Proof of an Algorithm to Compute Time-Optimal Third Order Polynomial S-curve Trajectories

Marc Gouttefarde, Ghina Hassan, Ahmed Chemori

Proof of an Algorithm to Compute Time-Optimal Third Order Polynomial S-curve Trajectories

Marc Gouttefarde, Ghina Hassan, Ahmed Chemori[1]

I. Introduction

This research report deals with time-optimal third order polynomial S-curve trajectories. Methods to generate time-optimal S-curve trajectories have been previously proposed. For instance, an algorithm to optimize an objective function composed of two terms, the execution time and the jerk, has been developed in [1]. A binary search method to optimize the S-curve velocity profile has been introduced in [2].

In the present work, a new algorithm is introduced and proved. This algorithm computes the maximum velocity and acceleration of a 3rd order polynomial S-curve trajectory such the total time of the S-curve trajectory is minimized for a given desired displacement. Upper bound constraints on the S-curve maximum velocity and acceleration are taken into account. The resulting optimization problem is solved by analyzing in detail the Karush-Kuhn-Tucker (KKT) conditions. This mathematical analysis allows to figure out and prove the validity of the proposed algorithm. While the analysis is slightly tedious, the obtained algorithm is very efficient and, to the best of our knowledge, it has never been proposed.

II. MINIMUM-TIME S-CURVE TRAJECTORY

A. Polynomial S-Curve Motion Profile

The third order polynomial S-curve is considered in this report. As shown in Figure 1, the motion profile of the third order polynomial S-curve consists of seven segments, among which the first three and the last three constitute the acceleration and deceleration phases, respectively, and the fourth segment constitutes the constant velocity stage. The jerk is defined as the following function of time:

$$j(t) = \begin{cases} J, & t_0 \le t \le t_1, t_6 \le t \le t_7 \\ 0, & t_1 \le t \le t_2, t_3 \le t \le t_4, t_5 \le t \le t_6 \\ -J, & t_2 \le t \le t_3, t_4 \le t \le t_5 \end{cases} \qquad (1)$$

where J is the maximum jerk value and the time instants t_i are shown in Figure 1. The acceleration, velocity and displacement as functions of time can be deduced by integration of (1) with appropriate initial and final conditions.

Referring to Figure 1, $d_j = t_1 - t_0$ is the time needed to increase the acceleration from zero to the maximum acceleration A and to decrease the acceleration from the maximum

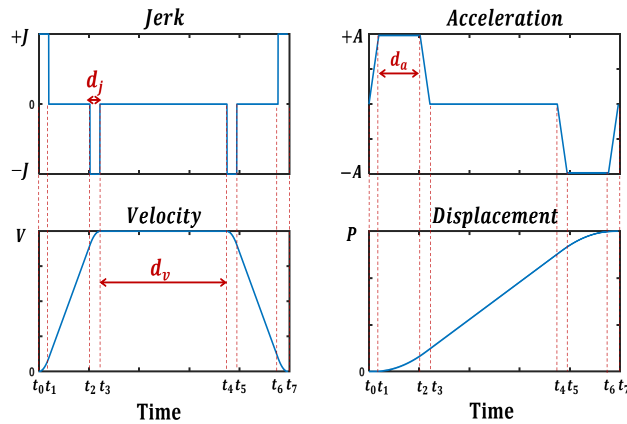

Fig. 1. Third order polynomial S-Curve model.

value A to zero, i.e., it is the time during which jerk stays constant at the maximum jerk value J. Besides, $d_a = t_2 - t_1$ corresponds to the time during which the acceleration profile remains constant and equal to the maximum acceleration A and $d_v = t_4 - t_3$ corresponds to the time during which the velocity profile remains constant and equal to V. Note that the symmetrical S-curve has been adopted in the present study (i.e. the acceleration and deceleration phases are symmetrical) and that d_j, d_a and d_v are all greater than or equal to zero. The three time intervals can be written as follows:

$$\begin{aligned} d_j &= \frac{A}{J} \\ d_a &= \frac{V}{A} - \frac{A}{J} \\ d_v &= \frac{P}{V} - \frac{V}{A} - \frac{A}{J} \end{aligned} \qquad (2)$$

A and V are the maximum velocity and acceleration achieved for a given displacement P. Both A and V should be less than or equal to the corresponding maximum actuator capabilities. The total time to travel a distance P is expressed as follows:

$$T = 4d_j + 2d_a + d_v \qquad (3)$$

B. Minimization Problem Formulation

In this report, the proof of a new algorithm to optimize the values of the acceleration A and velocity V is provided. For a given displacement P and taking into account upper bound constraints on maximum acceleration and velocity, i.e., $A \le A_{max}$ and $V \le V_{max}$, this algorithm efficiently computes the values of A and V of a 3rd order polynomial S-curve

[1] Marc Gouttefarde, Ghina Hassan and Ahmed Chemori are with LIRMM, Univ Montpellier, CNRS, Montpellier, France, {marc.gouttefarde}@lirmm.fr

trajectory with minimum total time T. By substitution of the three time intervals (2) in the total time (3), the function T to be minimized is expressed as follows:

$$T = \frac{P}{V} + \frac{V}{A} + \frac{A}{J} \tag{4}$$

Let us define $x = (x_1, x_2)$, where $x_1 = V$, $x_2 = A$, as well as $x_{1max} = V_{max}$ and $x_{2max} = A_{max}$. Hence, the objective function is:

$$f(x) = T = \frac{P}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{J} \tag{5}$$

The minimization problem of finding the minimum-time S-curve trajectory is then formulated as follows:

$$\min_x f(x) \quad \text{subject to} \begin{cases} C_1(x) = d_a = \frac{x_1}{x_2} - \frac{x_2}{J} \geq 0 \\ C_2(x) = d_v = \frac{P}{x_1} - \frac{x_1}{x_2} - \frac{x_2}{J} \geq 0 \\ C_3(x) = x_{1max} - x_1 \geq 0 \\ C_4(x) = x_{2max} - x_2 \geq 0 \\ C_5(x) = x_1 > 0 \\ C_6(x) = x_2 > 0 \end{cases} \tag{6}$$

C. Algorithm to Compute the Minimum-Time S-Curve Trajectory

The KKT first-order necessary optimality conditions of the optimization problem (6) yield 11 cases: One case of no active constraint, four cases of two active constraints (C_1, C_2, C_3 or C_4) and six cases of two active constrains ($C_1 = C_2 = 0$, $C_1 = C_3 = 0$, etc.), being given that the constraints C_5 and C_6 are always inactive since $V = x_1 > 0$ and $A = x_2 > 0$. A detailed analysis of these 11 cases (cf. Section III) shows that the following four are impossible at a (local) optimal solution:

- No active constraint
- One active constraint $C_1 = 0$
- One active constraint $C_4 = 0$
- Two active constraints $C_1 = C_4 = 0$.

It also shows that, at a local optimal solution:

- One active constraint $C_2 = 0 \Longleftrightarrow$ two active constraints $C_1 = C_2 = 0$
- One active constraint $C_3 = 0 \Longleftrightarrow$ two active constraints $C_1 = C_3 = 0$
- The case of two active constraints $C_2 = C_3 = 0$ turns out to be a particular case of the case one active constraint $C_3 = 0$.

Hence, out of the 11 cases, only four of them needs to be considered to determine the optimal solutions of the optimization problem (4):

- One active constraint $C_2 = 0$
- One active constraint $C_3 = 0$
- Two active constraints $C_2 = C_4 = 0$
- Two active constraints $C_3 = C_4 = 0$.

As detailed in the Section III, further analyzing these four cases lead to Algorithm 1 which allows to efficiently determine the values of V and A yielding the minimum-time polynomial S-curve trajectory.

Algorithm 1 Minimum-Time S-Curve Trajectory

Input: P, J, V_{max}, A_{max}
Output: A and V yielding the minimum total time T

if $(J^2 P \leq 2A_{max}^3)$ **then**
  if $(JP^2 \leq 4V_{max}^3)$ **then**
    $V = \sqrt[3]{\frac{JP^2}{4}}$ and $A = \sqrt[3]{\frac{J^2 P}{2}}$
  else
    $V = V_{max}$ and $A = \sqrt{JV_{max}}$
  end if
else
  if $(\sqrt{JV_{max}} \leq A_{max})$ **then**
    $V = V_{max}$ and $A = \sqrt{JV_{max}}$
  else
    $A = A_{max}$
    $V = \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2 P A_{max}}}{2J}$
    if $(V > V_{max})$ **then**
      $V = V_{max}$
    end if
  end if
end if

III. Proof of Algorithm 1

The KKT first-order necessary conditions [3] for $x^* = (x_1^*, x_2^*)$ to be a local solution to the optimization problem (6) can be stated as follows.

