prolog-like method (resp., a fact) together with its interpreting *MLL* formula (below) and the corresponding Andreoli's bipole in two ”graphical syntaxes” inspired to proof structures ($\nabla = \parr^n$, $\triangleright = \otimes^n$).



$$
\dfrac{\dfrac{\dfrac{\vdash 5,6 \quad \vdash 7,2 \quad \vdash 4,4^\perp \quad \vdash 1,1^\perp}{\vdash \beta_3, 1, 2, 4}\beta_3 \qquad \vdash 3 \qquad \vdash 0, 0^\perp}{\vdash \beta_2, \beta_3, 0, 4}\beta_2}{\vdash \beta_1, \beta_2, \beta_3}\beta_1
$$

$$
\mathcal{B}_\pi = \begin{cases} p_1 : (1) \ (0,2,3) \ (4) \\ p_2 : (2) \ (0,1,3) \ (4) \\ p_3 : (1) \ (2,3) \ (0,4) \\ p_4 : (2) \ (1,3) \ (0,4) \end{cases}
\quad
\mathcal{B}_\beta = \begin{cases} q_1 : (6) \ (5,7,1,4) \\ q_2 : (5) \ (6,7,1,4). \end{cases}
\quad
\mathcal{B}_{\pi \circ \beta} = \begin{cases} q_1.p_1 : (6) \ (5,7) \ (0,2,3) \\ q_2.p_1 : (5) \ (6,7) \ (0,2,3) \\ q_1.p_2 : (2) \ (6) \ (5,7,0,3) := q_1.p_4 \\ q_2.p_2 : (2) \ (5) \ (6,7,0,3) := q_2.p_4 \\ q_1.p_3 : (6) \ (5,7,0) \ (2,3) \\ q_2.p_3 : (5) \ (6,7,0) \ (2,3). \end{cases}
$$

Figure 2: three ways to perform the bipolar proof construction: by net expansion (on top l.h.s.), by sequential expansion (in the middle side) and by composition of orthogonal behaviors (on the bottom side); graphs of incidence on the top r.h.s..

on $\mathscr{U}$ having G as conclusion/output. A *bipolar proof structure* or *bipolar net* is built by juxtaposing/expanding bipoles through the common border; formally, the definition is given by induction: a single bipole is a bipolar net, then a bipolar proof structure π can be extended by a bipole β if *(i)* the output of β is included in the input of π and *(ii)* the gluing of π with β (the composition $\pi \circ \beta$) is s.t. every *Danos-Regnier switching*[1] of $\pi \circ \beta$ is acyclic (AC). E.g., the bipolar net π on the l.h.s. of Fig.2 is incrementally built starting by the expansion of bipole β_1 with bipole β_2, the result of which is finally expanded with bipole β_3. This graphical construction on nets sequentializes in the derivation on the middle top side of Fig.2.

More abstractly, a MLL bipolar net π can be described by the set of partitions of its border $B = I \cup O$ induced by all Danos-Regnier switchings of π; actually each

[1] A graph mutilation of one of the two premises of each $\parr$-link.

switching decomposes π into a constant number of connected components; fixed a switching, the elements of B belonging to a same connected component constitute a class or block of the partition of B induced by this switching. The set of partitions induced by all switchings of π is called *behavior* of the net π, denoted as $\mathscr{B}_\pi$. E.g., the behavior of the net β (a bipole, indeed) of Fig.1 is given by the following set of partitions over the border $B = I \cup O$ where $I = \{a,b,c,d\}$ and $O = \{h_1,h_2\}$ (we omit the method name "β" which is the handle of the method; it does not play any "gluing role" in the expansion since it is not consumed by the expansion process):

$$\mathscr{B}_\beta = \{\{(a,c,h_1,h_2),(b),(d)\},\{(a,d,h_1,h_2),(b),(c)\},\\ \{(b,c,h_1,h_2),(a),(d)\},\{(b,d,h_1,h_2),(a),(c)\}\}.$$

