



**HAL**  
open science

# On the approximation hardness of geodetic set and its variants

Tom Davot, Lucas Isenmann, Jocelyn Thiebaut

► **To cite this version:**

Tom Davot, Lucas Isenmann, Jocelyn Thiebaut. On the approximation hardness of geodetic set and its variants. COCOON 2021 - 27th International Computing and Combinatorics Conference, Oct 2021, Tainan, Taiwan. pp.76-88, 10.1007/978-3-030-89543-3\_7. lirmm-03328636v2

**HAL Id: lirmm-03328636**

**<https://hal-lirmm.ccsd.cnrs.fr/lirmm-03328636v2>**

Submitted on 24 Nov 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the approximation hardness of geodetic set and its variants

Tom Davot<sup>1</sup>, Lucas Isenmann<sup>2</sup>, and Jocelyn Thiebaut<sup>3</sup>

<sup>1</sup> LIRMM, Univ Montpellier, CNRS, Montpellier, France [davot@lirmm.fr](mailto:davot@lirmm.fr)

<sup>2</sup> Aix-Marseille Université, France [lucas.isenmann@laposte.net](mailto:lucas.isenmann@laposte.net)

<sup>3</sup> INSA Centre Val de Loire, Univ. Orléans, LIFO EA 4022, Orléans, France  
[jocelyn.thiebaut@univ-orleans.fr](mailto:jocelyn.thiebaut@univ-orleans.fr)

**Abstract.** Given a graph, a geodetic set (resp. edge geodetic set) is a subset of its vertices such that every vertex (resp. edge) of the graph is on a shortest path between two vertices of the subset. A strong geodetic set is a subset  $S$  of vertices and a choice of a shortest path for every pair of vertices of  $S$  such that every vertex is on one of these shortest paths. The geodetic number (resp. edge geodetic number) of a graph is the minimum size of a geodetic set (resp. edge geodetic set) and the strong geodetic number is the minimum size of a strong geodetic set. We first prove that, given a subset of vertices, it is  $\mathcal{NP}$ -hard to determine whether it is a strong geodesic set. Therefore, it seems natural to study the problem of maximizing the number of covered vertices by a choice of a shortest path for every pair of a provided subset of vertices. We provide a tight 2-approximation algorithm to solve this problem. Then, we show that there is no  $7^{81/780}$  polynomial-time approximation algorithm for edge geodetic number and strong geodetic number on subcubic bipartite graphs with arbitrarily high girth. We also prove that geodetic number and edge geodetic number are both  $\text{LOG-}\mathcal{APX}$ -hard, even on subcubic bipartite graphs with arbitrarily high girth. Finally, we disprove a conjecture of Iršič and Konvalinka by proving that the strong geodetic number can be computed in polynomial time in complete multipartite graphs.

## 1 Introduction

*Geodetic number and edge geodetic number.* A *geodesic* between two vertices of a graph  $G$  is a path of minimum length between  $x$  and  $y$ . The *geodetic number* of  $G$  is the minimum size of a subset  $X$  of the vertices such that, for every vertex  $v$ , there exists a geodesic between two vertices of  $X$  containing  $v$ . The geodetic number of a graph has been introduced by Harary et al in [13], where the authors showed that deciding whether a graph has a geodetic number less than an integer  $k$  is  $\mathcal{NP}$ -complete. The complexity of this problem has also been investigated in several classes of graphs, such as bipartite graphs [11] and chordal graphs [10]

---

Preprint Version

The final authenticated version is available online at [https://doi.org/10.1007/978-3-030-89543-3\\_7](https://doi.org/10.1007/978-3-030-89543-3_7)

where it remains  $\mathcal{NP}$ -complete. Recently, Chakraborty *et al.* proved that finding the geodetic number of a graph is  $\mathcal{NP}$ -hard on planar graphs with maximum degree six and line graphs [6]. They also proved in [7] that, unless  $\mathcal{P} = \mathcal{NP}$ , there is no polynomial time  $o(\log n)$ -approximation algorithm for computing the geodetic number of a graph, even on graphs that have a universal vertex and where  $n$  stands for the number of vertices in the input graph.

The edge version of the geodetic number has been introduced independently in [3] and in [22]. A subset  $X$  of the vertices is an *edge geodetic set* if, for every edge  $e$ , there is a geodesic between two vertices of  $X$  containing  $e$ . The *edge geodetic number* of  $G$  is the size of the smallest geodetic set of  $G$ . Note that, given a graph  $G$ , the geodetic number of  $G$  is smaller than its edge geodetic number. This edge version is also known to be  $\mathcal{NP}$ -hard [3]. This problem has been studied on several classes of graphs, such as Cartesian products [1, 21] and fuzzy graphs [20]. From a structural point of view, Santhakumaran and Ullas Chandran characterized graphs with a prescribed edge geodetic number [23]. For more results and motivations about geodetic sets, see [5].

*Strong geodetic number and strong edge geodetic number.* A subset of vertices  $X$  is a *strong geodetic set* if there exists a function  $\tilde{I}$  that associates a unique geodesic to each pair of vertices of  $X$  and such that every vertex  $v$  is contained in a geodesic  $\tilde{I}(a, b)$ , where  $\{a, b\} \subseteq X$ . In the following, we call such a function a *geodesic assignation* for  $X$ . The *strong geodetic number* of a graph  $G$  is the size of the smallest strong geodetic set. The strong geodetic number has been introduced recently by Arokiaraj *et al.* [2]. In their original paper, the authors motivate this variation by social network applications. Furthermore, they also prove that this problem is  $\mathcal{NP}$ -complete. Note that it remains  $\mathcal{NP}$ -hard even when restricted to bipartite graphs [15]. This problem has been studied on complete Apollonian networks [2], grids and cylinders [16], and on Cartesian product of graphs [12], on complete bipartite graphs [14], on complete multipartite graphs [15] and on outerplanar graphs [18]. Connections to the diameter of the graph were studied in [14] and to the isometric path problem [2].

Finally, the edge version of the strong geodetic problem, where we want to cover every edge of the graph, has been introduced by Manuel *et al.* [17] and were proved  $\mathcal{NP}$ -complete. Zmazek recently studied the values of the edge strong geodetic number on grids [25].

*Our results.* The results of our paper are divided in four sections. In Section 3, we propose a variant of the strong geodetic problem where, given a subset  $S$  of vertices, the question is to determine whether  $S$  is a strong geodesic set of the graph. Using a reduction from MONOTONE BALANCED 3-SAT-(4), we prove that this problem is  $\mathcal{NP}$ -hard. Then, we consider in Section 4 the problem of maximizing the number of covered vertices by a choice of a geodesic for each pair of a provided subset of vertices, and provide a tight 2-approximation algorithm to solve it. In Section 5, we reduce the geodetic problems from SET COVER. We first give it in the general case, and then we adapt the previous construction on bipartite graphs with arbitrarily high girth. Using the previous reductions, we show in Section 6 that there is no approximation of EDGE GEODETIC NUMBER

with an approximation factor better than  $781/780$ . We also prove that geodetic number and edge geodetic number are both LOG- $\mathcal{APX}$ -hard, even on subcubic bipartite graphs with arbitrarily high girth. Finally, in section 7, we give a polynomial time algorithm which computes the STRONG GEODETIC NUMBER of complete multipartite graphs, disproving the conjecture of [15] which states that STRONG GEODETIC NUMBER is  $\mathcal{NP}$ -hard on complete multipartite graphs. Due to space constraints, the proofs have been omitted. However, a full version can be found in <https://hal-lirmm.ccsd.cnrs.fr/lirmm-03328636>.

## 2 Notations

We first introduce some notations and formally define the problems. Given a set  $X$ , we denote by  $\mathcal{P}_2(X)$  the set of its pairs. Given two sets  $X$  and  $Y$ , we denote by  $X \sqcup Y$  the union  $X \cup Y$  when  $X$  and  $Y$  are disjoint. Let  $G$  be a graph, we denote by  $V(G)$  its set of vertices and by  $E(G)$  its set of edges. We denote by  $D_1(G)$  the set of vertices of degree one in  $G$ . Let  $X$  be a subset of vertices of  $G$  and  $x$  be a vertex, we say that  $x$  is *selected* by  $X$  if  $x \in X$  and that  $x$  is *covered* by  $X$  if  $x$  is contained in a geodesic between two vertices of  $X$  (or simply selected or covered if there is no ambiguity on  $X$ ). Likewise, let  $uv$  be an edge, we say that  $uv$  is *covered* by  $X$  (or simply covered) if  $uv$  is contained in a geodesic between two vertices of  $X$ .

To describe a path between two vertices  $u$  and  $v$ , we introduce the operator  $\sim$  as follows. Let  $p_1, p_2, \dots, p_k$  be some subgraphs such that  $u \in p_1, v \in p_k$  and for each  $i < k$ , there exists a unique vertex  $x_i \in p_i \cap p_{i+1}$ . The path described by  $p_1 \sim p_2 \sim \dots \sim p_k$  corresponds to  $(u, \dots, x_1, \dots, x_i, \dots, x_{k-1}, \dots, v)$  (following successively the paths  $p_1, p_2, \dots, p_k$ ). Let  $g$  be a path that contains the vertices  $u$  and  $v$ . We denote by  $g[u, v]$  the subpath of  $g$  with extremities  $u$  and  $v$ . Furthermore, we denote by  $V(g)$  the vertices of  $g$ . Similarly, given a geodesic assignation  $\tilde{I}$  for a set of vertices  $X$ , we denote by  $V(\tilde{I})$  the vertices covered by the geodesics of  $\tilde{I}$ .

We now introduce the problems studied in this work.

(STRONG) (EDGE) GEODETIC NUMBER

**Input:** a simple graph  $G$  and an integer  $k$ .

**Question:** is there a (strong) (edge) geodetic set  $X \subseteq V$  of size  $k$ ?

The following already known property will be fundamental in the proofs of our reductions as it helps to force some vertices to be part of a (strong) (edge) geodetic set.

*Property 1.* If  $G$  is a graph and  $X$  is a solution of any geodesic problem, then we have  $D_1(G) \subseteq X$ .