First-order necessary conditions: There is a Lagrange multiplier vector λ^*, with components λ_i^*, $1 \leq i \leq 6$, such that the following conditions are satisfied:

$$\nabla_x \mathscr{L}(x^*, \lambda^*) = 0 \tag{7}$$
$$C_i(x^*) \geq 0, \quad 1 \leq i \leq 6 \tag{8}$$
$$\lambda_i^* \geq 0, \quad 1 \leq i \leq 6 \tag{9}$$
$$\lambda_i^* C_i(x^*) = 0, \quad 1 \leq i \leq 6 \tag{10}$$

where the constraints $C_i(x)$ are defined in (6) and $\nabla_x \mathscr{L}(x, \lambda)$ is the gradient with respect to x of the Lagrangian function $\mathscr{L}(x, \lambda)$ defined as:

$$\mathscr{L}(x, \lambda) = f(x) - \sum_{i=1}^{6} \lambda_i C_i(x) \tag{11}$$

The first-order necessary conditions stated above in (7) to (10) are valid if the functions $f(x)$ and $C_i(x)$ are continuously differentiable and the so-called Linear Independence Constraint Qualification (LICQ) holds at x^* [3]. According to the definitions of $f(x)$ and $C_i(x)$, the continuous differentiability condition is true for problem (6). The fact that the LICQ holds is proved in Section III-P.

The well-known method to use the first-order necessary conditions to find local optimal solutions x^* consists in distinguishing all the possible cases of sets of active constraints which allows the determination of x^* and of the Lagrange multiplier vector λ^*.

A constraint $C_i(x)$ is defined as being active if $C_i(x) = 0$. According to (10), for a given i, either $C_i(x^*)$ is active, i.e. $C_i(x) = 0$, or $\lambda_i^* = 0$. Since $C_5(x) = x_1 = V > 0$ and $C_6(x) =$

$x_2 = A > 0$, $C_5(x)$ and $C_6(x)$ are never active, and according to (10), $\lambda_5 = 0$ and $\lambda_6 = 0$ at a local optimal solution. Then, only four constraints can be active, $C_1(x)$ to $C_4(x)$, and the problem (6) having two variables, x_1 and x_2, the following 11 cases of possible active constraint sets must be studied:

- No active constraint (one case)
- Four cases of one active constraint
- Six cases of two active constraints

These relatively small number of possible active constraint sets and the rather simple expressions of the functions $f(x)$ and $C_i(x)$ make the analysis of each of these eleven cases possible, as detailed in the following subsections.

A. Preliminaries: Expressions of the gradients

In (7), the gradient with respect to x of the Lagrangian function $\mathscr{L}(x,\lambda)$ in (11) is given by:

$$\nabla_x \mathscr{L}(x,\lambda) = \nabla_x f(x) - \sum_{i=1}^{4} \lambda_i \nabla_x C_i(x) \tag{12}$$

where the summation is taken for $i = 1$ to 4 since, as pointed above, $\lambda_5 = 0$ and $\lambda_6 = 0$ at a local optimal solution. According to (5) and (6), the gradients in (12) are as follows:

$$\nabla_x f(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\[2ex] \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_2} - \dfrac{P}{x_1^2} \\[2ex] \dfrac{1}{J} - \dfrac{x_1}{x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{x_1^2 - Px_2}{x_1^2 x_2} \\[2ex] \dfrac{x_2^2 - Jx_1}{Jx_2^2} \end{bmatrix} \tag{13}$$

$$\nabla_x C_1(x) = \begin{bmatrix} \dfrac{\partial C_1}{\partial x_1} \\[2ex] \dfrac{\partial C_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_2} \\[2ex] -\dfrac{1}{J} - \dfrac{x_1}{x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_2} \\[2ex] \dfrac{-x_2^2 - Jx_1}{Jx_2^2} \end{bmatrix} \tag{14}$$

$$\nabla_x C_2(x) = \begin{bmatrix} \dfrac{-P}{x_1^2} - \dfrac{1}{x_2} \\[2ex] \dfrac{x_1}{x_2^2} - \dfrac{1}{J} \end{bmatrix} = \begin{bmatrix} \dfrac{-Px_2 - x_1^2}{x_1^2 x_2} \\[2ex] \dfrac{Jx_1 - x_2^2}{Jx_2^2} \end{bmatrix} \tag{15}$$

$$\nabla_x C_3(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{16}$$

$$\nabla_x C_4(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{17}$$

B. Case 1: No active constraint

According to (10), $\lambda_i^* = 0$ for $i = 1 \ldots 4$ if there is no active constraint at a local optimal solution x^*. Then, (7) yields:

$$\begin{aligned} \nabla_x \mathscr{L}(x^*,\lambda^*) = \nabla_x f(x^*) = 0 \\ \iff \quad & \begin{cases} x_1^{*2} - Px_2^* = 0 \\ x_2^{*2} - Jx_1^* = 0 \end{cases} \\ \iff \quad & \begin{cases} x_1^{*3} = JP^2 \\ x_2^{*3} = J^2 P \end{cases} \end{aligned} \tag{18}$$

However, the constraint $C_2(x)$ is:

$$C_2(x) = \frac{P}{x_1} - \frac{x_1}{x_2} - \frac{x_2}{J} = \frac{J\left(Px_2 - x_1^2\right) - x_1 x_2^2}{Jx_1 x_2} \tag{19}$$

so that, with (18), the fact that $C_2(x)$ is not active leads to:

$$\begin{aligned} C_2(x^*) > 0 \quad &\iff \quad J\left(Px_2^* - x_1^{*2}\right) - x_1^* x_2^{*2} > 0 \\ &\iff \quad -x_1^* x_2^{*2} > 0 \end{aligned} \tag{20}$$

where the second equivalence comes from $Px_2^* - x_1^{*2} = 0$ which is a consequence of (18). The inequality (20) is impossible since $x_1^* > 0$ and $x_2^* > 0$ according to constraints C_5 and C_6. Consequently, this first case of no active constraint is not feasible and is discarded.

C. Case 2: One active constraint $C_1(x) = 0$

According to (10), we have $\lambda_i^* = 0$ for $i = 2, \ldots, 4$. Hence, x_1^*, x_2^* and λ_1^* can be determined from the following equation system obtained from (7) and $C_1(x^*) = 0$:

$$\begin{cases} \nabla_x \mathscr{L}(x^*,\lambda^*) = \nabla_x f(x^*) - \lambda_1^* \nabla_x C_1(x^*) = 0 \\ C_1(x^*) = \dfrac{x_1^*}{x_2^*} - \dfrac{x_2^*}{J} = 0 \end{cases} \tag{21}$$

From (13) and (14), the system (21) is equivalent to:

$$\begin{cases} x_1^{*2} - Px_2^* - \lambda_1^* x_1^{*2} = 0 \\ x_2^{*2} - Jx_1^* + \lambda_1^* \left(x_2^{*2} + Jx_1^*\right) = 0 \\ Jx_1^* - x_2^{*2} = 0 \end{cases} \tag{22}$$

This second equation system yields $\lambda_1^* = 0$, $x_1^{*3} = JP^2$ and $x_2^{*3} = J^2 P$. From these values of x_1^* and x_2^*, it follows that, similarly to Case 1 detailed in the previous subsection, the constraint C_2 is not feasible which discards Case 2.

D. Case 3: One active constraint $C_2(x) = 0$

When the only active constraint is C_2, (10) gives $\lambda_i^* = 0$ for all $i \neq 2$, and x_1^*, x_2^* and λ_2^* can then be determined from:

$$\begin{cases} \nabla_x \mathscr{L}(x^*,\lambda^*) = \nabla_x f(x^*) - \lambda_2^* \nabla_x C_2(x^*) = 0 \\ C_2(x^*) = \dfrac{J\left(Px_2^* - x_1^{*2}\right) - x_1^* x_2^{*2}}{Jx_1^* x_2^*} = 0 \end{cases} \tag{23}$$

which, from (13) and (15), is equivalent to:

$$\begin{cases} x_1^{*2} - Px_2^* + \lambda_2^* \left(Px_2^* + x_1^{*2}\right) = 0 \\ x_2^{*2} - Jx_1^* - \lambda_2^* \left(Jx_1^* - x_2^{*2}\right) = 0 \\ J\left(Px_2^* - x_1^{*2}\right) - x_1^* x_2^{*2} = 0 \end{cases} \tag{24}$$

The second equation of this system can be written $\left(x_2^{*2} - Jx_1^*\right)(1 + \lambda_2^*) = 0$ which implies $x_2^{*2} = Jx_1^*$ since $\lambda_2^* \geq 0$ according to (9). Then, the third equation of (24) gives $x_2^{*3} = \left(J^2 P\right)/2$, which in turn leads to $x_1^{*3} = \left(JP^2\right)/4$ and $\lambda_2^* = 1/2$.