Then, the correctness of the expansion step of a bipolar net π by a bipole β is guaranteed by the orthogonality of the respective behaviors restricted to their common border. We say that a MLL bipolar net π can be expanded by a bipole β iff *(i)* the (head) output H of β is included in the (body) input I of π and *(ii)* their respective behaviors, restricted to their common border H, are orthogonal, i.e. $(\mathscr{B}_\pi)^{\downarrow H} \perp (\mathscr{B}_\beta)^{\downarrow H}$. In general two partitions sets P,Q on the same set X are orthogonal, $P \perp Q$, iff they are pointwise orthogonal that is, $p \perp q$, $\forall p \in P$ and $\forall q \in Q$ where the "orthogonality $p \perp q$" is defined by a topological condition: the bipartite graph obtained by linking together classes (or blocks) of each partition sharing an element is acyclic and connected (ACC) see also [4, 5, 1]. E.g., $\{(1,2),(3)\}$ is both orthogonal to $\{(1,3),(2)\}$ and $\{(1),(2,3)\}$ but it is not orthogonal to $\{(1,2,3)\}$ as illustrated in the rightmost h.s. of Fig.2. Thus, the bipolar net π in the l.h.s. of Fig.2 is completely defined by its behavior $\mathscr{B}_\pi$ with respect to its input border $I = \{1,2,3,4\}$; similarly, bipole $\beta = \beta_3$ is completely defined by its behavior $\mathscr{B}_\beta$ over its border $\{1,4,5,6,7\}$ as in the bottom middle side of Fig.2. Now, since the head $H = \{1,4\}$ of β is included in the input $\{1,2,3,4\}$ of π and the restricted behaviors, $(\mathscr{B}_\pi)^{\downarrow H}$ and $(\mathscr{B}_\beta)^{\downarrow H}$, are orthogonal (i.e., $\{(1,4)\} \perp \{(1),(4)\}$), we can expand π by β and get the net $\pi \circ \beta$ with the behavior $\mathscr{B}_{\pi\circ\beta}$ as in Fig.2.

This abstraction suggests a new syntax for module/nets, based on behaviors (partitions sets of a border) allowing us to generalize the notion of bipolar nets exiting the standard MLL realm. A **multiplicative module** is a triple $\mu : \langle I = \{i_1,...,i_{n\geq 0}\}, O = \{o_1,...,o_{m\geq 1}\}, \mathscr{B}_\mu\rangle$ where I is an input set, O is an output set with $I \cap O = \emptyset$ and $\mathscr{B}_\mu$ is a set of partitions (the behavior of μ) over the border $B = I \cup O$ with $n,m \geq 1$. All partitions in $\mathscr{B}_\mu$ have the same size, moreover $\mathscr{B}_\mu$ is such that its orthogonal is not empty. A **multiplicative bipole** is a special case of module $\beta : \langle I = \{i_1,...,i_n\}, O = \{o_1,...,o_m\}, \mathscr{B}_\mu\rangle$ such that in each partition of $\mathscr{B}_\beta$ the output set O belongs to a single block (called, *head block/class*), with $n,m \geq 1$ (this is the "trigger" of the method).

Let μ be a multiplicative module with behavior $\mathscr{B}_\mu$ and border X and let β be a multiplicative bipole with behavior $\mathscr{B}_\beta$ over Y. If *(i)* the output O of β is included in the input I of μ and *(ii)* the restrictions $(\mathscr{B}_\mu)^{\downarrow O}$ and $(\mathscr{B}_\beta)^{\downarrow O}$ are orthogonal then, we can expand (by a **bipolar expansion**) μ with β through the border O and get the new multiplicative net $\mu \circ \beta$ s.t.:

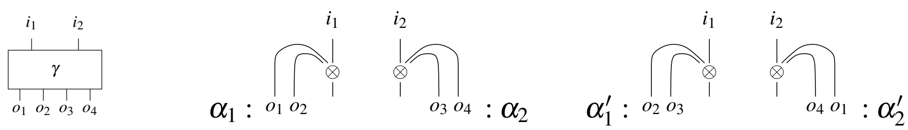
- the input border results by the union of the two input borders except O (the trigger of the expansion);
- the output border is that one of μ;
- the behavior $\mathscr{B}_{(\mu\circ\beta)}$ is built as follows: for every pair of partitions, $p \in \mathscr{B}_\mu$ and $q \in \mathscr{B}_\beta$, if χ is the head class (containing the trigger O) of q and $\sigma_1, ..., \sigma_m$ are the m classes of p containing separately the m elements of O then, we get the new partition $r \in \mathscr{B}_{(\mu\circ\beta)}$ consisting of:
  1. the class λ obtained by merging χ together with $\sigma_1, ..., \sigma_m$;
  2. the remaining classes of p and q.

A **multiplicative (bipolar) net** is a multiplicative module built by only composing multiplicative bipoles via bipolar expansion. We say that a multiplicative net μ is **MLL-definable** or **MLL-decomposable**, iff the behavior of μ is also the behavior of some MLL bipolar net.

2 Extending the paradigm with ”unfoldable” modules

Programming with general multiplicative nets takes the advantage of extending Andreoli's proof construction paradigm by introducing new methods which cannot be expressed in the standard MLL setting: these methods are not MLL definable or decomposable by the focused (positive/negative) alternation of connectives $\otimes/\parr$. Nevertheless, it is possible to build new multiplicative modules simply by taking the intersection of the behaviors of a family of bipoles having the same output border and the same “logical skeleton”, that is the same logical structure up to cyclic permutation of the input border. Using specific generalized connectives (tipically, ”undecomposable Girard connectives” [4, 5, 1]), we may define special multiplicative methods which behave either as intersection of a set of MLL bipoles or as an union of pairs of MLL bipoles. How we will see in next sections, these new links are able to model specific instances of “additive” computational behaviors (e.g., *slices*) and allow us to define an ”unfolding” operation.

We illustrate how to unfold a net by MLL slices by means of an example. Consider e.g., the following multiplicative module γ equipped with the behavior $\mathscr{B}_\gamma$ below (on the l.h.s.). It is not difficult to convince that this behavior does not correspond to the behavior of any MLL proof structure. Neverthless, this behavoir can be seen as obtained by the sum (an union) of the behaviors of the two pairs of concurrent bipoles below, $\gamma_1 = (\alpha_1, \alpha_2)$ and $\gamma_2 = (\alpha'_1, \alpha'_2)$ (their omitted method names can be easily interpreted as $\gamma_1 = \alpha_1 \parr \alpha_2$ and $\gamma_2 = \alpha'_1 \parr \alpha'_2$):



$$\mathscr{B}_\gamma = \{ \begin{array}{l}\{(i_1,o_1,o_2),(i_2,o_3,o_4)\},\\ \{(i_1,o_2,o_3),(i_2,o_4,o_1)\},\end{array} \} = \{\mathscr{B}_{\gamma_1} = \{\{(i_1,o_1,o_2),(i_2,o_3,o_4)\}\} \cup \mathscr{B}_{\gamma_2} = \{\{(i_1,o_2,o_3),(i_2,o_4,o_1)\}\}\}$$

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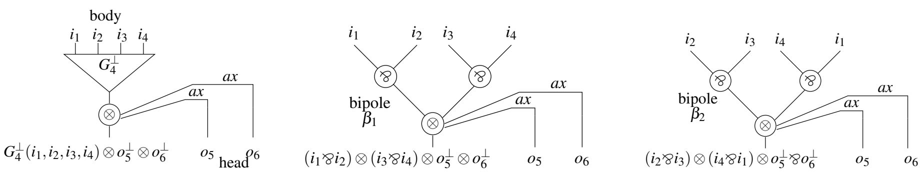
Dually, consider the module β with the behavior $\mathscr{B}_\beta$ as below (on the l.h.s.). Observe that this behavior, restricted to the input $I_\beta = \{i_1, i_2, i_3, i_4\}$, is orthogonal to the behavior of γ restricted to the output $O_\gamma = \{o_1, o_2, o_3, o_4\}$, once we assume the following correspondence between the indexes of I_β and O_γ: $i_1 = o_1, i_2 = o_2, i_3 = o_3, i_4 = o_4$. As before, it is not difficult to convince that $\mathscr{B}_\beta$ is not the behavior of any MLL proof structure, nevertheless, it can be obtained by the intersection of the behaviors of two MLL bipoles, β_1 and β_2, whose input borders are, respectively obtained by the cyclic permutation of the sequence (i_1, i_2, i_3, i_4), that is $\mathscr{B}_\beta = \mathscr{B}_{\beta_1} \cap \mathscr{B}_{\beta_2}$, as illustrated below ($\beta$ contains the first undecomposable connective discovered by Girard [4], denoted by $G_4^\perp$, thus the method name of β is $G_4^\perp(i_1, i_2, i_3, i_4) \otimes o_5^\perp \otimes o_6^\perp$).