## 3 Hardness to find a geodesic assignation

In the proof that computing the strong geodesic number is  $\mathcal{NP}$ -complete, the geodesic assignation is rather trivial [2]. In this section, we show that determining

if a set of vertices is a strong geodesic set (*i.e.* computing a geodesic assignment) is in itself  $\mathcal{NP}$ -complete. To do so, we reduce from a special case of 3-SAT called MONOTONE BALANCED 3-SAT-(4). In this variant, the boolean formula is composed of *monotone* clauses, that is, clauses that contains only positive literals or only negative literals. MONOTONE BALANCED 3-SAT-(4) is defined as follows.

MONOTONE BALANCED 3-SAT-(4)

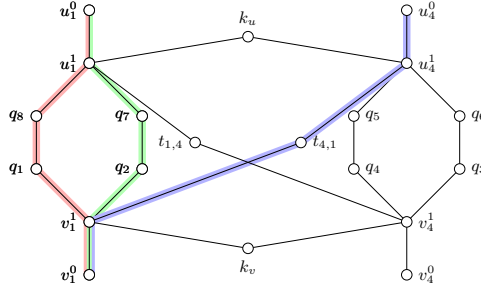
**Input:** a monotone 3-SAT formula  $\varphi$  where each variable occurs exactly two times positively and two times negatively.

**Question:** is  $\varphi$  satisfiable?

Darman and Döcker showed that this problem is  $\mathcal{NP}$ -complete [8]. We introduce the following construction.

**Construction 1.** Let  $\varphi$  be a MONOTONE BALANCED 3-SAT-(4) formula, we construct the following graph  $G$ :

- For each clause  $C_j$ , introduce a vertex  $q_j$ .
- For each variable  $x_i$ , introduce two edges  $v_i^0 v_i^1$  and  $u_i^0 u_i^1$ . Furthermore, let  $C_j$  and  $C_{j'}$ , with  $j < j'$  be the two clauses where  $x_i$  occurs with the same polarity (*i.e.* it appears positively in both clauses, or negatively in both), construct a path  $(v_i^1, q_j, q_{j'}, u_i^1)$ .
- For each pair of vertices  $v_i^1$  and  $u_{i'}^1$ , with  $i \neq i'$ , introduce a vertex  $t_{i,i'}$  and construct the path  $(v_i^1, t_{i,i'}, u_{i'}^1)$ .
- Finally, construct two vertices  $k_v$  and  $k_u$ , and for each variable  $x_i$ , introduce the edges  $v_i^1 k_v$  and  $u_i^1 k_u$ .



**Fig. 1.** Example of a subgraph induced by Construction 1. In the Boolean formula, the variable  $x_1$  appears positively in  $C_2$  and  $C_7$  and negatively in  $C_1$  and  $C_8$ . The variable  $x_4$  appears positively in  $C_3$  and  $C_6$  and negatively in  $C_4$  and  $C_5$ . The paths  $p_1$ ,  $\bar{p}_1$  and  $p_{1,4}$  are depicted in green, red and blue, respectively.

For each variable  $x_i$ , let  $C_j$  and  $C_{j'}$  (resp.  $C_k$  and  $C_{k'}$ ) with  $j < j'$  be the clauses where  $x_i$  occurs positively (resp. negatively). We denote by  $p_i$  the path  $(v_i^0, v_i^1, q_j, q_{j'}, u_i^1, u_i^0)$ , by  $\bar{p}_i$  the path  $(v_i^0, v_i^1, q_k, q_{k'}, u_i^1, u_i^0)$  and by  $p_{i,i'}$  the path  $(v_i^0, v_i^1, t_{i,i'}, u_{i'}^1, u_{i'}^0)$ , for any  $i \neq i'$ . An example of a graph produced by Construction 1 is depicted in Figure 1.

**Lemma 1.** Let  $\varphi$  be a MONOTONE BALANCED 3-SAT-(4) formula and  $G$  its graph resulting from Construction 1. Let  $\bar{I}$  be a geodesic assignment for  $D_1(G)$ .

It is possible to construct a geodesic assignment  $\tilde{I}'$  for  $D_1(G)$  such that  $|V(\tilde{I}')| \leq |V(\tilde{I})|$ , and:

- (1) for any  $i \neq i'$ , the geodesic between  $v_i^0$  and  $u_{i'}^0$  in  $\tilde{I}'$  is  $p_{i,i'}$ , and
- (2) for any  $i$ , the geodesic between  $v_i^0$  and  $u_i^0$  in  $\tilde{I}'$  is either  $p_i$  or  $\bar{p}_i$ .

*Proof.* First, note that every geodesic with extremity  $v_i^0$  (resp.  $u_i^0$ ) covers the vertex  $v_i^1$  (resp.  $u_i^1$ ). Second, the geodesic between two vertices  $u_i^0$  and  $u_{i'}^0$  is  $(u_i^0, u_i^1, k_u, u_{i'}^1, u_{i'}^0)$ . Similarly, the geodesic between two vertices  $v_i^0$  and  $v_{i'}^0$  is  $(v_i^0, v_i^1, k_v, v_{i'}^1, v_{i'}^0)$ . Hence, it remains to cover the  $q_j$  vertices and the  $t_{i,i'}$  vertices. Let  $V_1 = \{v_i^1 \mid x_i \in \varphi\}$  and  $U_1 = \{u_i^1 \mid x_i \in \varphi\}$ . Since there is no edge between  $V_1$  and  $U_1$ , the distance between any vertex of  $V_1$  and any vertex of  $U_1$  is at least two. From that, we can deduce that no geodesic  $g$  between two vertices  $v_i^0$  and  $u_{i'}^0$  contains the vertex  $k_v$  or the vertex  $k_u$  since the length of  $g$  would be at least six, and  $p_{i,i'}$  or  $p_i$  (if  $i = i'$ ) are shorter paths. Let us now prove the two items of the statement.

- (1) Since the distance between  $V_1$  and  $U_1$  is at least two, the path  $p_{i,i'}$  is a geodesic between  $v_i^0$  and  $u_{i'}^0$ . Suppose that there exists another geodesic  $g$  between  $v_i^0$  and  $u_{i'}^0$ . We have  $g = (v_i^0, v_i^1, q_j, u_{i'}^1, u_{i'}^0)$ , where the clause  $C_j$  contains  $x_i$  and  $x_{i'}$  in  $\varphi$ . Suppose, without loss of generality that  $C_j$  contains positive literals. Toward a contradiction, suppose that  $t_{i,i'}$  is covered by  $\tilde{I}$  and let  $g'$  be the geodesic that covers  $t_{i,i'}$ . Since  $g'$  cannot have the same extremities as  $g$ ,  $g'$  contains either the vertex  $k_v$  or the vertex  $k_u$ , contradicting  $g'$  being a geodesic. Hence, the vertex  $t_{i,i'}$  is not covered by  $\tilde{I}$ , then we can replace  $g$  by  $p_{i,i'}$  in it to obtain a solution as thought.
- (2) Let  $g$  be a geodesic between  $v_i^0$  and  $u_i^0$ . Since,  $g$  does not contain  $k_v$ , then  $v_i^1$  is adjacent to a vertex  $q_j$  in  $g$ . Likewise,  $u_i^1$  is adjacent to a vertex  $q_{j'}$  in  $g$ . By contradiction, suppose that  $x_i$  appears positively in  $C_j$  and negatively in  $C_{j'}$ . Then, since the edge  $q_j q_{j'}$  exists, by construction, there is a path  $p_{i'}$  or  $\bar{p}_{i'}$  containing  $q_j$  and  $q_{j'}$ . But then, the variable  $x_{i'}$  appears either positively in  $C_j$  and  $C_{j'}$  or negatively in  $C_j$  and  $C_{j'}$ . In any case, we reach a contradiction since either  $C_j$  or  $C_{j'}$  is not monotone. Hence,  $x_i$  appears only positively in  $C_j$  and  $C_{j'}$  or only negatively in  $C_j$  and  $C_{j'}$ , and then  $g$  corresponds to either  $p_i$  or  $\bar{p}_i$ .

**Theorem 1.** *It is  $\mathcal{NP}$ -hard to determine if a set of vertices  $V'$  is a strong geodetic set even if, for every strong geodetic set  $V_{strong}$ , we have  $V' \subseteq V_{strong}$ .*

*Proof.* Let  $\varphi$  be a MONOTONE BALANCED 3-SAT-(4) formula and  $G$  its graph resulting from Construction 1. We show that  $\varphi$  is satisfiable if and only if  $D_1(G)$  is a strong geodetic set.

- Let  $\beta$  be a satisfying assignment of  $\varphi$ , we construct a geodesic assignment  $\tilde{I}$  for  $D_1(G)$  as follows. For each  $i \neq i'$ , we set  $\tilde{I}(v_i^0, u_{i'}^0) = p_{i,i'}$ ,  $\tilde{I}(v_{i'}^0, u_i^0) = p_{i',i}$ ,  $\tilde{I}(v_i^0, v_{i'}^0) = (v_i^0, k_v, v_{i'}^0)$ , and  $\tilde{I}(u_i^0, u_{i'}^0) = (u_i^0, k_u, u_{i'}^0)$ . Hence, every vertex is covered except the  $q_j$  vertices. Now, for each variable  $x_i$ , if  $x_i$  is assigned to *true*, we set  $\tilde{I}(v_i^0, u_i^0) = p_i$  and we set  $\tilde{I}(v_i^0, u_i^0) = \bar{p}_i$ , otherwise.

Suppose, there is a vertex  $q_j$  that is not covered by  $\tilde{I}$ . Let  $x_i, x_k$  and  $x_\ell$  be the three variables that occur in  $C_j$  and suppose that  $C_j$  contains positive literals. We have  $\text{img}(\tilde{I}) \cap \{p_i, p_k, p_\ell\} = \emptyset$ , and then  $\beta(x_i) = \beta(x_k) = \beta(x_\ell) = \text{false}$ , contradicting  $\beta$  being a satisfying assignment for  $\varphi$ . Hence, every vertex of  $G$  is covered and  $D_1(G)$  is a strong geodetic set of  $G$ .

- Let  $\tilde{I}$  be a geodesic assignment for  $D_1(G)$  that covers every vertex of  $G$ , and such that  $\tilde{I}$  respects properties of Lemma 1. We construct a satisfying assignment  $\beta$  for  $\varphi$  as follows. For each variable  $x_i$ , if  $p_i \in \text{img}(\tilde{I})$ , we set  $\beta(x_i) = \text{true}$  and  $\beta(x_i) = \text{false}$ , otherwise.