Since $\lambda_2^* = 1/2 > 0$ and $\lambda_i^* = 0$ for all $i \neq 2$, (9) is true. For the first order necessary conditions to be all true, it remains to verify that (8) is satisfied for $i = 1$, 2 and 3, i.e., $C_1(x^*) \geq 0$, $C_3(x^*) \geq 0$ and $C_4(x^*) \geq 0$.

$C_1(x^*) \geq 0$ is equivalent to $Jx_1^* - x_2^{*2} \geq 0$ which is true since $x_2^{*2} = Jx_1^*$. Finally, since $x_1^{*3} = \left(JP^2\right)/4$ and $x_2^{*3} = \left(J^2P\right)/2$, $C_3(x^*) \geq 0$ and $C_4(x^*) \geq 0$ yield the two following conditions on J, P, $x_{1max} = V_{max}$ and $x_{2max} = A_{max}$:

$$x_1^* \leq x_{1max} \Longleftrightarrow JP^2 \leq 4V_{max}^3 \tag{25}$$

$$x_2^* \leq x_{2max} \Longleftrightarrow J^2P \leq 2A_{max}^3 \tag{26}$$

To conclude this third case, if the two conditions (25) and (26) are verified, the following vector is a local minimum candidate:

$$x^* = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ \sqrt[3]{\dfrac{J^2P}{2}} \end{bmatrix} \tag{27}$$

E. Case 4: One active constraint $C_3(x) = 0$

When the only active constraint is C_3, (10) gives $\lambda_i^* = 0$ for all $i \neq 3$ and $C_3(x^*) = x_{1max} - x_1^* = 0$ gives $x_1^* = x_{1max} = V_{max}$. Moreover, (7) is $\nabla_x f(x^*) - \lambda_3^* \nabla_x C_3(x^*) = 0$, i.e.:

$$\begin{cases} x_1^{*2} - Px_2^* + \lambda_3^* x_1^{*2} x_2 = 0 \\ x_2^{*2} = Jx_1^* \end{cases} \tag{28}$$

With $x_1^* = V_{max}$ and the second equation of (28), after some calculations, the first equation of (28) leads to:

$$\lambda_3^* = \frac{P\sqrt{J} - \sqrt{V_{max}}^3}{V_{max}^2\sqrt{J}} \tag{29}$$

Then, since $\lambda_3^* \geq 0$ according to (9), the following condition must hold:

$$\sqrt{V_{max}}^3 \leq P\sqrt{J} \iff V_{max}^3 \leq JP^2 \tag{30}$$

For the first order necessary conditions to be true, it remains to verify that (8) is satisfied for $i = 1$, 2 and 4, i.e., $C_1(x^*) \geq 0$, $C_2(x^*) \geq 0$ and $C_4(x^*) \geq 0$. First, since $x_2^{*2} = Jx_1^*$, we have $C_1(x^*) = 0$ so that $C_1(x^*) \geq 0$, and $C_2(x^*) \geq 0$ is equivalent to $x_1^{*3} \leq JP^2/4$, i.e., from $x_1^* = V_{max}$, the following condition is obtained:

$$V_{max}^3 \leq \frac{JP^2}{4} \tag{31}$$

Note that condition (31) implies condition (30) so that only (31) is to be retained. Finally, $C_4(x^*) \geq 0$, $x_2^{*2} = Jx_1^*$ and $x_1^* = V_{max}$ yields the following condition:

$$\sqrt{JV_{max}} \leq A_{max} \tag{32}$$

To summarize this fourth case, if the two conditions (31) and (32) are verified, the following vector is a local minimum candidate:

$$x^* = \begin{bmatrix} V_{max} \\ \sqrt{JV_{max}} \end{bmatrix} \tag{33}$$

F. Case 5: One active constraint $C_4(x) = 0$

When the only active constraint is C_4, (10) gives $\lambda_i^* = 0$ for all $i \neq 4$ and $C_4(x^*) = x_{2max} - x_2^* = 0$ gives $x_2^* = x_{2max} = A_{max}$. Moreover, (7) is $\nabla_x f(x^*) - \lambda_4^* \nabla_x C_4(x^*) = 0$ which is a system of two equations. The first equation of this system yields $x_1^{*2} = Px_2^*$. Then, according to (19), $C_2(x^*) \geq 0$ is equivalent to $-x_1^* x_2^{*2} \geq 0$ which is impossible since $x_1^* > 0$ and $x_2^* > 0$. In conclusion, this fifth case is not feasible and is thus discarded.

G. Case 6: Two active constraints $C_1(x) = C_2(x) = 0$

When only C_1 and C_2 are active, $C_3(x^*) > 0$ and $C_4(x^*) > 0$ imply with (10) that $\lambda_3^* = 0$ and $\lambda_4^* = 0$. Then, x_1^*, x_2^*, λ_1^* and λ_2^* have to be determined. The determination of λ_1^* and λ_2^* shall be done to verify that (9) is satisfied.

First, the x_1^* and x_2^* are calculated from the following system of two equations obtained from $C_1(x^*) = C_2(x^*) = 0$:

$$\begin{cases} Jx_1^* - x_2^{*2} = 0 \\ J\left(Px_2^* - x_1^{*2}\right) - x_1^* x_2^{*2} = 0 \end{cases} \tag{34}$$

whose solution is:

$$x^* = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ \sqrt[3]{\dfrac{J^2P}{2}} \end{bmatrix} \tag{35}$$

Then, Eq. (7) with $\lambda_3^* = 0$ and $\lambda_4^* = 0$ is used to determine λ_1^* and λ_2^*:

$$\begin{aligned} & \nabla_x f(x^*) - \lambda_1^* \nabla_x C_1(x^*) - \lambda_2^* \nabla_x C_2(x^*) = 0 \\ \Longleftrightarrow \; & \begin{cases} x_1^{*2} - Px_2^* - \lambda_1^* x_1^{*2} + \lambda_2^* \left(Px_2^* + x_1^{*2}\right) = 0 \\ x_2^{*2} - Jx_1^* + \lambda_1^* \left(x_2^{*2} + Jx_1^*\right) - \lambda_2^* \left(Jx_1^* - x_2^{*2}\right) = 0 \end{cases} \\ \Longleftrightarrow \; & \begin{bmatrix} -x_1^{*2} & Px_2^* + x_1^{*2} \\ x_2^{*2} + Jx_1^* & x_2^{*2} - Jx_1^* \end{bmatrix} \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \end{bmatrix} = \begin{bmatrix} Px_2^* - x_1^{*2} \\ Jx_1^* - x_2^{*2} \end{bmatrix} \end{aligned}$$

With the expressions of x^* in (35), the determinant of the matrix in the last equation is equal to $\left(-3J^2P^2\right)/2$. and the equation system possesses a unique solution. Once solved, e.g. with Cramer's rule, it yields $\lambda_1^* = 0$ and $\lambda_2^* = 1/3$ which verify (9).

It remains to verify (8), i.e., $C_3(x^*) \geq 0$ and $C_4(x^*) \geq 0$ which leads to the two following conditions:

$$JP^2 \leq 4V_{max}^3 \quad \text{and} \quad J^2P \leq 2A_{max}^3 \tag{36}$$

To conclude on this sixth case, if conditions (36) are true, x^* in (35) is a local minimum candidate. Moreover, it appears that Case 6 is equivalent to Case 3 since x^* in (35) is the same as x^* in (27) and conditions (36) are the same as those in (25) and (26) (which is a consequence of x^* being the same).

H. Case 7: Two active constraints $C_1(x) = C_3(x) = 0$

$C_1(x^*) = 0$ and $C_3(x^*) = 0$ give:

$$x^* = \begin{bmatrix} V_{max} \\ \sqrt{JV_{max}} \end{bmatrix} \tag{37}$$

Moreover, $C_2(x^*) > 0$ and $C_4(x^*) > 0$ imply with (10) that $\lambda_2^* = 0$ and $\lambda_4^* = 0$, and (7) yields the following equation system in λ_1^* and λ_3^*

$$\begin{aligned} & \nabla_x f(x^*) - \lambda_1^* \nabla_x C_1(x^*) - \lambda_3^* \nabla_x C_3(x^*) = 0 \\ \iff & \left\{ \begin{array}{c} x_1^{*2} - P x_2^* - \lambda_1^* x_1^{*2} + \lambda_3^* \left(x_1^{*2} x_2^* \right) = 0 \\ x_2^{*2} - J x_1^* + \lambda_1^* \left(x_2^{*2} + J x_1^* \right) = 0 \end{array} \right. \end{aligned}$$

With (37), the second equation yields $\lambda_1^* = 0$ and then the first equation gives $\lambda_3^* = \left(P x_2^* - x_1^{*2}\right) / \left(x_1^{*2} x_2^*\right)$. With (37), for (9) to be true, i.e. $\lambda_3^* \geq 0$, the following condition must hold:

$$V_{max}^3 \leq J P^2 \tag{38}$$

It remains to verify that (8) is satisfied for $i = 2$ and $i = 4$, i.e., $C_2(x^*) \geq 0$ and $C_4(x^*) \geq 0$, which leads to the two following conditions:

$$V_{max}^3 \leq \frac{J P^2}{4} \quad \text{and} \quad \sqrt{J V_{max}} \leq A_{max} \tag{39}$$

where the first condition in (39) is stronger than (38), i.e., (39) implies (38).