$$\mathscr{B}_\beta = \left\{ \begin{array}{l} \{(i_1,i_3,o_5,o_6),(i_2),(i_4)\}, \\ \{(i_2,i_4,o_5,o_6),(i_1),(i_3)\} \end{array} \right\} = \begin{array}{c} \mathscr{B}_{\beta_1} : \{\{(i_1,i_3,o_5,o_6),(i_2),(i_4)\},\{(i_2,i_4,o_5,o_6),(i_1),(i_3)\},\{(i_1,i_4,o_5,o_6),(i_2),(i_3)\},\{(i_2,i_3,i_4,o_5),(i_1),(i_4)\}\} \\ \cap \\ \mathscr{B}_{\beta_2} : \{\{(i_1,i_3,o_5,o_6),(i_2),(i_4)\},\{(i_2,i_4,o_5,o_6),(i_1),(i_3)\},\{(i_1,i_2,o_5,o_6),(i_3),(i_4)\},\{(i_3,i_4,o_5,o_6),(i_1),(i_2)\}\} \end{array}$$

Multiplicative bipoles like γ (resp., β) are called **unfoldable bipoles** and $\{\gamma_i, \gamma_2\}$ (resp., $\{\beta_1, \beta_2\}$) is the **unfolding trace** (or the *unfolding family*) of γ (resp., β).

Now, assume we built a multiplicative net T which contains an expansion step of a bipole β by a bipole γ as in the T_2 module enclosed by a blue dashed line in Fig.3 (on the l.h.s.). We may **unfold** T_2 by two modules, T_2' and T_2'', and get two correct nets, $T' = T_1 \circ T_2'$ and $T'' = T_1 \circ T_2''$, as in the middle and r.h.s. of Fig.3.

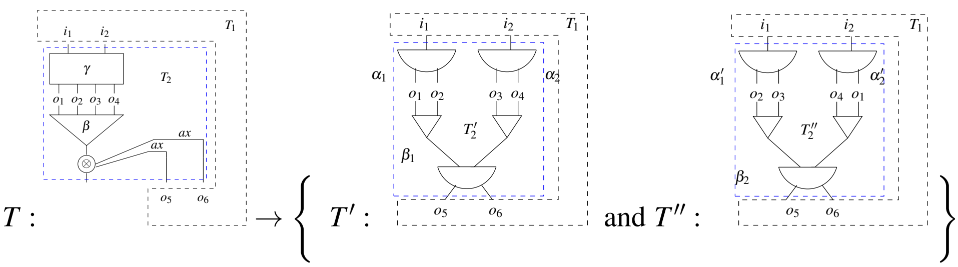
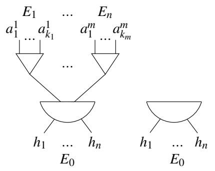


Figure 3: The unfolding of the $T_1 \circ T_2$ net by $T_1 \circ T_2'$ and $T_1 \circ T_2''$ nets.