Suppose there is a clause  $C_j$  that is not satisfied by  $\beta$  and suppose by symmetry that  $C_j$  contains positive literals. Let  $x_i, x_k$  and  $x_\ell$  be the three literals of  $C_j$ . We have  $\text{img}(\tilde{I}) \cap \{p_i, p_k, p_\ell\} = \emptyset$  and then  $q_j$  is not covered by  $\tilde{I}$  which is a contradiction. Hence,  $\beta$  is a satisfying assignment for  $\varphi$ .

Finally, by Property 1, the set  $D_1(G)$  belongs to any strong geodetic set.

From this theorem, we can derive a result about the residue variant of STRONG GEODETIC NUMBER. The *residue variant* of an optimisation problem has been defined recently in [24] and consists of, given a partial solution  $P$  for an instance  $I$ , finding an optimal partial solution  $R$  such that  $P \cup R$  is a solution for  $I$ . The complexity class  $\mathcal{RAPX}$  contains the residue variant optimisation problems such that the score of the residue can be approximated by a constant.

**Corollary 1.** STRONG GEODETIC NUMBER  $\notin \mathcal{RAPX}$

*Proof.* Let  $\varphi$  be a MONOTONE BALANCED 3-SAT-(4) formula and  $G$  be its graph resulting from Construction 1. Then, given a partial solution  $P = D_1(G)$ ,  $G$  has a residue solution  $R = \emptyset$  if and only if  $\varphi$  is satisfiable. Hence, the residual variant of STRONG GEODETIC NUMBER cannot be approximated to any constant factor unless  $\mathcal{P} = \mathcal{NP}$ .

## 4 Approximation

Since it is hard to determine if a subset of vertices is a strong geodetic set, a natural question that arises is to find, given a subset of vertices, a geodetic assignment that maximizes the number of covered vertices. We call this problem MAX GEODESIC ASSIGNATION. By Theorem 1, this problem is also  $\mathcal{NP}$ -hard and we show that this problem belongs to  $\mathcal{APX}$ , *i.e.* approximable within a constant ratio. In this part, we show that this problem is 2-approximable using a simple greedy algorithm, defined in Algorithm 1.

**Theorem 2.** *Algorithm 1 computes in polynomial time a solution for MAX GEODESIC ASSIGNATION with an approximation ratio of 2 and this ratio is tight.*

*Proof.* Let  $\tilde{I}_{app}$  be the geodesic assignment computed by Algorithm 1 and let  $\tilde{I}_{opt}$  be an optimal geodesic assignment. We show that there exists an application  $f : V(\tilde{I}_{opt}) \rightarrow V(\tilde{I}_{app})$  such that for each  $u \in V(\tilde{I}_{app})$ ,  $|f^{-1}(u)| \leq 2$ .

---

**Algorithm 1: Greedy Algorithm**


---

**Data:** A graph  $G$  and a set of vertices  $V' \subseteq V(G)$ .

**Result:** A geodetic assignment  $\tilde{I}$  for  $V'$ .

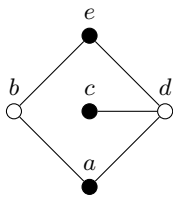
```

1  $A \leftarrow \mathcal{P}_2(V')$  ;
2 while  $A \neq \emptyset$  do
3    $\{u, v\} \leftarrow$  first element of  $A$ ;
4    $g \leftarrow$  geodesic between  $u$  and  $v$  that maximizes  $|V(g) \setminus V(\tilde{I})|$ ;
5   Set  $\tilde{I}(u, v) := g$ ;  $A \leftarrow A \setminus \{\{u, v\}\}$ ;
6 end
7 return  $\tilde{I}$ ;

```

---

First, for each vertex  $v \in V(\tilde{I}_{opt}) \cap V(\tilde{I}_{app})$ , we set  $f(v) = v$ . Further, let  $g_{app}^i$  be the geodesic chosen by the greedy algorithm at step  $i$  and let  $g_{opt}^i$  be the geodesic with the same extremities in  $\tilde{I}_{opt}$ . For each  $i$ , let  $V_{opt}^i = V(g_{opt}^i) \setminus \bigcup_{j < i} V(g_{opt}^j)$  and  $V_{app}^i = V(g_{app}^i) \setminus \bigcup_{j < i} V(g_{app}^j)$  (*i.e.* the set of vertices newly covered by  $g_{app}^i$ ). We have  $|V_{opt}^i \setminus V(\tilde{I}_{app})| \leq |V_{app}^i|$  since otherwise, the greedy algorithm would have chosen  $g_{opt}^i$  at step  $i$ . Thus, there exists an injective function  $f' : V_{opt}^i \setminus V(\tilde{I}_{app}) \rightarrow V_{app}^i$  and, for each vertex  $v \in V_{opt}^i \setminus V(\tilde{I}_{app})$ , we set  $f(v) = f'(v)$ . Since each vertex  $u \in V(\tilde{I}_{app})$  belongs to a unique  $V_{app}^i$ , we have  $|f^{-1}(u)| \leq 2$ . Moreover, since each vertex  $v \in V(\tilde{I}_{opt})$  belongs either to  $V(\tilde{I}_{opt}) \cap V(\tilde{I}_{app})$  or to a set  $V_{opt}^i \setminus V(\tilde{I}_{app})$ , we defined a function  $f$  as thought. It follows that  $|V(\tilde{I}_{opt})| \leq 2 \cdot |V(\tilde{I}_{app})|$ , proving the approximation ratio. Furthermore, the ratio is tight, as shown by Figure 2.

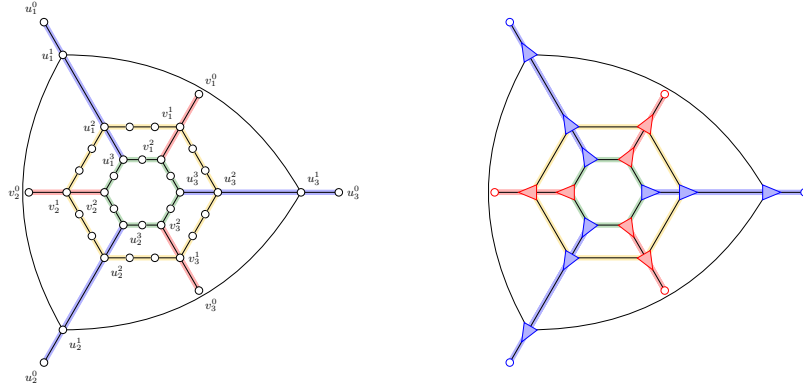


**Fig. 2.** Tightness of the approximation ratio of Algorithm 1. Consider  $a, c$  and  $e$  as selected (in black in the graph). The optimal solution consists in taking the geodesics  $(a, b, e)$ ,  $(c, d, e)$  and  $(a, c, d)$  which cover the non-selected vertices  $b$  and  $d$ . The greedy algorithm can start by taking the geodesic  $(a, d, e)$  between  $a$  and  $e$ . Then the algorithm will choose  $(c, d, e)$  and  $(a, c, d)$  for the last two pairs. This leads to a set of geodesics which only covers  $d$ .

## 5 Reduction from Set Cover

In this part, we prove preliminary results that will be used in the next section. More specifically, we reduce the geodetic problems from the classic  $\mathcal{NP}$ -complete problem SET COVER described as follows.





**Fig. 3.** Example of a graph produced by Construction 2 (left) and Construction 4 (right) on the collection containing  $S_1 = \{E_1, E_3\}$ ,  $S_2 = \{E_1, E_2\}$  and  $S_3 = \{E_2, E_3\}$ . Element paths, set paths, cut paths and long paths are coloured in blue, red, yellow and green, respectively. In the right graph, black edges, green edges and yellow edges represent paths of length  $h$ ,  $2h$  and  $3h$ , respectively

#### SET COVER (SC)

**Input:** A collection  $C = \{S_1, \dots, S_{m'}\}$  of finite sets over the universe  $U = \{E_1, \dots, E_{n'}\}$ .

**Question:** Find a minimum  $C' \subseteq C$  such that every element of  $U$  is contained in a set of  $C'$ .

For the strong versions, we use a version of SET COVER, denoted  $(k, k')$ -SET COVER, where the size of the intersection between two sets is at most  $k$  and the set sizes are bounded by  $k'$ . Notice that since VERTEX COVER is a particular case of  $(1, k')$ -SET COVER, then  $(k, k')$ -SET COVER is  $\mathcal{NP}$ -complete. In the following, we first show how this reduction works in the general case and then, we adapt it in subcubic bipartite graphs with arbitrary high girth.

#### 5.1 On general case

**Construction 2.** Let  $(C, U)$  be an instance of SET COVER. We construct a graph  $G$  as follows:

- For each set  $S_i$ , create a 3-path  $sp_i = (v_i^0, v_i^1, v_i^2)$ .
- For each element  $E_j$ , create a 4-path  $ep_j = (u_j^0, u_j^1, u_j^2, u_j^3)$ . We denote the edge  $u_j^2 u_j^3$  as  $e_j$ .
- For each set  $S_i$  and each element  $E_j \in S_i$ , introduce a 3-path  $cp_j^i$  between  $v_i^1$  and  $u_j^2$  and a 2-path  $lp_j^i$  between  $v_i^2$  and  $u_j^3$ .
- For each pair of elements  $E_j$  and  $E_{j'}$ , introduce the edge  $t_{j,j'} = u_j^1 u_{j'}^1$ .

The paths  $ep_j$ ,  $sp_i$ ,  $cp_j^i$  and  $lp_k^i$  are called *element paths*, *set paths*, *cut paths* and *long paths*, respectively. An example of a graph produced by Construction 2 is depicted in Figure 3.

Clearly, the construction can be carried in polynomial time. In order to show that Construction 2 constitutes a reduction, we introduce the following lemmas.

**Lemma 2.** *Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) and let  $G$  be its graph resulting from Construction 2. The set  $D_1(G)$  covers (resp. strongly covers) every edge of  $G'$  except the edges in  $\{e_j \mid E_j \in U\}$ .*

*Proof.* We start by describing the geodesics between each pair of vertices of  $D_1(G)$  and how we can make the assignation them when there is some choice.