To conclude on case 7, if conditions (39) are true, x^* in (37) is a local minimum candidate. Moreover, it turns out that Case 7 is equivalent to Case 4 since x^* in (37) is the same as x^* in (33) and conditions (39) are the same as those in (31) and (32) (which is a consequence of x^* being the same).

I. Case 8: Two active constraints $C_1(x) = C_4(x) = 0$

$C_1(x^*) = 0$ and $C_4(x^*) = 0$ give:

$$x^* = \left[\begin{array}{c} \frac{A_{max}^2}{J} \\ A_{max} \end{array} \right] \tag{40}$$

Moreover, $C_2(x^*) > 0$ and $C_3(x^*) > 0$ imply with (10) that $\lambda_2^* = 0$ and $\lambda_3^* = 0$, and (7) yields the following equation system in λ_1^* and λ_4^*:

$$\begin{aligned} & \nabla_x f(x^*) - \lambda_1^* \nabla_x C_1(x^*) - \lambda_4^* \nabla_x C_4(x^*) = 0 \\ \iff & \left\{ \begin{array}{c} x_1^{*2} - P x_2^* - \lambda_1^* x_1^{*2} = 0 \\ x_2^{*2} - J x_1^* + \lambda_1^* \left(x_2^{*2} + J x_1^* \right) + \lambda_4^* J x_2^{*2} = 0 \end{array} \right. \end{aligned}$$

From the first equation and (40), we have:

$$\lambda_1^* = \frac{x_1^{*2} - P x_2^*}{x_1^{*2}} = 1 - \frac{J^2 P}{A_{max}^3} \tag{41}$$

and, from the second equation:

$$\lambda_4^* = \frac{J x_1^* - x_2^{*2} - \lambda_1^* \left(x_2^{*2} + J x_1^* \right)}{J x_2^{*2}} = -\frac{2}{J} \lambda_1^* \tag{42}$$

since, according to (40), $J x_1^* - x_2^{*2} = 0$ and $x_2^{*2} + J x_1^* = 2 x_2^{*2}$. Eq. (42) and (9) imply that $\lambda_1^* = \lambda_4^* = 0$ so that (41) gives:

$$J^2 P = A_{max}^3 \tag{43}$$

However, $C_2(x^*) \geq 0$, which must hold true according to (8), leads to:

$$\begin{aligned} & J \left(P x_2^* - x_1^{*2} \right) - x_1^* x_2^{*2} \geq 0 \\ \implies & J P A_{max} - 2 \frac{A_{max}^4}{J} \geq 0 \\ \implies & \frac{J^2 P}{2} \geq A_{max}^3 \end{aligned}$$

which is impossible in view of (43). In conclusion, Case 8 is impossible and thus discarded.

J. Case 9: Two active constraints $C_2(x) = C_3(x) = 0$

$C_3(x^*) = 0$ means that $x_1^* = V_{max}$ while $C_2(x^*) = 0$ yields a second-order polynomial equation in x_2^*. The latter equation can be solved to obtain two solutions for x_2^*, but it is better to first consider (7). Since, with (10), $C_1(x^*) > 0$ and $C_4(x^*) > 0$ imply that $\lambda_1^* = 0$ and $\lambda_4^* = 0$, (7) yields:

$$\begin{aligned} & \nabla_x f(x^*) - \lambda_2^* \nabla_x C_2(x^*) - \lambda_3^* \nabla_x C_3(x^*) = 0 \\ \iff & \left\{ \begin{array}{c} x_1^{*2} - P x_2^* + \lambda_2^* \left(x_1^{*2} + P x_2^* \right) + \lambda_3^* \left(x_1^{*2} x_2^* \right) = 0 \\ x_2^{*2} - J x_1^* + \lambda_2^* \left(x_2^{*2} - J x_1^* \right) = 0 \end{array} \right. \end{aligned} \tag{44}$$

The second equation implies that either $x_2^{*2} = J x_1^*$ or $\lambda_2^* = -1$. Since $\lambda_2^* \geq 0$ according to (9), $\lambda_2^* = -1$ is not possible so that $x_2^{*2} = J x_1^*$, i.e., $x_2^* = \sqrt{J V_{max}}$. Then, with $x_1^* = V_{max}$ and $x_2^* = \sqrt{J V_{max}}$, $C_2(x^*) = 0$ is equivalent to:

$$V_{max}^3 = \frac{J P^2}{4} \tag{45}$$

Going back to the equation system (44), note that in the second equation, since $x_2^{*2} = J x_1^*$, λ_2^* is undetermined so that any $\lambda_2^* \geq 0$ satisfies this equation and (9) as well. Now, let us verify if there exist $\lambda_2^* \geq 0$ and $\lambda_3^* \geq 0$ such that the first equation of (44) is verified. With $x_1^* = V_{max}$ and $x_2^* = \sqrt{J V_{max}}$, this first equation is equivalent to:

$$\lambda_3^* = \frac{P \sqrt{J V_{max}} - V_{max}^2 - \lambda_2^* \left(P \sqrt{J V_{max}} + V_{max}^2 \right)}{V_{max}^2 \sqrt{J V_{max}}} \tag{46}$$

so that $\lambda_3^* \geq 0$ if and only if:

$$\lambda_2^* \leq \frac{P \sqrt{J V_{max}} - V_{max}^2}{P \sqrt{J V_{max}} + V_{max}^2} \tag{47}$$

Since λ_2^* must be non-negative, (47) is possible if and only if:

$$P \sqrt{J V_{max}} - V_{max}^2 \geq 0 \iff J P^2 \geq V_{max}^3 \tag{48}$$

which, according to (45), is true. Hence, taking λ_2^* equal to the right-hand side of the inequality (47), we have $\lambda_2^* \geq 0$ and λ_3^* in (46) is also non-negative which proves that there exist $\lambda_2^* \geq 0$ and $\lambda_3^* \geq 0$ such that (44) is verified.

Finally, the inequalities $C_1(x^*) \geq 0$ and $C_4(x^*) \geq 0$ should be verified for (8) to be true. First, we have $C_1(x^*) = 0$ since $x_2^{*2} = J x_1^*$. Second, $C_4(x^*) \geq 0$ is $x_2^* \leq A_{max}$ which yields:

$$\sqrt{J V_{max}} \leq A_{max} \tag{49}$$

To conclude Case 9, if conditions (45) and (49) are verified, the following vector is a local minimum candidate:

$$x^* = \left[\begin{array}{c} V_{max} \\ \sqrt{J V_{max}} \end{array} \right] \tag{50}$$

Comparing (45) with (31), (49) with (32) and (50) with (33), Case 9 appears to be a particular case of Case 4 where (31) is verified as an equality.

K. Case 10: Two active constraints $C_2(x) = C_4(x) = 0$

$C_4(x^*) = x_{2max} - x_2^* = 0$ gives $x_2^* = x_{2max} = A_{max}$ while $C_2(x^*) = 0$ yields the following quadratic equation in x_1^*:

$$Jx_1^{*2} + x_2^{*2} x_1^* - JPx_2^* = 0 \tag{51}$$

The discriminant of this equation is:

$$\Delta = x_2^* \left(x_2^{*3} + 4J^2 P \right) > 0 \tag{52}$$

and (51) possesses the following two solutions:

$$x_1^* = \frac{-x_2^{*2} \pm \sqrt{\Delta}}{2J} \tag{53}$$

Since $x_1^* > 0$ according to (6), the only possible solution is:

$$x_1^* = \frac{-x_2^{*2} + \sqrt{\Delta}}{2J} = \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2 P A_{max}}}{2J} \tag{54}$$

since the other one is negative. Note that x_1^* in (54) is positive because $-x_2^{*2} + \sqrt{\Delta} > 0$, i.e., $x_2^{*4} < \Delta$ which can be deduced from (52) and $4J^2P > 0$.