3 Probabilistic logic programming with modules

In the MLL case, it appears "natural" [6] to associate, respectively, a *conditional probability* to a MLL method/bipole and an *a-priori probability* to a MLL fact represented as in the following figure, with the implicit assumption that we interpret $E_0 = [h_1, ..., h_n]$ and $E_1 = [a_1^1, ..., a_k^1] \otimes \cdots \otimes E_m = [a_1^m, ..., a_k^m]$ as independent observable events E_i with $1 \leq i \leq m$.



$$\left|\begin{array}{ll} \texttt{method:} & [h_1,...,h_n] : -[a^1_1,...,a^1_{k_1}],...,[a^m_1,...,a^m_{k_m}] \\ \texttt{fact:} & [h_1,...,h_n] : -. \end{array}\right|$$

Note that in general, the probability of the product of two independent events is equal to the product of the probabilities of these events, $\boldsymbol{p}(A \cap B) = \boldsymbol{p}(A).\boldsymbol{p}(B)$ or, in general, for m (independent) events, $E_1,...,E_m$, we get

$$\boldsymbol{p}(\prod_{i=1}^{m} E_i) = \prod_{i=1}^{m} P(E_i).$$

Thus, in case β is a MLL fact "$E_0 = [h_1,...,h_n] : -$" we may associate an a-priori probability to β as $\boldsymbol{p}(\beta) = \boldsymbol{p}(E_0)$ otherwise, in case β is a MLL method/bipole "$E_0 : -E_1,...,E_m$" we associate a conditional probability $\boldsymbol{p}(\beta) = \boldsymbol{p}(E_0 | \prod_{i=1}^{m} E_i)$ to the MLL bipole β. Now, since MLL bipoles are special cases of general unfoldable multiplicative bipoles we extend probabilities to unfoldable bipoles as follows.

Definition 1 (**probability distribution of unfoldable bipoles).** *Let β be a multiplicative unfoldable bipole with border $I = \{i_1,...,i_n\} \uplus O = \{o_1,...,o_m\}$and behavior $\mathscr{B}_\beta$; let $\beta_1,...,\beta_k$ be the unfolding trace (or family of MLL bipoles) s.t. $\bigcup_{i=1}^{k} \beta_i = \mathscr{B}_\beta$. We call a* **probability distribution** *for β a (finite) set of real values, $\boldsymbol{P}_\beta(O|I) = \{v_1,...,v_{k\geq 1}\}$, with $0 < v_i \in \mathbb{R} \leq 1$ and k is the size (cardinality) of the trace of the unfolding family of β, built as follows:*

- *If β is a MLL bipole then (the trace of β is the singleton $\{\beta\}$ so) $\boldsymbol{P}(O|I) = \{\boldsymbol{p}(\beta)\}$ where $\boldsymbol{p}(\beta)$ is a real number s.t. $0 < \boldsymbol{p}(\beta) \leq n$; in particular, if β is a method with $I \neq \emptyset$ then $\boldsymbol{p}(\beta)$ is the conditional probability $\boldsymbol{p}(O|I)$, otherwise, in case β is a fact (i.e., $I = \emptyset$) then we associate to β an a-priori probability $\boldsymbol{p}(O)$.*

- *Otherwise, β is an unfoldable multiplicative bipole whose trace $\beta_1,...,\beta_k$ has size $k \leq 2$ then we associate to each element of the trace β_i a probability value $\boldsymbol{p}(\beta_i)$ (i.e. a real number) s.t. $0 < \boldsymbol{p}(\beta_i) \leq n$ with the condition that in case that $\bigcup_{i=1}^{k} \beta_i = \mathscr{B}_\beta$ then $\sum_{i=1}^{k} \boldsymbol{p}(\beta_i) = 1$; thus $P(O|I) = \{\boldsymbol{p}(\beta_i) \mid \beta_i$ is in the trace of $\beta\}$.*

Now, how do we propagate/compose probability information while building a a net? There are two directions of the information flow in our net construction model:

1. **net expansion ↑:** the first direction consists in the bottom-up construction of the net, by module expansions;

2. **info propagation ↓:** the second direction intervenes when the net construction is successfully completed; in fact, if successful, we can invert the direction of the information and propagate (by composition) the probability

information starting from the top (that is, the a-priori probabilities associated to the axiom-bipoles/facts) towards to the bottom (via composition of the conditional probabilities associated to the bipoles/methods). It is in the information propagation phase that the unfolding plays its decisive role.