- (a) For any pair of elements  $E_j$  and  $E_{j'}$  of  $U$ , the distance between  $u_j^0$  and  $u_{j'}^0$  is 3 and the unique geodesic between these two vertices is described by  $ep_j \sim t_{j,j'} \sim ep_{j'}$ .
- (b) For any set  $S_i$  of  $C$  and any element  $E_j \in S_i$ , the distance between  $u_j^0$  and  $v_i^0$  is 6 and the unique geodesic between these two vertices is described by  $ep_j \sim cp_j^i \sim sp_i$ .
- (c) For any set  $S_i$  of  $C$  and any element  $E_j \notin S_i$ , the distance between  $u_j^0$  and  $v_i^0$  is 7 and the geodesics are described by  $ep_j \sim t_{j,j'} \sim ep_{j'} \sim cp_{j'}^i \sim sp_i$ , for any element  $E_{j'} \in S_i$ . Since these geodesics contain only edges that are covered by the cases (a) or (b), we can assign any geodesic for this case.
- (d) For any pair of sets  $S_i$  and  $S_{i'}$  of  $C$  such that  $S_i \cap S_{i'} \neq \emptyset$ , the distance between  $v_i^0$  and  $v_{i'}^0$  is 8 and the geodesics between these two vertices are described by  $sp_i \sim cp_j^i \sim cp_{j'}^{i'} \sim sp_{i'}$  and  $sp_i \sim lp_j^i \sim lp_{j'}^{i'} \sim sp_{i'}$ , for each  $E_j \in S_i \cap S_{i'}$ . Since, the edges of  $cp_j^i$  and  $cp_{j'}^{i'}$  are already covers by the case (b), it is better to assign a geodesic  $sp_i \sim lp_j^i \sim lp_{j'}^{i'} \sim sp_{i'}$ , note that if  $|S_i \cap S_{i'}| = 1$ , then there only one such geodesic.
- (e) For any pair of sets  $S_i$  and  $S_{i'}$  of  $C$  such that  $S_i \cap S_{i'} = \emptyset$ , the distance between  $v_i^0$  and  $v_{i'}^0$  is 11 and the geodesics between these two vertices are described by  $sp_i \sim cp_j^i \sim ep_j \sim t_{j,j'} \sim ep_{j'} \sim cp_{j'}^{i'} \sim sp_{i'}$ , for any  $E_j \in S_i$  and  $E_{j'} \in S_{i'}$ .

Remark that the  $e_j$  edges are not covered by any geodesic between the vertices of  $D_1(G)$ . We conclude by showing that every other edges are covered by  $D_1(G)$ , even if we fix a unique geodesic in the case where the intersection between two sets is at most one.

Let  $e$  be an edge in  $E(G) \setminus \{e_j \mid E_j \in U\}$ . If  $e$  belongs to a set path  $sp_i$  or a long path  $lp_i$ , then  $e$  is covered by the geodesic between  $v_i^0$  and  $v_j^0$ . If  $e$  belongs to an element path  $ep_j$  or a cut path  $cp_j^i$ , then  $e$  is covered by the geodesic between  $v_i^0$  and  $u_{i,j}^0$ . Finally, if  $e$  is a  $t_{j,j'}$  edge, then  $e$  is covered by the geodesic between  $u_j^0$  and  $u_{j'}^0$ . Hence, every edge of  $G$  is covered except the  $e_j$  edges.

In the following, let  $Y_i^S \subset V(G)$  denote the set containing  $sp_i \setminus \{v_i^0\}$  and every long path  $lp_j^i$  and cut path  $cp_j^i$  incident to  $cp_i$  minus vertices of every element path  $ep_j$ . Formally,  $Y_i = (sp_i \setminus \{v_i^0\}) \cup \{(cp_j^i \cup lp_j^i) \setminus ep_j \mid \forall E_j \in S_i\}$ . For each element  $E_j \in U$ , we also denote  $Y_j^E = \{Y_i \mid E_j \in S_i\} \cup ep_j \setminus \{u_j^0\}$ .

**Lemma 3.** *Let  $(C, U)$  be an instance of SET COVER and let  $G$  be its graph resulting from Construction 2. For each element  $E_j$ , every geodesic containing the edge  $e_j$  has an extremity in  $Y_j^E$ .*

*Proof.* Toward a contradiction, we suppose that there exists an edge  $e_j$  such that there is a geodesic  $g$  with extremities  $x \notin Y_j^E$  and  $y \notin Y_j^E$  that contains  $e_j$ . For simplicity, we denote the subpath  $g[x, u_j^3]$  as  $g_x$  (we suppose that  $e_j \notin g_x$ ). By hypothesis,  $g_x$  contains a long path  $lp_j^i$  such that  $E_j \in S_i$ . Since  $g_x$  can not have an extremity in  $Y_i^S$ ,  $g_x$  contains a vertex  $x'$  in some element path  $ep_k$ , such that  $E_k \in S_i$ . The subpath  $g[x', u_j^2]$  can be described either by  $cp_k^i \sim sp_i \sim cp_j^i \sim ep_j$  or by  $lp_k^i \sim sp_i \sim lp_j^i \sim ep_j$ . Let  $g'$  be the path between  $x'$  and  $u_j^2$  described by  $ep_k \sim t_{k,j} \sim ep_j$ . By construction, we have  $|g'| < |g[x', u_j^2]|$ . Thus replacing  $g[x', u_j^2]$  by  $g'$  in  $g$  constructs a path between  $x$  and  $y$  that is shorter than  $g$ , contradicting that  $g$  is a geodesic. Hence,  $X$  contains at least one vertex in  $Y_j^E$ .

In order to easily produce a set cover in  $G$  from a (strong) edge geodetic set  $X$  of  $G'$ , we need  $X$  to respect a certain property. Hence, we use the following lemma.

**Lemma 4.** *Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) and  $G$  its graph resulting from Construction 2. Let  $X \subseteq V(G)$  be an edge geodetic set (resp. strong edge geodetic set) of  $G$ . It is possible to construct an edge geodetic set (resp. a strong edge geodetic set)  $X'$  of  $G$  such that  $|X'| \leq |X|$  and*

$$X' \subseteq \{v_i^2 \mid S_i \in C\} \cup D_1(G)$$

.

*Proof.* First by Property 1,  $X$  contains any vertices of  $D_1(G)$ . Therefore, by Lemma 2, every edge is (strongly) covered by  $D_1(G)$  except the  $e_j$  edges. Note that selecting a vertex  $v_i^2$  ensures to cover every  $e_j$  edges (with  $E_j \in S_i$ ) since the unique geodesic between  $v_i^1$  and  $u_j^0$  is described by  $lp_j^i \sim ep_j$ .

We show how to transform  $X$  so that it respects lemma's property. For each edge  $e_j$ , by Lemma 3, there is a vertex  $x$  in the intersection  $X \cap Y_j^E$ . If  $x \in Y_i^S$  for some  $S_i$  such that  $E_j \in S_i$ , we replace  $x$  in  $X$  by  $v_i^2$ . Since  $x$  belongs to any set  $Y_k^E$  such that  $S_k \in E_i$ , every edge previously covered by  $x$  is still covered. If  $x \in ep_j$  then it is only used to cover  $e_j$ . Thus, we can replace  $x$  arbitrarily by any vertex  $v_i^2$  such that  $E_j \in S_i$ . Finally, there exists a vertex that does not belong to  $\{v_i^2 \mid S_i \in C\} \cup D_1(G)$ , we can remove it from  $X$ . Hence, we obtain a solution as thought.

**Lemma 5.** *Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) and  $G$  its graph resulting from Construction 2. Then the instance  $(C, U)$  contains a set cover of size  $k$  if and only if  $G$  contains an edge geodetic set (resp. strong edge geodetic set) of size  $k + |C| + |U|$ .*

*Proof.* – Let  $C' \subseteq C$  be a set cover of size  $k$  of  $(C, U)$  and consider the (strong) edge geodetic set  $X = D_1(G) \cup \{v_i^2 \mid S_i \in C'\}$ . By Lemma 2, every edges

- in  $E(G) \setminus \{e_j \mid E_j \in U\}$  are (strongly) covered. Let  $E_j$  be an element of  $U$ , there is a set  $S_i \in S'$  such that  $E_j \in S_i$ . Thus,  $v_i^2 \in X$  and therefore the edge  $e_j$  is covered by the unique geodesic between  $v_i^2$  and  $u_{i,j}^0$ . Since every element of  $U$  appears in  $S'$ , every  $e_j$  edges of  $G$  are also (strongly) covered. Hence, we produce a (strong) edge geodetic set of size  $k + |C| + |U|$ .
- Let  $X \subset V(G)$  be a (strong) edge geodetic set of size  $k + |C| + |U|$  of  $G$  that respects the property of Lemma 4. Consider the set cover  $C' = \{S_i \mid v_i^2 \in X\}$ . Let  $E_j$  be an element of  $U$  and suppose that it does not belong to a set of  $C'$ . In that case, no vertex  $v_i^2$  such that  $E_j \in S_i$  belongs to  $X$ . By Lemma 3, the edge  $e_j$  is not covered by  $X$ , contradicting  $X$  being a (strong) edge geodetic set. Hence, every element of  $U$  is contained in a set of  $C'$  and, we construct a set cover of size  $k$ .

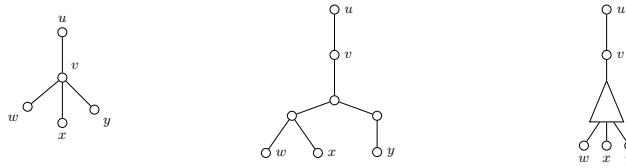
## 5.2 On subcubic bipartite graphs

We now extend the previous result to subcubic and bipartite graphs. First, we show that the result holds in graph with maximum degree three. We introduce the following construction.

**Construction 3.** Given a graph  $G$ , a vertex  $u \in V(G)$ , a set of non-adjacent neighbours  $N^0 = \{v_0^0, \dots, v_{k-1}^0\} \subseteq N(u)$  and an integer  $h > \log k$ , emplace a  $h$ -pyramid  $Py(h, u, N^0)$  consists of removing all edges between  $u$  and  $N^0$  and replacing them with the following subgraph. For each  $0 < i < h$ , construct recursively the sets  $N^i$ :

- create  $t = \lceil |N^{i-1}|/2 \rceil$  vertices  $v_0^i, \dots, v_t^i$ , and
- introduce the edges  $v_t^i, v_{2t'}^{i-1}$ , and  $v_t^i, v_{2t'+1}^{i-1}$  (if  $v_{2t'+1}^{i-1}$  exists) for each  $t' < t$ .