Now, let us determine the Lagrange multipliers λ_2^* and λ_4^* and establish the conditions for λ_2^* and λ_4^* to be non-negative and thus to satisfy (9). Since, with (10), $C_1(x^*) > 0$ and $C_3(x^*) > 0$ imply that $\lambda_1^* = 0$ and $\lambda_3^* = 0$, (7) yields:

$$\nabla_x f(x^*) - \lambda_2^* \nabla_x C_2(x^*) - \lambda_4^* \nabla_x C_4(x^*) = 0$$
$$\iff \begin{cases} x_1^{*2} - Px_2^* + \lambda_2^* \left(x_1^{*2} + Px_2^* \right) = 0 \\ Jx_1^* - x_2^{*2} + \lambda_2^* \left(Jx_1^* - x_2^{*2} \right) = \lambda_4^* \end{cases} \tag{55}$$

Since $x_1^{*2} + Px_2^* > 0$, we have from the first equation:

$$\lambda_2^* = \frac{Px_2^* - x_1^{*2}}{Px_2^* + x_1^{*2}} \tag{56}$$

With the expression of x_1^* in (54), one can verify that $Px_2^* - x_1^{*2} \geq 0$ is equivalent to $\Delta \geq x_2^{*4}$ which is true according to (52) and $4J^2P > 0$. Consequently, λ_2^* in (56) is non-negative. Besides, from the second equation of (55), we have:

$$\lambda_4^* = \left(Jx_1^* - x_2^{*2} \right) \left(1 + \lambda_2^* \right) \tag{57}$$

Since $\lambda_2^* \geq 0$, $\lambda_4^* \geq 0$ is equivalent to $Jx_1^* - x_2^{*2} \geq 0$. With (54) and (52), the latter inequality is equivalent to $x_2^{*3} \leq \dfrac{J^2P}{2}$. Hence, $\lambda_4^* \geq 0$ leads to the following condition:

$$A_{max}^3 \leq \frac{J^2 P}{2} \tag{58}$$

Finally, the inequalities $C_1(x^*) \geq 0$ and $C_3(x^*) \geq 0$ should be verified for (8) to be true. $C_1(x^*) \geq 0$ is equivalent to $Jx_1^* - x_2^{*2} \geq 0$ i.e. to $\lambda_4^* \geq 0$ which is verified if (58) is true. $C_3(x^*) \geq 0$ is:

$$x_1^* = \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2 P A_{max}}}{2J} \leq V_{max} \tag{59}$$

To conclude Case 10, if conditions (58) and (59) are verified, the following vector is a local minimum candidate:

$$x^* = \begin{bmatrix} \dfrac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2 P A_{max}}}{2J} \\ A_{max} \end{bmatrix} \tag{60}$$

L. Case 11: Two active constraints $C_3(x) = C_4(x) = 0$

In this last case, the two active constraints $C_3(x) = C_4(x) = 0$ yield directly the following local minimum candidate:

$$x^* = \begin{bmatrix} V_{max} \\ A_{max} \end{bmatrix} \tag{61}$$

Since, with (10), $C_1(x^*) > 0$ and $C_2(x^*) > 0$ imply that $\lambda_1^* = 0$ and $\lambda_2^* = 0$, (7) gives:

$$\nabla_x f(x^*) - \lambda_3^* \nabla_x C_3(x^*) - \lambda_4^* \nabla_x C_4(x^*) = 0$$
$$\iff \begin{cases} \lambda_3^* = \dfrac{Px_2^* - x_1^{*2}}{x_1^{*2} x_2^*} \\ \lambda_4^* = \dfrac{Jx_1^* - x_2^{*2}}{Jx_2^{*2}} \end{cases} \tag{62}$$

so that, with (61), the following equality should hold for (9) to be true:

$$\lambda_3^* \geq 0 \iff Px_2^* - x_1^{*2} \geq 0 \iff PA_{max} \geq V_{max}^2 \tag{63}$$

and also:

$$\lambda_4^* \geq 0 \iff Jx_1^* - x_2^{*2} \geq 0 \iff JV_{max} \geq A_{max}^2 \tag{64}$$

Finally, the inequalities $C_1(x^*) \geq 0$ and $C_2(x^*) \geq 0$ should be verified for (8) to hold. $C_1(x^*) \geq 0$ is equivalent to $Jx_1^* - x_2^{*2} \geq 0$, i.e., to (64). $C_2(x^*) \geq 0$ is equivalent to:

$$J \left(Px_2^* - x_1^{*2} \right) - x_1^* x_2^{*2} \geq 0 \iff JPA_{max} - JV_{max}^2 \geq V_{max} A_{max}^2 \tag{65}$$

In summary, if the inequalities (63), (64) and (65) are satisfied, x^* in (61) is a local minimum candidate.

M. Synthesis of the 11 cases and Algorithm 2

Let us now summarize all the 11 cases:

- Cases 1, 2, 5 and 8 are impossible in the sense that they do not verify the KKT first-order necessary conditions.
- Case 6 is equivalent to Case 3 and Case 7 is equivalent to Case 4.
- Case 9 is a particular case of Case 4.

Hence, only four cases, namely Cases 3, 4, 10 and 11, need to be considered to find (local) solutions $x^* = (x_1^*, x_2^*)$ to the optimization problem (6).

In order to ease the analysis of the relationships between these cases, the conditions to be fulfilled for each one of these four cases are summarized below.

Conditions for Case 3:

$$JP^2 \leq 4V_{max}^3 \tag{66}$$

$$J^2 P \leq 2A_{max}^3 \tag{67}$$

Conditions for Case 4:

$$4V_{max}^3 \leq JP^2 \tag{68}$$

$$JV_{max} \leq A_{max}^2 \tag{69}$$

Conditions for Case 10:

$$2A_{max}^3 \leq J^2 P \tag{70}$$

$$JPA_{max} - JV_{max}^2 \leq V_{max}A_{max}^2 \quad (71)$$

where (71) is equivalent (59) as can be shown with some elementary calculations. Moreover, again after some calculations, it turns out that (70) and (71) imply that:

$$A_{max}^2 \leq JV_{max} \quad (72)$$

Conditions for Case 11:

$$V_{max}^2 \leq PA_{max} \quad (73)$$

$$A_{max}^2 \leq JV_{max} \quad (74)$$

$$V_{max}A_{max}^2 \leq JPA_{max} - JV_{max}^2 \quad (75)$$

Note that, from (74) and (75), we have:

$$2A_{max}^3 \leq J^2P \quad (76)$$

Indeed, the sum of (76) and (74) multiplied by V_{max} gives $2V_{max}A_{max} \leq JP$. Using the latter inequality and multiplying (74) by $2A_{max}$ lead to (76).

Carefully analyzing all these conditions leads to the following relationships between the four remaining cases (Cases 3, 4, 10 and 11) and in turn to Algorithm 1.

Let us first assume that condition (67) of Case 3 is satisfied as a strict inequality. Then, Cases 10 and 11 are not possible because of (70) and (76), respectively, and there are two possibles cases:

- If (66) of Case 3 is verified (as a strict inequality), Case 4 is not possible because of (68). The only possible case is Case 3 which means that the sole local minimum candidate is x^* in (27) which corresponds to line 3 of Algorithm 1.
- If (66) of Case 3 is not verified, Case 3 is not possible and condition (68) of Case 4 is verified. Case 4 is then the only possible case provided that condition (69) is true. It turns out that (67) and (68) imply that condition (69) is verified. Indeed, (67) is equivalent to:

$$\sqrt[3]{\frac{J^4P^2}{4}} \leq A_{max}^2 \quad (77)$$

(68) is equivalent to:

$$\sqrt[3]{\frac{4}{JP^2}} \leq \frac{1}{V_{max}} \quad (78)$$

and (77) and (78) imply that:

$$\sqrt[3]{J^3} \leq \frac{A_{max}^2}{V_{max}} \Longleftrightarrow JV_{max} \leq A_{max}^2 \quad (79)$$

which is (69). Hence, Case 4 is the only possible case meaning that the sole local minimum candidate is x^* in (33) which corresponds to line 5 of Algorithm 1.

Let us now assume that condition (67) of Case 3 is not satisfied. Case 3 is then not possible and:

- If (69) is satisfied (as a strict inequality), Cases 10 and 11 are not possible because of (72) and (74), respectively. Moreover, since (67) is not satisfied, we have:

$$2A_{max}^3 < J^2P \Longleftrightarrow A_{max} < \sqrt[3]{\frac{J^2P}{2}} \quad (80)$$

and using the latter in (69) (satisfied as a strict inequality) yields:

$$J < \frac{A_{max}^2}{V_{max}} < \frac{1}{V_{max}}\sqrt[3]{\frac{J^4P^2}{4}} \quad (81)$$

which implies that:

$$V_{max} < \sqrt[3]{\frac{JP^2}{4}} \Longleftrightarrow V_{max}^3 < \frac{JP^2}{4} \quad (82)$$

i.e. (68) is satisfied. The two conditions for Case 4 are then satisfied and Case 4 is the only possible case meaning that the unique local minimum candidate is x^* in (33) which corresponds to line 9 of Algorithm 1.
- If (69) is not satisfied, Case 4 is not possible and the only possible cases are Cases 10 and 11. Let us then consider the two following complementary situations:
  - If (71) is satisfied (as a strict inequality), Case 11 is not possible because of (75). Case 10 is possible since (70) is true because (67) is not satisfied. Case 10 is then the only possible case and the unique local minimum candidate is x^* in (60) which corresponds to lines 11 and 12 of Algorithm 1.
  - If (71) is not satisfied, Case 10 is not possible but Case 11 is then feasible. Indeed (75) is true since (71) is not satisfied, (74) is true since (69) is not satisfied, and (73) is true since it turns out to be implied by (75). The latter result comes from the fact that (75) is equivalent to:

$$V_{max}A_{max}^2 + JV_{max}^2 \leq JPA_{max} \quad (83)$$

which implies that:

$$JV_{max}^2 \leq JPA_{max} \Longleftrightarrow V_{max}^2 \leq PA_{max} \quad (84)$$

the latter inequality being (73). Hence, Case 11 is the only possible case and the unique local minimum candidate is x^* in (61) which corresponds to lines 11 and 14 of Algorithm 1.