We show by an example, inspired to Bayes' Theorem, how the unfolding of the net allows to propagate alternative distributions of probabilities.

3.1 An example inspired to Bayes' Theorem

In probability theory and statistics, Bayes' Theorem describes the probability of an event, based on prior knowledge of conditions that might be related to this event. It is used to calculate the probability of a cause which triggered the observed event and also in Machine Learning for training Naive Bayesian Classificators.
Bayes' Theorem: let $A_1,...,A_n$ be independent events and a distinct event E then

$$\boldsymbol{p}(A_i|E) = \frac{\boldsymbol{p}(E|A_i)P(A_i)}{\boldsymbol{p}(E)} = \frac{\boldsymbol{p}(E|A_i)\boldsymbol{p}(A_i)}{\sum_{i=1}^{n}\boldsymbol{p}(E|A_i)\boldsymbol{p}(A_i)} \quad \text{where:}$$

- $\boldsymbol{p}(A_i)$ is the *a-priori probability* of A_i, where "a priori" means that it does not account for any information about the event E; we assume $\Sigma_{i=1}^{n}\boldsymbol{p}(A_i) = 1$;
- $\boldsymbol{p}(E|A_i)$ is the *conditional probability* of E given A_i is true;
- $\boldsymbol{p}(A_i|E)$ is a *conditional probability*: the likelihood of event A_i occurring given that E is true (i.e., the *a-posteriori probability* of A_i given E is true);
- $\boldsymbol{p}(E) = \Sigma_{i=1}^{n}\boldsymbol{p}(E|A_i)P(A_i)$ is the *absolute probability* of E.

The theorem describes how opinions in observing A_i are enriched by observing the event E. Let us use it in our syntax. Assume we built by bottom-up expansion the net T on the l.h.s. of Fig. 3. For the sake of simplicity, assume that module γ is a "multi-facts" that is, a multi-facts or concurrent facts (i.e., its input set $\{i_1,i_2\}$ is empty). We know that T is unfoldable, since both γ and β are so then assume that the trace of γ is $\gamma_1 = (\alpha_1 \parr \alpha_2)$ and $\gamma_2 = (\alpha_1', \parr \alpha_2')$ and the trace of β is β_1 and β as in Fig.3. According to Definition 1, we may assign a probability distribution to the traces of γ and β as follows:

- (*a-priori probabilities*): $\boldsymbol{P}_\gamma(O|I) = \{\boldsymbol{p}(\gamma_1 = (\alpha_1 \parr \alpha_2)), \boldsymbol{p}(\gamma_2 = (\alpha_1' \parr \alpha_2'))\}$ where $\boldsymbol{p}(\gamma_1) = 2/5$ denotes e.g. the a-priori probability that "a student is female" ($\boldsymbol{p}(F)$) while $\boldsymbol{p}(\gamma_2) = 3/5$ denotes e.g. the a-priori probability that "a student is male" ($\boldsymbol{p}(M)$). Their sum, $\boldsymbol{p}(\gamma_1) + \boldsymbol{p}(\gamma_2) = 1$, satisfies Def. 1.
- (*conditional probabilities*): $\boldsymbol{P}_\beta(O|I) = \{\boldsymbol{p}(\beta_1), \boldsymbol{p}(\beta_2)\}$ where $\boldsymbol{p}(\beta_1)$ denotes e.g. the conditional probability that "a female student wears pants" $\boldsymbol{p}((o_5,o_6)|(i_1,i_2,i_3,i_4)) = \boldsymbol{p}(P|F) = 1/2$ while $\boldsymbol{p}(\beta_2)$ is the conditional probability that a "a male student wears pants" $\boldsymbol{p}((o_5,o_6)|(i_2,i_3,i_4,i_1)) = \boldsymbol{p}(P|M) = 1$.