Finally, introduce the edge  $w_0^{h-1}$  ( $N^{h-1}$  consists of a single vertex since  $h > \log k$ ).



**Fig. 4.** Example of a 3-pyramid  $Py(3, v, \{w, x, y\})$  produced by Construction 3. **Left:**  $v$  and its neighbours in the original graph. **Center:** emplaced 3-pyramid. **Right:** Representation of the pyramid used in Figure 3.

Let  $Py(h, v, N)$  be a  $h$ -pyramid. We can make the following observations.

- The maximum degree of  $Py_h(h, v, N)$  is three.
- Let  $n_1, n_2 \in N$ , the distance between  $n_1$  and  $n_2$  in  $Py_h(h, v, N)$  is between 2 and  $2h$  and the distance between  $v$  and  $n_1$  or  $n_2$  is  $h$ .

We now use the previous structure to modify Construction 2 as follows.

**Construction 4.** Let  $(C, U)$  be an instance of SET COVER and  $G$  be its graph resulting from Construction 2. Let  $h > \log \Delta(G)$  be an integer. We modify  $G$  as follows:

- for each set  $S_i$  and each integer  $k \in \{1, 2\}$ , emplace a  $h$ -pyramid  $Py(h, v_i^k, N(v_i^k) \setminus \{v_i^{k-1}\})$ ,
- for each element  $E_j$  and each integer  $k \in \{1, 2, 3\}$ , emplace a  $h$ -pyramid  $Py(h, u_j^k, N(u_j^k) \setminus \{u_j^{k-1}\})$ , and
- replace each edge of  $G$  that does not belong to a  $h$ -pyramid by a path of length  $h$ .

Note that the resulting graph has maximum degree three. Moreover, if  $k$  is odd then the resulting graph is bipartite. Finally, by taking an arbitrary high value of  $k$ , the resulting graph has an arbitrary high girth. We use a similar vocabulary than for Construction 2: an *element tree*  $et_j$  is the tree induced by the vertices of  $ep_j$  in the  $h$ -pyramids emplaced in it in the original graph. A *set tree*  $st_i$  is defined the same way. For each element  $E_j$ , the  $h$ -path that replaces the edge  $e_j$  is denoted  $p_j$ . An example of a graph produced by Construction 4 is depicted in Figure 3. Since Construction 4 multiplies the length of every path of Construction 2 by  $h$ , we can adapt Lemmas 2 to 4 to it by replacing  $ep_j$  by  $et_j$  and  $sp_i$  by  $st_i$  in the geodesics descriptions.

**Lemma 6.** Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) be a connected graph and  $G$  its graph resulting from Construction 4. The set  $D_1(G)$  covers (resp. strongly cover) every edge and vertex of  $G$  except edges and vertices in  $\{p_j \mid E_j \in U\}$ .

*Proof.* By construction, geodesics between vertices of  $D_1(G)$  can be described the same way as for Lemma 2. The result follows.

In the following  $Y_i^S$  and  $Y_j^E$  are defined in the same way as for Construction 2 (by taking  $st_i$  instead of  $sp_i$  and  $et_j$  instead of  $ep_j$ ). Notice that the geodesic between a vertex  $v^1$  and a vertex  $u_j^0$  is described by  $st_i \sim lp_j^i \sim et_j$  and contains the path  $p_j$ . Hence, using the same arguments as in Lemmas 3 and 4, we can show the two following results.

**Lemma 7.** Let  $(C, U)$  be an instance of SET COVER and  $G$  be its graph resulting from Construction 4. For each element  $E_j$ , every geodesic containing a vertex of  $p_j$  has an extremity in  $Y_j^E$ .

**Lemma 8.** Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) and  $G$  be its graph resulting from Construction 4. Let  $X \subseteq V(G)$  be a geodesic or an edge geodesic set of  $G'$  (resp. strong geodesic or a strong edge geodesic set). It is possible to construct a geodesic or an edge geodesic set (resp. strong geodesic or a strong edge geodesic set)  $X'$  of  $G$  such that  $|X'| \leq |X|$  and

$$X' \subseteq \{v_i^2 \mid S_i \in C\} \cup D_1(G).$$

Using the same idea as for Lemma 5, we can now show that Construction 4 constitutes a reduction: if a path  $p_j$  is (strongly) covered, then there is a vertex  $v_i^2$ , such that  $E_j \in S_i$ , is selected. Thus, given a solution for a geodetic problem  $X$  the set  $\{S_i \mid v_i^2 \in X\}$  is a set cover of  $G$ . Hence, we obtain the following result.

**Lemma 9.** *Let  $(C, U)$  be an instance of SET COVER (resp.  $(1, k')$ -SET COVER) and  $G$  its graph resulting from Construction 2. Then the instance  $(C, U)$  contains a set cover of size  $k$  if and only if  $G$  contains an edge geodetic set and a geodetic set (resp. strong edge geodetic set and a strong geodetic set) of size  $k + |C| + |U|$ .*

## 6 Non-approximability

In this section, we use the results of the previous section to find hardness of approximation results for the geodetic problems.

### 6.1 Strong geodetic set and strong edge geodetic set

First, recall the definition of  $L$ -reduction between two hard problems  $\Pi$  and  $\Pi'$  (with respective cost functions  $val_\Pi$  and  $val_{\Pi'}$ ), as described by Papadimitriou and Yannakakis [19]. Let  $OPT_\Pi(x)$  and  $OPT_{\Pi'}(x)$  be the optimal value of  $val_\Pi$  and  $val_{\Pi'}$  on an instance  $x$ , respectively. An  $L$ -reduction consists of polynomial-time computable functions  $f$  and  $g$  such that, for each instance  $x$  of  $\Pi$ ,  $f(x)$  is an instance of  $\Pi'$  and for each feasible solution  $y'$  for  $f(x)$ ,  $g(y')$  is a feasible solution for  $x$ . Moreover, there are constants  $\alpha_1, \alpha_2 > 0$  such that:

1.  $OPT_{\Pi'}(f(x)) \leq \alpha_1 \cdot OPT_\Pi(x)$  and
2.  $|val_\Pi(g(y')) - OPT_\Pi(x)| \leq \alpha_2 \cdot |val_{\Pi'}(y') - OPT_{\Pi'}(f(x))|$ .

Using Construction 4, we obtain an  $L$ -reduction with  $\alpha_1 = (2k' + 2)$  and  $\alpha_2 = 1$ .

**Lemma 10.** *Let  $\rho_{k'}$  be the best possible polynomial time approximation factor of  $(1, k')$ -MINIMUM SET COVER. Then STRONG GEODETIC NUMBER and STRONG EDGE GEODETIC NUMBER cannot be approximated with a factor better than*

$$1 + \frac{\rho_{k'} - 1}{2k' + 2},$$

*in subcubic bipartite graphs with arbitrary high girth.*

*Proof.* Let  $(C, U)$  be an instance of  $(1, k')$ -MINIMUM SET COVER and  $G$  its graph resulting from Construction 4. Suppose there is a polynomial-time approximation algorithm for STRONG GEODETIC NUMBER (resp. STRONG EDGE GEODETIC NUMBER) in subcubic bipartite graphs with arbitrary high girth and let  $X_{app}$  be a strong geodetic set (resp. strong edge geodetic set) computed by this algorithm in  $G$ . We suppose that  $X_{app}$  respects the property of Lemma 8. Let  $C'_{app} = \{S_i \mid v_i^2 \in X_{app}\}$ . Using the same argument as in Lemma 9, we can show that  $C'_{app}$  is a set cover of  $(C, U)$  and  $|X_{app}| = |C| + |U| + |C'_{app}|$ . Let  $C'_{opt} \subset C$  be a set cover of  $(C, U)$  and let  $X_{opt}$  be a minimum strong geodetic set (resp. strong edge

geodetic set) in  $G$ . Similarly, we can show that  $|X_{opt}| = |C| + |U| + |C'_{opt}|$ . Hence, we have

$$|X_{app}| - |X_{opt}| = |C'_{app}| - |C'_{opt}|. \quad (1)$$

Moreover, since the intersection between two sets contains at least two elements and the size of each set is bounded by  $k'$ , we deduce

$$|U| \leq k' \cdot |C'_{opt}| \quad \text{and} \quad |C| \leq (k' + 1) \cdot |C'_{opt}|$$

which leads to

$$|X_{opt}| \leq (2k' + 2) \cdot |C'_{opt}|. \quad (2)$$

Thus we construct a  $L$ -reduction with  $\alpha_1 = (2k' + 2)$  and  $\alpha_2 = 1$ . We conclude,

$$\begin{aligned} |X_{app}| &\stackrel{(1)}{=} |C'_{app}| - |C'_{opt}| + |X_{opt}| \\ &\geq (\rho_{k'} - 1) \cdot |C'_{opt}| + |X_{opt}| \\ &\stackrel{(2)}{\geq} \left(1 + \frac{\rho_{k'} - 1}{2k' + 2}\right) \cdot |X_{opt}|. \end{aligned}$$

Since MINIMUM VERTEX COVER with bounded maximum degree  $k'$  is a particular case of  $(1, k')$ -MINIMUM SET COVER, we can pick the value of  $k'$  (and so the corresponding best-known value  $\rho_{k'}$ ) that maximize the previous inapproximation ratio. Thus, since Berman and Karpinski showed that MINIMUM VERTEX COVER cannot be approximated with a factor better than  $79/78$  in graphs with maximum degree four [4], we obtain the following result.

**Corollary 2.** STRONG GEODETIC NUMBER *and* STRONG EDGE GEODETIC NUMBER *cannot be approximated with a factor better than  $781/780$  in subcubic bipartite graphs with arbitrary high girth.*

## 6.2 Geodetic set and edge geodetic set

Now, we provide approximation lower bounds for GEODETIC NUMBER and STRONG GEODETIC NUMBER. We apply the following modification to Construction 4.