Algorithm 2 summarizes the above analysis. It is exactly the same as Algorithm 1 which proves that the latter is correct. Note that line 12 of Algorithm 2 corresponds to lines 11 and 12 of Algorithm 1 as can be seen from the expression of x^* in (60). Moreover, line 14 of Algorithm 2 corresponds to lines 11 and 14 of Algorithm 1 since, with $V = \frac{-A_{max}^2+\sqrt{A_{max}^4+4J^2PA_{max}}}{2J}$ as calculated at line 12 of Algorithm 1, the condition $V > V_{max}$ at line 13 of Algorithm 1 means that (59) is not satisfied. Hence, since (59) and (71) are equivalent (as can be shown with some elementary calculations), (71) is also not satisfied which corresponds to the condition for line 14 of Algorithm 2.

It is important to note that, according to the above analysis, Algorithm 1 and Algorithm 2 computes in fact the global minimum of the minimization problem (6).

N. Particular cases

For the proof to be complete, it remains to be shown that Algorithm 2, and hence Algorithm 1, is correct in a number of

Algorithm 2 Minimum-Time S-Curve Trajectory Rewritten

1: **if** (67) **then**
2:   **if** (66) **then**
3:     x^* in (27) % Case 3
4:   **else**
5:     x^* in (33) % Case 4
6:   **end if**
7: **else**
8:   **if** (69) **then**
9:     x^* in (33) % Case 4
10:   **else**
11:     **if** (71) **then**
12:       x^* in (60) % Case 10
13:     **else**
14:       x^* in (61) % Case 11
15:     **end if**
16:   **end if**
17: **end if**

particular cases. The latter were overlooked at various places in the proof presented in Section III-M where conditions (66), (67), (69) and (71) were supposed to be verified as strict inequalities. In fact, the particular cases to be considered to complete the proof are the following ones:

- (67) and (66) satisfied as equalities;
- (67) satisfied as an equality and (66) satisfied as a strict inequality;
- (67) satisfied as an equality and (66) not satisfied;
- (67) not satisfied and (69) satisfied as an equality;
- (67) and (69) not satisfied and (71) satisfied as an equality.

First particular case: (67) *and* (66) *satisfied as equalities*

$$J^2P = 2A_{max}^3 \tag{85}$$

$$JP^2 = 4V_{max}^3 \tag{86}$$

These two equalities imply that (27) (line 3 of Algorithm 2) is:

$$x^* = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ \sqrt[3]{\dfrac{J^2P}{2}} \end{bmatrix} = \begin{bmatrix} V_{max} \\ A_{max} \end{bmatrix} \tag{87}$$

Now, let us consider x^* in (33) (line 5 of Algorithm 2):

$$x^* = \begin{bmatrix} V_{max} \\ \sqrt{JV_{max}} \end{bmatrix} = \begin{bmatrix} V_{max} \\ A_{max} \end{bmatrix} \tag{88}$$

To show that the second equality in the previous equation holds, consider $P = \dfrac{2A_{max}^3}{J^2}$ from (85) and $P^2 = \dfrac{4V_{max}^3}{J}$ from (86). Then, we have:

$$\frac{4A_{max}^6}{J^4} = \frac{4V_{max}^3}{J} \iff A_{max} = \sqrt{JV_{max}} \tag{89}$$

Next, (60) (line 12 of Algorithm 2) is:

$$x^* = \begin{bmatrix} \dfrac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \\ A_{max} \end{bmatrix} = \begin{bmatrix} V_{max} \\ A_{max} \end{bmatrix} \tag{90}$$

since, with $J^2P = 2A_{max}^3$ in (85):

$$\begin{aligned} \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} &= \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 8A_{max}^4}}{2J} \\ &= \frac{-A_{max}^2 + \sqrt{9A_{max}^4}}{2J} \\ &= \frac{A_{max}^2}{J} = V_{max} \end{aligned} \tag{91}$$

where the last equality is obtained from (89).

Hence, in this first particular case, it turns out that x^* in (27), (33) and (60) are all equal to x^* in (61), i.e, all possible local minima are equal. Then, Algorithm 2 necessarily computes the optimal value $x^* = [V_{max}, A_{max}]^T$ at line 3.

Second particular case: (67) *satisfied as an equality and* (66) *satisfied as a strict inequality*

$$J^2P = 2A_{max}^3 \tag{92}$$

$$JP^2 < 4V_{max}^3 \tag{93}$$

First, note that Case 3 is possible since both (67) and (66) are satisfied and Case 4 is not possible since (68) is not satisfied. Moreover, Algorithm 2 computes x^* at line 3 according to (27):

$$x^* = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ \sqrt[3]{\dfrac{J^2P}{2}} \end{bmatrix} = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ A_{max} \end{bmatrix} \tag{94}$$

where $\sqrt[3]{\dfrac{JP^2}{4}} < V_{max}$ since $JP^2 < 4V_{max}^3$.

Now, let us check whether or not Cases 10 and 11 are possible and, if they are, verify that x^* in (94) also corresponds to their minimum.

The first condition for Case 10 to be possible is (70). This condition is verified since $J^2P = 2A_{max}^3$. The second condition for Case 10 is (71) which can equivalently be written as:

$$V_{max}A_{max}^2 + JV_{max}^2 \geq JPA_{max} \tag{95}$$

Since $V_{max} > \sqrt[3]{\dfrac{JP^2}{4}}$ and $J^2P = 2A_{max}^3$, one can write:

$$\begin{aligned} V_{max}A_{max}^2 + JV_{max}^2 &> A_{max}^2\sqrt[3]{\frac{JP^2}{4}} + J\sqrt[3]{\frac{J^2P^4}{16}} \\ &= \sqrt[3]{\frac{J^4P^2}{4}}\sqrt[3]{\frac{JP^2}{4}} + J\sqrt[3]{\frac{J^2P^4}{16}} \\ &= \sqrt[3]{\frac{J^5P^4}{16}} + \sqrt[3]{\frac{J^5P^4}{16}} \\ &= 2\sqrt[3]{\frac{J^5P^4}{16}} = \sqrt[3]{\frac{J^5P^4}{2}} = JP\sqrt[3]{\frac{J^2P}{2}} \end{aligned}$$

Hence, $V_{max}A_{max}^2 + JV_{max}^2 > JP\sqrt[3]{\frac{J^2P}{2}} = JPA_{max}$, i.e., (95) is verified as a strict inequality so that (71) is also verified as a strict inequality and Case 10 is possible. Furthermore, the minimum x^* in Case 10 is given in (60):

$$x^* = \left[\begin{array}{c} \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \\ A_{max} \end{array}\right] = \left[\begin{array}{c} \frac{A_{max}^2}{J} \\ A_{max} \end{array}\right] \tag{96}$$

where the second equality is from (91). Since $J^2P = 2A_{max}^3$, we have:

$$\sqrt[3]{\frac{JP^2}{4}} = \sqrt[3]{\frac{J}{4}\frac{4A_{max}^6}{J^4}} = \frac{A_{max}^2}{J} \tag{97}$$

so that the minimum x^* of Case 10 is equal to the minimum x^* of Case 3 given in (94).

Finally, Case 11 is not possible because (71) is verified as a strict inequality and, hence, (75) is not verified.

Summarizing, only Cases 3 and 10 are possible, the minimum x^* of these two cases are equal and Algorithm 2 indeed computes this x^* at line 3.