Let us "translate" the MLL bipoles occurring in the trace of T in to the corresponding Prolog-like program (4 facts and 2 methods) as follows:

$\alpha_1 : [1,2] : -.$ $\quad \alpha_2 : [3,4] : -.$ $\quad \beta_1 : [5,6] : -[1,2],[3,4]$

$\alpha'_1 : [2,3] : -.$ $\quad \alpha'_2 : [4,1] : -.$ $\quad \beta_2 : [5,6] : -[2,3],[4,1].$

Now, let us consider the following pair of independent events $\{F,M\}$ where:

- $F = (F_1 \otimes F_2)$ with $F_1 = [1,2], F_2 = [3,4]$ is the event F="a female student";
- $M = (M_1 \otimes M_2)$ with $M_1 = [2,3], M_2 = [4,1]$ is the event M="a male student",
- $E = [5,6]$ is the event E="a student wears pants".

We then associate the following a-priori probability distribution:

- $\boldsymbol{p}(F) = \boldsymbol{p}(F_1).\boldsymbol{p}(F_2) = \boldsymbol{p}(\gamma_1) = 2/5$ and
- $\boldsymbol{p}(M) = \boldsymbol{p}(M_1).\boldsymbol{p}(M_2) = \boldsymbol{p}(\gamma_2) = 3/5$ (clearly, $\boldsymbol{p}(F) + \boldsymbol{p}(M) = 1$).

Finally, we associate the conditional probabilities to the set of methods as follows:

- $\boldsymbol{p}(E|F = (F_1 \otimes F_2)) = \boldsymbol{p}(\beta_1)$ associated to bipole β_1
- $\boldsymbol{p}(E|M = (M_1 \otimes M_2)) = \boldsymbol{p}(\beta_2)$ associated to bipole β_2

From which we can calculate the absolute probability that a student wears trousers: $\boldsymbol{p}(E) = \sum_{i=1}^{2} \boldsymbol{p}(E|E_i)\boldsymbol{p}(E_i)$, for $E_1 = F$ and $E_2 = M$.

The unfolding of the net $T_1 \circ T_2$ of Fig. 3, allows to propagate the probabilities information top-down and so to calculate, via the Bayes' Theorem, the a-posterior probability that "a student wearing trousers is female" $\boldsymbol{p}(F|E)$ (resp., "a student wearing trousers is male", $\boldsymbol{p}(M|E)$) by using T'_2 (resp., by using T''_2).

$$\boldsymbol{p}(F|E) = \frac{\boldsymbol{p}(E|F)\boldsymbol{p}(F)}{\boldsymbol{p}(E)} = 1/4 \qquad\qquad \boldsymbol{p}(M|E) = \frac{\boldsymbol{p}(E|M)\boldsymbol{p}(M)}{\boldsymbol{p}(E)} = 3/4.$$

Thus the Naive Bayesian Classificator, trained on this model, will classify next student wearing pants as "male" since $\boldsymbol{p}(M|E) > \boldsymbol{p}(F|E)$.

References

[1] Acclavio, M. and Maieli, R.: Generalized Connectives for Multiplicative Linear Logic. In LIPIcs, Vol. 152, Proc. of CSL200 Conference. 3, 4

[2] Andreoli, J.-M.: Focussing proof-nets construction as a middleware paradigm. LNCS, vol. 2392, pp. 501-516, Springer, Heidelberg (2002). 1

[3] Danos, V. and Regnier, L.: The structure of multiplicatives. Archive for Mathematical Logic, vol. 28, pg. 181-203, Springer-Verlag, 1989. 1

[4] Girard, J.-Y.: Multiplicatives. Rendiconti del Seminario Matematico dell'Università e Policlinico di Torino, pp. 11–33, 1987. 1, 3, 4, 5

[5] Maieli, R.: Non decomposable connectives of linear logic. In APAL, Vol. 170, Issue 11, November 2019. 3, 4

[6] Nguembang Fadja, A. and and Riguzzi, F.: Probabilistic Logic Programming in Action. LNCS, vol. 10344, Springer 2015. 5