**Construction 5.** *Let  $(C, U)$  be an instance of SET COVER,  $G$  be its graph produced by Construction 4 and  $k > |V(G)|$  be an integer. We construct a graph  $G'$  as follows:*

- create  $k$  disjoint copies  $\{G_1, \dots, G_k\}$  of  $G$ ,
- for each vertex  $x$  of  $D_1(G)$ ,
  - create an edge  $s_x^0 s_x^1$ ,
  - for each  $G_\ell \in \{G_1, \dots, G_k\}$  and for each vertex  $x \in D_1(G_\ell)$ , construct a  $k$ -path  $p_x^\ell$  between  $x$  and  $s_x^1$ , and
  - emplace a  $k$ -pyramid  $Py(k, s_x^1, N(s_x^1) \setminus \{s_x^0\})$ .

Notice that the resulting graph has maximum degree three. For simplicity, we denote the  $k$ -pyramide  $Py(k, s_x^1, N(s_x^1) \setminus \{s_x^0\})$  as  $Py[x]$ .

**Lemma 11.** *Let  $(C, U)$  be an instance of SET COVER and let  $G'$  be its graphs resulting from Construction 5. The set  $D_1(G')$  covers every edge and vertex of  $G'$  except edges and vertices in  $\{p_j \mid E_j \in U, G_\ell \in \{G_1, \dots, G_k\}, p_j \in G_\ell\}$ .*

*Proof.* Let  $G$  be the graph of  $(C, U)$  produced Construction 4. Let  $x \in D_1(G)$  and  $G_\ell$  in  $\{G_1, \dots, G_k\}$ . The shortest path between  $G_\ell$  and  $s_x^0$  has length  $2k$  and walks through  $p_k^\ell$  and  $Py[x]$ . Let  $x_1$  and  $x_2$  be two vertices of  $D_1(G)$  and  $x'_1$  and  $x'_2$  be their corresponding vertices in  $G_\ell$ . Let  $g_\ell$  be a geodesic between  $x'_1$  and  $x'_2$ , note that  $g_\ell$  is entirely contained in  $G_\ell$  since every path leaving  $G_\ell$  has length  $k > |V(G_\ell)|$ . Suppose that  $g = s_{x_1}^0 s_{x_1}^1 \sim Py[x_1] \sim p_{x_1}^\ell \sim g_\ell \sim p_{x_2}^\ell \sim Py[x_2] \sim s_{x_2}^0 s_{x_2}^1$  is not a geodesic between  $s_{x_1}^0$  and  $s_{x_2}^0$ . Then, let  $g'$  be a geodesic between  $s_{x_1}^0$  and  $s_{x_2}^0$ . If  $g'$  contains two subgraph  $G_\ell$  and  $G_{\ell'}$ , then since the distance between  $G_\ell$  and  $G_{\ell'}$  is  $2k$ , the length of  $g'$  is at least  $6k$  and then  $g$  is shorter than  $g'$ . Thus,  $g'$  contains only one subgraph  $G_\ell$  and then, it can be described by  $s_{x_1}^0 s_{x_1}^1 \sim Py[x_1] \sim p_{x_2}^\ell \sim g'_\ell \sim p_{x_2}^\ell \sim Py[x_2] \sim s_{x_2}^0 s_{x_2}^1$ , where  $g'_\ell$  is a path between  $x'_1$  and  $x'_2$  in  $G_\ell$ , but then it contradicts  $g_\ell$  being a geodesic. Hence,  $g$  is a geodesic between  $s_{x_1}^0$  and  $s_{x_2}^0$ . From that, we can conclude that  $D_1(G')$  covers exactly the same edges and vertices than  $D_1(G)$  in each  $G_\ell$ . Hence, by Lemma 6, every edge and vertex in any  $G_\ell$  is covered, except edges and vertices in  $\{p_j \mid E_j \in U\}$ .

It remains to show that edges and vertices in every  $Py[x]$  and  $p_x^\ell$  are also covered. First, since for every  $x \in D_1(G)$  and every  $G_\ell$ , there is a geodesic  $g$  with extremity  $s_x^0$  that walks through  $G_\ell$ , then  $g$  contains  $p_x^\ell$  and thus every edge and vertex of  $p_x^\ell$  is covered. Moreover, since every  $p_x^\ell$  is adjacent to a distinct leaf of  $Py[x]$ , the set of geodesic with extremity  $s_x^0$  covers every edge and vertex of  $Py[x]$ . The result follows.

**Lemma 12.** *Let  $(C, U)$  be an instance of SET COVER and let  $G'$  be its graph resulting from Construction 5. Let  $X \subseteq V(G')$  be a geodetic set or an edge geodetic set of  $G'$ . It is possible to construct a geodetic set or an edge geodetic set  $X'$  of  $G'$  such that  $|X'| \leq |X|$  and*

$$X' \subseteq \{v_i^2 \mid S_i \in C, G_\ell \in \{G_1, \dots, G_k\}, v_i^2 \in V(G_\ell)\} \cup D_1(G').$$

*Proof.* Since for each  $G_\ell \in \{G_1, \dots, G_k\}$ , the result of Lemma 7 holds in  $G_\ell$ , we can use the same technique as for Lemma 8 to obtain a set  $X'$  such that  $X' \cap V(G_\ell) = \{v_i^2 \mid S_i \in C\}$ . Then, the result follows.

SET COVER is hard to approximate with a factor better than a logarithmic function [9]. Therefore, we can transfer the lower bounds of approximation of SET COVER to GEODETIC NUMBER and EDGE GEODETIC NUMBER. This result is in addition to the one proved by Chakraborty *et al.* [7].

**Theorem 3.** *GEODETIC NUMBER and EDGE GEODETIC NUMBER are LOG-APX-hard, even in bipartite subcubic graphs with arbitrary high girth.*



*Proof.* Let  $(C, U)$  be an instance SET COVER graph and  $G'$  be its graph produced by Construction 5. Suppose that we have a polynomial-time approximation algorithm  $A$  to compute a geodetic set or an edge geodetic set of a graph. We denote by  $X$  the geodetic set obtained by  $A$  in  $G'$  and we suppose that  $X$  respects Lemma 12 property. We can suppose that  $|X \cap V(G_\ell)|$  has the same value for any  $G_\ell \in \{G_1, \dots, G_k\}$ , since otherwise it suffices to transpose the solution with the minimum  $|X \cap V(G_\ell)|$  value to the every other  $G_\ell$ . Let  $C' = \{S_i \mid v_i^2 \in X \cap V(G_1)\}$ . Using the same argument as in the proof of Lemma 5, we can show that  $C$  is a set cover of  $(C, U)$ , and that  $|X| = k|C'| + |C| + |U|$ .

Let  $X_{opt}$  be a minimum geodetic set or a minimum edge geodetic set of  $G'$  and  $C'_{opt}$  be a minimum set cover of  $(C, U)$ . Similarly, we can show that  $|X_{opt}| = k|C'_{opt}| + |U| + |C|$ . We have:

$$\frac{|X|}{|X_{opt}|} = \frac{k|C'| + |U| + |C|}{k|C'_{opt}| + |U| + |C|} \quad (3)$$

As  $k$  tends to  $+\infty$  we deduce that  $\frac{|X|}{|X_{opt}|}$  converges to  $|C'|/|C'_{opt}|$ . Since SET COVER cannot be approximated with a factor better than a logarithmic function, we deduce that  $A$  cannot have an approximation factor better than a logarithmic function.

## 7 Strong Geodetic Number on complete multipartite graphs

First, remark that geodesics of complete multipartite graphs are easy to determine: for any pair of vertices which are not in the same part, the edge between them is the unique shortest path between them. For a pair of vertices which are in the same part, the shortest paths between them are all the paths of length two between them with all the vertices not in this part as middle vertices.

In this section, we develop a polynomial algorithm which computes the strong geodetic number of a complete multipartite graph. The algorithm is based on dynamic programming where we not only look after a minimum strong geodetic set of vertices covering all the graph, but we look after all sets of vertices maximizing the number of pairs not used to cover other vertices among sets with some fixed parameters.

Let  $K_{n_1, \dots, n_r}$  denotes a complete multipartite graph whose parts are noted  $X_1, \dots, X_r$  such that  $|X_i| = n_i$  for every  $i \in \{1, \dots, r\}$ . We denote  $N_i = \sum_{j=1}^i n_j$  for every  $i \in \{1, \dots, r\}$  and  $K_{n_1, \dots, n_i}$  by  $G_i$ .

**Definition 1.** A selection of  $K_{n_1, \dots, n_r}$  is a set of selected vertices  $S$  in which we pick a set of pairs of non-adjacent vertices  $C$  to cover some non-selected vertices. Formally, a selection is a triplet  $(S, C, f)$ , where

- $S \subseteq V$ ,
- $C \subseteq \bigcup_{j=1}^r \mathcal{P}_2(S \cap X_j)$  and,

- $f : C \rightarrow V \setminus S$  is an injective map such that  $\forall c \in C \cap \mathcal{P}_2(S \cap X_i), f(c) \notin X_i$  (i.e. two vertices of  $X_i$  can not cover another vertex of  $X_i$ ).

Given a selection  $s(S, C, f)$ , we denote by

- $s(S, C, f) = |S|$ , the number of selected vertices,
- $r(S, C, f) = |V \setminus (S \sqcup f(C))| = n - s(S, C, f) - |C|$ , the number of vertices that are neither selected nor covered, and
- $d(S, C, f) = |\cup_{j=1}^n \mathcal{P}_2(S \cap X_j) \setminus C| = \sum_{j=1}^n \binom{|S \cap X_j|}{2} - |C|$ , the number of pairs of non-adjacent vertices that are not in  $C$ .

We say that a selection  $(S, C, f)$  is nice if it maximizes the number  $d(S, C, f)$  among all selection with the same size  $s(S, C, f)$  and the same number  $r(S, C, f)$ . In our dynamic programming approach, we construct every nice selections in every subgraph  $G_i$ , successively. To construct a selection in  $G_i$  from a selection of  $G_{i-1}$ , we use the following lemma.