Third particular case: (67) *satisfied as an equality and* (66) *not satisfied*

$$J^2P = 2A_{max}^3 \tag{98}$$

$$JP^2 > 4V_{max}^3 \tag{99}$$

In this particular case, Case 3 is not possible since (66) is not satisfied and Algorithm 2 computes x^* at line 5 according to (33):

$$x^* = \left[\begin{array}{c} V_{max} \\ \sqrt{JV_{max}} \end{array}\right] \tag{100}$$

First, note that Case 4 is possible since (68) is true since $J^2P = 2A_{max}^3$. Moreover, (69) is also verified since:

$$J^2P = 2A_{max}^3 \Longleftrightarrow A_{max}^2 = \sqrt[3]{\frac{J^4P^2}{4}} = J\sqrt[3]{\frac{JP^2}{4}} \tag{101}$$

and:

$$JP^2 > 4V_{max}^3 \Longleftrightarrow V_{max} < \sqrt[3]{\frac{JP^2}{4}} \tag{102}$$

imply that:

$$JV_{max} < J\sqrt[3]{\frac{JP^2}{4}} = A_{max}^2 \tag{103}$$

Furthermore, let us examine Cases 10 and 11. Case 10 is not feasible since (71) is not verified. Indeed, in the previous paricular case, it was shown that $J^2P = 2A_{max}^3$ and $JP^2 < 4V_{max}^3$ leads to (71) being verified as a strict inequality, i.e., $JPA_{max} - JV_{max}^2 < V_{max}A_{max}^2$. Then, the same reasoning allows to conclude that $J^2P = 2A_{max}^3$ and $JP^2 > 4V_{max}^3$ imply $JPA_{max} - JV_{max}^2 > V_{max}A_{max}^2$ i.e. (71) is not verified in the present particular case. Besides, Case 11 is also not feasible since $A_{max}^2 > JV_{max}$ from (103) so that (74) is not true.

Hence, only Case 4 is possible and the corresponding minimum x^* is indeed computed at line 5 of Algorithm 2.

Fourth particular case: (67) *not satisfied and* (69) *satisfied as an equality*

$$J^2P > 2A_{max}^3 \tag{104}$$

$$JV_{max} = A_{max}^2 \tag{105}$$

First, note that Case 3 is not possible since (67) is not satisfied and that Algorithm 2 computes x^* at line 9 according to (33):

$$x^* = \left[\begin{array}{c} V_{max} \\ \sqrt{JV_{max}} \end{array}\right] = \left[\begin{array}{c} V_{max} \\ A_{max} \end{array}\right] \tag{106}$$

which is also the minimum x^* of Case 11 as given in (61) and where the second equality in (106) comes from (105). Then, let us check whether or not Cases 4, 10 and 11 are possible.

Case 4 is possible since (69) is truce according to (105). Moreover, (104) and (105) imply that:

$$4V_{max}^3 = 4\frac{A_{max}^6}{J^3} < \frac{4}{J^3}\frac{J^4P^2}{4} = JP^2 \tag{107}$$

so that (68) is satisfied.

Case 10 is not possible since (71) is not satisfied. Indeed, with (105), we have:

$$V_{max}A_{max}^2 + JV_{max}^2 = \frac{2A_{max}^4}{J} \tag{108}$$

and (104) then implies that:

$$\frac{2A_{max}^4}{J} < J^2P\frac{A_{max}}{J} = JPA_{max} \tag{109}$$

so that:

$$V_{max}A_{max}^2 + JV_{max}^2 < JPA_{max} \Longleftrightarrow JPA_{max} - JV_{max}^2 > V_{max}A_{max}^2 \tag{110}$$

Case 11 turns out to be possible since (110) shows that (75) is true and (74) is satisfied according to (105). Moreover, (104) and (105) imply that:

$$V_{max}^2 = \frac{A_{max}^4}{J^2} < \frac{A_{max}}{J^2}\frac{J^2P}{2} = \frac{A_{max}P}{2} < PA_{max} \tag{111}$$

i.e. (73) is satisfied.

In conclusion of this fourth particular case, only Cases 4 and 11 are possible, their minimum x^* are equal and indeed computed by Algorithm 2 at line 9.

Fifth particular case: (67) *and* (69) *not satisfied and* (71) *satisfied as an equality*

$$J^2P > 2A_{max}^3 \tag{112}$$

$$JV_{max} > A_{max}^2 \tag{113}$$

$$JPA_{max} - JV_{max}^2 = V_{max}A_{max}^2 \tag{114}$$

Cases 3 and 4 are then not possible since (67) and (69) are not satisfied, respectively.

Case 10 is possible since (112) implies (70) and (71) is true according to (114).

Then, about the feasibility of Case 11, it is not straightforward to verify whether or not (73) is true from (112), (113) and (114). Fortunately, this verification is in fact not required. Indeed, in the present particular case, Algorithm 2 computes

x^* of Case 10 at line 12 and, as proved below, this x^* is equal to the one of Case 11, $x^* = [V_{max}, A_{max}]^T$. Hence, if Case 11 is not possible, Algorithm 2 computes x^* of Case 10 which is the only possible minimum and, if Case 11 is possible, x^* of Cases 10 and 11 are equal and Algorithm 2 indeed computes this minimum at line 12.

It remains to prove that the x^* of Case 10, which according to (60) is:

$$x^* = \begin{bmatrix} \dfrac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \\ A_{max} \end{bmatrix} \tag{115}$$

is equal to the one of Case 11 which is $x^* = [V_{max}, A_{max}]^T$, i.e., to prove that

$$\frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} = V_{max} \tag{116}$$

To this end, let us consider (114) as a quadratic equation in V_{max}:

$$JV_{max}^2 + A_{max}^2 V_{max} - JPA_{max} = 0 \tag{117}$$

whose two solutions are:

$$V_{max} = \frac{-A_{max}^2 \pm \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \tag{118}$$

The only possible solution for V_{max} among those two is the following one since the other one is negative:

$$V_{max} = \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \tag{119}$$

which shows that x^* of Case 10 in (115) is equal to x^* of Case 11.

O. Second-Order Sufficient Conditions

Algorithm 2 (and thus Algorithm 1) is based on the KKT first-order necessary conditions. Hence, the vectors x^* computed in the various cases in Algorithm 2 are local minima candidates and it remains to be verified that these vectors x^* are indeed local minima. To this end, the following second-order sufficient conditions can be used [3].

Second-Order Sufficient Conditions: Suppose that at some feasible vector x^*, there exists a Lagrange multiplier vector λ^* satisfying the KKT conditions and that the Lagrangian Hessian $\nabla_{xx}^2 \mathscr{L}(x^*, \lambda^*)$ is positive definite. Then, x^* is a strict local minimum to the optimization problem (6).

From (12), the Lagrangian Hessian is:

$$\nabla_{xx}^2 \mathscr{L}(x, \lambda) = \nabla_{xx}^2 f(x) - \sum_{i=1}^{4} \lambda_i \nabla_{xx}^2 C_i(x) \tag{120}$$

The individual Hessian matrices appearing in the above expression of $\nabla_{xx}^2 \mathscr{L}$ are obtained from (13) to (17) as follows.

$$\nabla_{xx}^2 f(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2P}{x_1^3} & -\dfrac{1}{x_2^2} \\ -\dfrac{1}{x_2^2} & \dfrac{2x_1}{x_2^3} \end{bmatrix} \tag{121}$$

$$\nabla_{xx}^2 C_1(x) = \begin{bmatrix} 0 & -\dfrac{1}{x_2^2} \\ -\dfrac{1}{x_2^2} & \dfrac{2x_1}{x_2^3} \end{bmatrix} \tag{122}$$

$$\nabla_{xx}^2 C_2(x) = \begin{bmatrix} \dfrac{2P}{x_1^3} & \dfrac{1}{x_2^2} \\ \dfrac{1}{x_2^2} & -\dfrac{2x_1}{x_2^3} \end{bmatrix} \tag{123}$$

$$\nabla_{xx}^2 C_3(x) = 0 \tag{124}$$

$$\nabla_{xx}^2 C_4(x) = 0 \tag{125}$$

Therefore, the Lagrangian Hessian is:

$$\nabla_{xx}^2 \mathscr{L}(x, \lambda) = \begin{bmatrix} (1-\lambda_2)\dfrac{2P}{x_1^3} & (\lambda_1 - \lambda_2 - 1)\dfrac{1}{x_2^2} \\ (\lambda_1 - \lambda_2 - 1)\dfrac{1}{x_2^2} & (1 - \lambda_1 + \lambda_2)\dfrac{2x_1}{x_2^3} \end{bmatrix} \tag{126}$$

Let us now verify the second-order sufficient conditions at the local minimum candidate x^* of Case 3 in (27), Case 4 in (33), Case 10 in (60) and Case 11 in (61).