**Lemma 13.** *Let  $(S', C', f')$  a selection of  $G_{i-1}$ . Let three integers  $k, u$  and  $q$  such that  $0 \leq k \leq n_i$ ,  $0 \leq u \leq \min(n_i - k, d(S', C', f'))$  and  $0 \leq q \leq \min(\binom{k}{2}, r(S', C', f'))$ . There exists a selection  $(S, C, f)$  of  $G_i$  such that*

$$\begin{cases} s(S, C, f) &= s(S', C', f') + k \\ r(S, C, f) &= r(S', C', f') + n_i - k - q - u \\ d(S, C, f) &= d(S', C', f') + \binom{k}{2} - q - u. \end{cases}$$

In other words, we select  $k$  vertices in  $X_i$ , cover  $u$  vertices in  $X_i$  and cover  $q$  vertices in  $G_{i-1}$ .

*Proof.* Let  $T$  be any subset of  $X_i$  of size  $k$ . Let  $Q$  be any subset of  $\mathcal{P}_2(T)$  of size  $q$ . Let  $U$  be any subset of  $\bigcup_{j=1}^{i-1} \mathcal{P}_2(S' \cap X_j) \setminus C'$  of size  $u$ . Let  $Q'$  be any subset of  $\bigcup_{j=1}^{i-1} X_j \setminus (S' \cup f'(C'))$  of size  $q$ . Let  $U'$  be any subset of  $X_i \setminus T$  of size  $u$ .

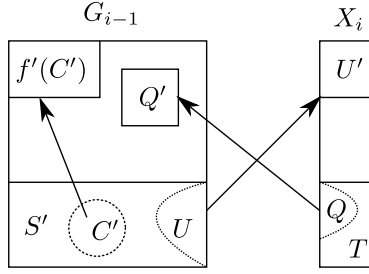
We define

$$\begin{aligned} S &= S' \cup T \\ C &= C' \cup Q \cup U \\ f : C &\rightarrow V \setminus S \end{aligned}$$

where the function  $f$  is defined as follows. On  $C'$ ,  $f|_{C'} = f'$ . As  $U$  (resp.  $Q$ ) and  $U'$  (resp.  $Q'$ ) are of the same size, we can define  $f|_U$  (resp.  $f|_Q$ ) as any bijection between  $U$  and  $U'$  (resp.  $Q$  and  $Q'$ ).

Let us show that  $(S, C, f)$  is a selection of  $G_i$ . As  $S' \subseteq \bigcup_{j=1}^{i-1} X_j$  and  $T \subseteq X_i$ , we deduce that  $S \subseteq \bigcup_{j=1}^i X_j$ . As  $C' \subseteq \bigcup_{j=1}^{i-1} \mathcal{P}_2(S \cap X_j)$ ,  $Q \subseteq \mathcal{P}_2(S \cap X_i)$  and  $U \subseteq \bigcup_{j=1}^{i-1} \mathcal{P}_2(S \cap X_j)$ , we deduce that  $C \subseteq \bigcup_{j=1}^i \mathcal{P}_2(S \cap X_j)$ .

By definition of  $f$ ,  $f|_{C'}$ ,  $f|_Q$  and  $f|_U$  are injective. Let  $x, x' \in C$  such that  $f(x) = f(x')$ . As  $f(U) \subseteq X_i$  and  $f(Q)$  and  $f(C') \subseteq \bigcup_{j=1}^{i-1} X_j$ , we deduce that we can suppose that  $x \in Q$  and  $x' \in C'$ . As  $f(x) \in Q'$ , we have  $f(x) \notin f'(C')$ . It contradicts the fact that  $f(x) = f(x') \in f'(C')$  because  $x' \in C'$ . We conclude that  $x = x'$  and that  $f$  is injective.



**Fig. 5.** Description of the creation of selection  $(S, C, f)$  of  $G_i$  from the selection  $(S', C', f')$  of  $G_{i-1}$ . An arrow between two set indicates that pairs of vertices of the source set are used to cover vertices of the target set.

Let  $a \in C \cap \mathcal{P}_2(S \cap X_i) = Q$ . Then  $f(a) \in Q' \subseteq \bigcup_{j=1}^{i-1} X_j$ . Thus,  $f(a) \notin X_i$ . Let  $a \in C \cap \mathcal{P}_2(S \cap X_j) = C' \cup U$  for any  $j < i$ . If  $a \in C'$ , then  $f(a) = f'(a) \notin X_j$  because  $(S', C', f')$  is a selection of  $G_{i-1}$ . Otherwise,  $a \in U$  and  $f(a) \in X_i$  and thus  $f(a) \notin X_j$ .

We conclude that  $(S, C, f)$  is a selection of  $G_i$ .

As  $S = S' \sqcup T$ , we deduce that  $s(S, C, f) = s(S', C', f') + |T| = s(S', C', f') + k$ . As  $C = C' \sqcup Q \sqcup U$ , we deduce that  $r(S, C, f) = N_i - |S| - |C| = n_i + N_{i-1} - |S'| - k - |C'| - |Q| - |U| = n_i - k - q - u + r(S', C', f')$ . Furthermore

$$\begin{aligned}
 d(S, C, f) &= \sum_{j=1}^i \binom{|S \cap X_j|}{2} - |C| \\
 &= \sum_{j=1}^{i-1} \binom{|S' \cap X_j|}{2} + \binom{|T|}{2} - |C'| - q - u \\
 &= d(S', C', f') + \binom{k}{2} - q - u
 \end{aligned}$$

Since the number of selections can be exponential, we only keep nice selections. In order to do that, we adapt the result of Lemma 13 as follows.

**Lemma 14.** *Let  $(S, C, f)$  be a nice selection of  $G_i$ , then there exists a nice selection  $(S', C', f')$  of  $G_{i-1}$  and numbers  $k, u, q$  such that  $0 \leq k \leq n_i$ ,  $0 \leq q \leq \min(\binom{k}{2}, r(S', C', f'))$  and  $0 \leq u \leq \min(n_i - k, d(S', C', f'))$  and*

$$\begin{cases}
 s(S, C, f) = s(S', C', f') + k \\
 r(S, C, f) = r(S', C', f') + n_i - u - q - k \\
 d(S, C, f) = d(S', C', f') + \binom{k}{2} - q - u
 \end{cases}$$

*Proof.* We define  $S' = S \cap G_{i-1}$ ,  $C' = C \cap G_{i-1} \setminus f^{-1}(X_i)$  and  $f' = f|_{C'}$ . Let us prove that  $(S', C', f')$  is a nice selection of  $G_{i-1}$  satisfying the announced equalities.

Let us show that  $\forall c \in C' \cap \mathcal{P}_2(S \cap X_i), f'(c) \notin X_i$ . Let  $c \in C' \cap \mathcal{P}_2(S \cap X_i)$ , suppose by contradiction that  $f'(c) \in X_i$ . As  $f'(c) = f(c)$ , we deduce that  $c \in f^{-1}(X_i)$ . This contradicts the definition of  $C'$ . Thus,  $f'(c) \notin X_i$ . As  $f$  is injective,  $f'$  is also injective and, we deduce that  $(S', C', f')$  is a selection of  $G_{i-1}$ .

We define  $k = |S \cap X_i|$ ,  $q = |C' \cap \mathcal{P}_2(S \cap X_i)|$  and  $u = |f^{-1}(X_i)|$ . As  $S = S' \sqcup (S \cap X_i)$ , we have  $s(S) = s(S') + k$ . As we have  $C = C' \sqcup f^{-1}(X_i) \sqcup (C' \cap \mathcal{P}_2(S \cap X_i))$ , thus  $|C| = |C'| + u + q$ . We deduce that  $r(S) = N_i - |S| - |C| = n_i + N_{i-1} - s(S', C', f') - k - |C'| - u - q = r(S', C', f') + n_i - k - u - q$ . Furthermore,

$$\begin{aligned} d(S, C, f) &= \sum_{j=1}^i \binom{|S \cap X_j|}{2} - |C| \\ &= \sum_{j=1}^{i-1} \binom{|S' \cap X_j|}{2} + \binom{k}{2} - |C'| - q - u \\ &= d(S', C', f') + \binom{k}{2} - q - u \end{aligned}$$

We have  $k = |S \cap X_i| \leq |X_i| \leq n_i$ . We have  $q \leq |\mathcal{P}_2(S \cap X_i)| \leq \binom{k}{2}$ . As  $r(S, C, f) = r(S', C', f') + n_i - k - u - q$ , we have  $q = r'(S', C', f') - r(S, C, f) + n_i - k - u \leq r'(S', C', f')$  as we have  $n_i \leq r(S, C, f) + k + u$  as  $r(S, C, f)$  is the size of the set  $\bigcup_{j=1}^i X_j \setminus (S \sqcup f(C))$  and as  $X_i \setminus ((S \cap X_i) \sqcup f(f^{-1}(X_i))) \subseteq \bigcup_{j=1}^i X_j \setminus (S \sqcup f(C))$ . Thus,  $q \leq r(S', C', f')$ .

As  $(S \cap X_i) \sqcup f(f^{-1}(X_i)) \subseteq X_i$ , we deduce that  $k + u \leq n_i$  (as  $f$  is injective), thus  $u \leq n_i - k$ . As  $d(S, C, f)$  is the size of the set  $\bigcup_{j=1}^i \mathcal{P}_2(S \cap X_j) \setminus C$ , we have  $\mathcal{P}_2(S \cap X_i) \setminus (C' \cap \mathcal{P}_2(S \cap X_i)) \subseteq \bigcup_{j=1}^i \mathcal{P}_2(S \cap X_j) \setminus C$ . Thus,  $\binom{k}{2} - q \leq d(S, C, f)$  and so,  $u = d(S', C', f') - d(S, C, f) + \binom{k}{2} - q \leq d(S', C', f')$ .