In Case 3, we have $\lambda_1^* = 0$, $\lambda_2^* = 1/2$, $x_2^{*2} = Jx_1^*$ and:

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \sqrt[3]{\dfrac{JP^2}{4}} \\ \sqrt[3]{\dfrac{J^2P}{2}} \end{bmatrix} \tag{127}$$

so that:

$$\nabla_{xx}^2 \mathscr{L}(x^*, \lambda^*) = \begin{bmatrix} \dfrac{P}{x_1^{*3}} & -\dfrac{3}{2x_2^{*2}} \\ -\dfrac{3}{2x_2^{*2}} & \dfrac{3x_1^*}{x_2^{*3}} \end{bmatrix} \tag{128}$$

According to Sylvester's criterion, since $\dfrac{P}{x_1^{*3}} > 0$, the Hessian $\nabla_{xx}^2 \mathscr{L}(x^*, \lambda^*)$ is positive definite if and only if its determinant is strictly positive. Being given that $x_2^{*2} = Jx_1^*$ and with (127), the determinant is:

$$\begin{aligned} \det\left(\nabla_{xx}^2 \mathscr{L}(x^*, \lambda^*)\right) &= \frac{3P}{x_1^{*2} x_2^{*3}} - \frac{9}{4x_2^{*4}} && (129) \\ &= \frac{3}{x_2^{*4}}\left(\frac{J^2P}{x_2^{*3}} - \frac{3}{4}\right) && (130) \\ &= \frac{3}{x_2^{*4}}\left(\frac{2J^2P}{J^2P} - \frac{3}{4}\right) && (131) \\ &= \frac{3}{x_2^{*4}}\frac{5}{4} > 0 && (132) \end{aligned}$$

which is positive since $x_2^* = \sqrt[3]{\frac{J^2 P}{2}} > 0$. Hence, the second-order sufficient conditions are satisfied and x^* of Case 3, given in (27) and computed at line 3 of Algorithm 2, is indeed a strict local minimum.

In Case 4, we have $\lambda_1^* = \lambda_2^* = 0$, $x_2^{*2} = Jx_1^*$ and:

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} V_{max} \\ \sqrt{JV_{max}} \end{bmatrix} \tag{133}$$

so that:

$$\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*) = \begin{bmatrix} \frac{2P}{x_1^{*3}} & -\frac{1}{x_2^{*2}} \\ -\frac{1}{x_2^{*2}} & \frac{2x_1^*}{x_2^{*3}} \end{bmatrix} \tag{134}$$

whose determinant is:

$$\begin{aligned} \det\left(\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*)\right) &= \frac{4P}{x_1^{*2}x_2^{*3}} - \frac{1}{x_2^{*4}} &(135)\\ &= \frac{4J^2P}{x_2^{*4}x_2^{*3}} - \frac{1}{x_2^{*4}} &(136)\\ &= \frac{1}{x_2^{*4}}\left(\frac{4J^2P}{x_2^{*3}} - 1\right) &(137)\\ &= \frac{1}{x_2^{*4}}\left(\frac{4J^2P}{\sqrt{J^3V_{max}^3}} - 1\right) &(138)\end{aligned}$$

Since $x_2^* > 0$, this determinant is positive if $4J^2P > \sqrt{J^3V_{max}^3}$ and hence if $16JP^2 > V_{max}^3$. The latter inequality is true since it is a consequence of (68):

$$JP^2 \geq 4V_{max}^3 \Longleftrightarrow 16JP^2 \geq 64V_{max}^3 \tag{139}$$

Hence, $\det\left(\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*)\right) > 0$ and since $\frac{2P}{x_1^{*3}} > 0$, Sylvester's criterion implies that $\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*)$ is positive definite. The second-order sufficient conditions are thus satisfied and x^* of Case 4, given in (33) and computed at lines 5 and 9 of Algorithm 2, is a strict local minimum.

In Case 10, we have $\lambda_1^* = 0$ and:

$$\lambda_2^* = \frac{Px_2^* - x_1^{*2}}{Px_2^* + x_1^{*2}} \tag{140}$$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \frac{-A_{max}^2 + \sqrt{A_{max}^4 + 4J^2PA_{max}}}{2J} \\ A_{max} \end{bmatrix} \tag{141}$$

With $\lambda_1^* = 0$, the Hessian in (126) becomes:

$$\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*) = \begin{bmatrix} (1-\lambda_2^*)\frac{2P}{x_1^{*3}} & -(1+\lambda_2^*)\frac{1}{x_2^{*2}} \\ -(1+\lambda_2^*)\frac{1}{x_2^{*2}} & (1+\lambda_2^*)\frac{2x_1^*}{x_2^{*3}} \end{bmatrix} \tag{142}$$

and with the expression of λ_2^* in (140):

$$\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*) = \frac{1}{Px_2^* + x_1^{*2}} \begin{bmatrix} \frac{4P}{x_1^*} & -\frac{2P}{x_2^*} \\ -\frac{2P}{x_2^*} & \frac{4Px_1^*}{x_2^{*2}} \end{bmatrix} \tag{143}$$

Referring to Section III-K, $x_1^* > 0$ and $x_2^* > 0$ so that $Px_2^* + x_1^{*2} > 0$ and $\frac{4P}{x_1^*} > 0$ hold. Then, according to Sylvester's criterion, $\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*)$ is positive definite if and only if the determinant of the matrix in (143) is positive. This determinant is positive since it is calculated as follows:

$$\begin{aligned} \begin{vmatrix} \frac{4P}{x_1^*} & -\frac{2P}{x_2^*} \\ -\frac{2P}{x_2^*} & \frac{4Px_1^*}{x_2^{*2}} \end{vmatrix} &= \frac{16P^2}{x_2^{*2}} - \frac{4P^2}{x_2^{*2}} &(144)\\ &= \frac{12P^2}{x_2^{*2}} &(145)\end{aligned}$$

Hence, the second-order sufficient conditions are satisfied and x^* of Case 10, given in (60) and computed at line 12 of Algorithm 2, is a strict local minimum.

Finally, in Case 11, we have $\lambda_1^* = 0$ and $\lambda_2^* = 0$ and:

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} V_{max} \\ A_{max} \end{bmatrix} \tag{146}$$

so that:

$$\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*) = \begin{bmatrix} \frac{2P}{x_1^{*3}} & -\frac{1}{x_2^{*2}} \\ -\frac{1}{x_2^{*2}} & \frac{2x_1^*}{x_2^{*3}} \end{bmatrix} \tag{147}$$

whose determinant is:

$$\begin{aligned} \det\left(\nabla^2_{xx}\mathscr{L}(x^*,\lambda^*)\right) &= \frac{4P}{x_1^{*2}x_2^{*3}} - \frac{1}{x_2^{*4}} &(148)\\ &= \frac{1}{x_2^{*3}}\left(\frac{4P}{x_1^{*2}} - \frac{1}{x_2^*}\right) &(149)\\ &= \frac{1}{A_{max}^3}\left(\frac{4P}{V_{max}^2} - \frac{1}{A_{max}}\right) &(150)\\ &= \frac{1}{A_{max}^3}\frac{4PA_{max} - V_{max}^2}{V_{max}^2A_{max}} &(151)\end{aligned}$$

This determinant is positive since $PA_{max} \geq V_{max}^2$ according to (73). Hence, the second-order sufficient conditions are satisfied and x^* of Case 11, given in (61) and computed at line 14 of Algorithm 2, is a strict local minimum.

P. LICQ

In the KKT first-order necessary conditions stated at the beginning of Section III, the LICQ should hold. As defined in Definition 12.4 of [3], the LICQ holds at a given x^* if the set of active constraint gradients are linearly independent at x^*.

For completeness of the proof of Algorithm 2 (and thus of Algorithm 1), let us verify that the LICQ holds at the local minimum x^* of Case 3 in (27), Case 4 in (33), Case 10 in (60) and Case 11 in (61).

For Cases 3 and 4, only one constraint is active, $C_2(x) = 0$ and $C_3(x) = 0$, respectively. Hence, the LICQ holds at x^* of Case 3 if $\nabla_x C_2(x^*)$ in (15) is nonzero which is necessarily the case since $-Px_2 - x_1^2$ is strictly negative for any x. The LICQ holds at x^* of Case 4 if $\nabla_x C_3(x^*)$ in (16) is nonzero which is always the case.

In Case 10, there are two active constraints, $C_2(x) = C_4(x) = 0$. From (15) and (17), $\nabla_x C_2(x)$ and $\nabla_x C_4(x)$ are easily seen to be linearly independent whatever x so that the LICQ holds.

Finally, in Case 11, there are two active constraints, $C_3(x) = C_4(x) = 0$, and, according to (16) and (17), $\nabla_x C_3(x)$ and $\nabla_x C_4(x)$ are trivially linearly independent whatever x so that the LICQ holds.

References

[1] A. Gasparetto and V. Zanotto, "A technique for time-jerk optimal planning of robot trajectories," *Robotics and Computer-Integrated Manufacturing*, vol. 24, no. 3, pp. 415–426, 2008.

[2] X. Li, "Optimizing s-curve velocity for motion control," 2010.

[3] J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd ed. Springer, 2006.