We conclude that  $k, q, u$  satisfy the following inequalities:

$$\begin{aligned} 0 &\leq k \leq n_i \\ 0 &\leq q \leq \min\left(\binom{k}{2}, r(S', C', f')\right) \\ 0 &\leq u \leq \min(n_i - k, d(S', C', f')) \end{aligned}$$

Let us show that  $(S', C', f')$  is a nice selection of  $G_{i-1}$ . Suppose by contradiction that there exists a selection  $(S'', C'', f'')$  of  $G_{i-1}$  such that  $s(S'', C'', f'') = s(S', C', f')$  and  $r(S'', C'', f'') = r(S', C', f')$  and that  $d(S'', C'', f'') > d(S', C', f')$ . According to Lemma 13, there exists a selection  $(Sp, Cp, fp)$  of  $G_i$  such that

$$\begin{cases} s(Sp, Cp, fp) &= s(S'', C'', f'') + k \\ r(Sp, Cp, fp) &= r(S'', C'', f'') + n_i - k - q - u \\ d(Sp, Cp, fp) &= d(S'', C'', f'') + \binom{k}{2} - q - u \end{cases}$$

As  $s(S'', C'', f'') = s(S', C', f')$ ,  $r(S'', C'', f'') = r(S', C', f')$  and  $d(S'', C'', f'') > d(S', C', f')$ , we obtain

$$\begin{aligned}
s(Sp, Cp, fp) &= s(S, C, f) \\
r(Sp, Cp, fp) &= r(S, C, f) \\
d(Sp, Cp, fp) &> d(S, C, f)
\end{aligned}$$

Which contradicts that  $(S, C, f)$  is a nice selection of  $G_i$ . We conclude that  $(S', C', f')$  is a nice selection of  $G_{i-1}$ .

We denote by  $d(i, j, r)$  the maximum of  $d(S, C, f)$  for any selection  $(S, C, f)$  of  $G_i$  such that  $s(S, C, f) = j$  and  $r(S, C, f) = r$ . This quantity is set to  $-\infty$  if no such selection of  $G_i$  exists.

**Lemma 15.** *For any integers  $i, s, r$  and integers  $k, u, q$  we define the following quantities:*

$$s' = s - k, \quad r' = r - n_i + u + q + k \quad \text{and} \quad d' = d(i - 1, s', r')$$

$$\text{We deduce that: } d(i, s, r) = \max \left\{ (d') + \binom{k}{2} - q - u \left| \begin{array}{l} 0 \leq k \leq n_i \\ 0 \leq u \leq \min(n_i - k, d') \\ 0 \leq q \leq \min(\binom{k}{2}, r') \end{array} \right. \right\}$$

*Proof.* According to Lemma 13, we have

$$d(i, s, r) \leq \max \left\{ (d') + \binom{k}{2} - q - u \left| \begin{array}{l} 0 \leq k \leq n_i \\ 0 \leq u \leq \min(n_i - k, d') \\ 0 \leq q \leq \min(\binom{k}{2}, r') \end{array} \right. \right\}$$

as for any selection  $(S', C', f')$  of  $G_{i-1}$  such that  $s(S', C', f') = s', r(S', C', f') = r'$  and  $d(S', C', f') = d'$ , we can create a selection  $(S, C, f)$  of  $G_i$  such that  $s(S, C, f) = s, r(S, C, f) = r$  and  $d(S, C, f) = d' + \binom{k}{2} - q - u$ .

According to Lemma 14, we have

$$d(i, s, r) \geq \max \left\{ (d') + \binom{k}{2} - q - u \left| \begin{array}{l} 0 \leq k \leq n_i \\ 0 \leq u \leq \min(n_i - k, d') \\ 0 \leq q \leq \min(\binom{k}{2}, r') \end{array} \right. \right\}$$

as for any selection  $(S, C, f)$  of  $G_i$  such that  $s(S, C, f) = s, r(S, C, f) = r$  and  $d(S, C, f) = d(i, s, r)$  we can create a nice selection  $(S', C', f')$  of  $G_{i-1}$  such that  $s(S', C', f') = s', r(S', C', f') = r'$  and  $d(S', C', f') = d(S, C, f) - \binom{k}{2} + q + u$ . We deduce the equality between the two quantities.

From previous lemma we deduce the following theorem.

**Theorem 4.** *There exists an algorithm in  $O(n^8)$  computing the geodetic number of a complete multipartite graph with  $n$  vertices.*

*Proof.* The algorithm 2 consists mainly in six independent “for” loops. Four of them are of length at most  $n$  and two of them are of length at most  $n^2$ . The complexity of the algorithm is therefore in  $O(n^8)$ .

Notice that a complete multipartite graph can be described with the list of integers  $n_1, \dots, n_k$ . In that case, the dynamic programming that we described is not polynomial if the values of the  $n_i$  are exponential. Thus, if we formulate EDGE GEODETIC NUMBER on complete multipartite graph as a specific problem on this class, the question whether such a problem is weak  $\mathcal{NP}$ -hard or not is open.

## 8 Conclusion

In this paper, we investigated the hardness of the approximation of the geodetic set problems. Given our approximation lower bound for GEODETIC NUMBER and EDGE GEODETIC NUMBER, the question of the existence of a  $O(\log(n))$ -approximation algorithm seems natural. We also proved that deciding whether a set admits a geodesic assignment NP-hard. Therefore, a second question arises: is it also hard to decide whether a set of vertices is a strong geodetic set. We also give a tight 2-approximation of this problem. Finding a lower bound for this problem is probably a good question for further work. Finally, for STRONG GEODETIC NUMBER, we proved that it was polynomial on complete multipartite graphs. What about other graph classes?

## References

1. Anand, B.S., Changat, M., Chandran, S.U.: The edge geodetic number of product graphs. In: Conference on Algorithms and Discrete Applied Mathematics. pp. 143–154. Springer (2018)
2. Arokiaraj, A., Klavzar, S., Manuel, P.D., Thomas, E., Xavier, A.: Strong geodetic problems in networks. *Discuss. Math. Graph Theory* **40**(1), 307–321 (2020)
3. Atici, M.: On the edge geodetic number of a graph. *International Journal of Computer Mathematics* **80**(7), 853–861 (2003)
4. Berman, P., Karpinski, M.: On some tighter inapproximability results (extended abstract). In: Automata, Languages and Programming, 26th International Colloquium, ICALP’99, Prague, Czech Republic, July 11–15, 1999, Proceedings. pp. 200–209 (1999)
5. Bresar, B., Kovse, M., Tepeh, A.: Geodetic sets in graphs. In: Dehmer, M. (ed.) *Structural Analysis of Complex Networks*, pp. 197–218. Birkhäuser / Springer (2011)
6. Chakraborty, D., Das, S., Foucaud, F., Gahlawat, H., Lajou, D., Roy, B.: Algorithms and Complexity for Geodetic Sets on Planar and Chordal Graphs. In: Cao, Y., Cheng, S.W., Li, M. (eds.) *31st International Symposium on Algorithms and Computation (ISAAC 2020)*. Leibniz International Proceedings in Informatics (LIPIcs), vol. 181, pp. 7:1–7:15. Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2020)
7. Chakraborty, D., Foucaud, F., Gahlawat, H., Ghosh, S.K., Roy, B.: Hardness and approximation for the geodetic set problem in some graph classes. In: Changat, M., Das, S. (eds.) *Algorithms and Discrete Applied Mathematics - 6th International Conference, CALDAM 2020, Hyderabad, India, February 13–15, 2020, Proceedings*. Lecture Notes in Computer Science, vol. 12016, pp. 102–115. Springer (2020)
8. Darmann, A., Döcker, J.: On simplified NP-complete variants of monotone 3-sat. *Discret. Appl. Math.* **292**, 45–58 (2021)

9. Dinur, I., Steurer, D.: Analytical approach to parallel repetition. In: Proceedings of the Forty-Sixth Annual ACM Symposium on Theory of Computing, p. 624–633. STOC '14, Association for Computing Machinery, New York, NY, USA (2014)
10. Doughat, A., Kong, M.: Computing geodetic bases of chordal and split graphs. *Journal Of Combinatorial Mathematics And Combinatorial Computing* **22**, 67–78 (1996)
11. Douthat, A., Kong, M.: Computing the geodetic number of bipartite graphs. *Congressus Numerantium* pp. 113–120 (1995)
12. Gledel, V., Irsic, V., Klavzar, S.: Strong geodetic cores and cartesian product graphs. *Appl. Math. Comput.* **363** (2019)
13. Harary, F., Loukakis, E., Tsouros, C.: The geodetic number of a graph. *Mathematical and Computer Modelling* **17**(11), 89–95 (1993)
14. Iršič, V.: Strong geodetic number of complete bipartite graphs and of graphs with specified diameter. *Graphs Comb.* **34**(3), 443–456 (2018)
15. Iršič, V., Konvalinka, M.: Strong geodetic problem on complete multipartite graphs. *Ars Math. Contemp.* **17**(2), 481–491 (2019)
16. Klavzar, S., Manuel, P.D.: Strong geodetic problem in grid-like architectures. *Bulletin of the Malaysian Mathematical Sciences Society* **41**(3), 1671–1680 (2018)
17. Manuel, P., Klavžar, S., Xavier, A., Arokiaraj, A., Thomas, E.: Strong edge geodetic problem in networks. *Open Mathematics* **15**(1), 1225–1235 (2017)
18. Mezzini, M.: An  $O(mn^2)$  algorithm for computing the strong geodetic number in outerplanar graphs. *Discuss. Math. Graph Theory* (2020)
19. Papadimitriou, C.H., Yannakakis, M.: Optimization, approximation, and complexity classes **43**(3), 425–440 (1991)
20. Rehmani, S., Sunitha, M.: Edge geodesic number of a fuzzy graph. *Journal of Intelligent & Fuzzy Systems* **37**(3), 4273–4286 (2019)
21. Santhakumaran, A., Ullas Chandran, S.: The edge geodetic number and cartesian product of graphs. *Discussiones Mathematicae Graph Theory* **30**(1), 55–73 (2010)
22. Santhakumaran, A., John, J.: Edge geodetic number of a graph. *Journal of Discrete Mathematical Sciences and Cryptography* **10**(3), 415–432 (2007)
23. Santhakumaran, A., Ullas Chandran, S.: Comment on” edge geodetic covers in graphs. *Proyecciones (Antofagasta)* **34**(4), 343–350 (2015)
24. Weller, M., Chateau, A., Giroudeau, R., König, J., Pollet, V.: On residual approximation in solution extension problems. *J. Comb. Optim.* **36**(4), 1195–1220 (2018)
25. Zmazek, E.: Strong edge geodetic problem on grids. *Bulletin of the Malaysian Mathematical Sciences Society* pp. 1–20 (2021)

---

**Algorithm 2:** Strong Geodetic Number Algorithm For Complete Multipartite Graphs

---

**Data:**  $L$  the list of the size of the parts of the complete multipartite graph  $G$   
**Result:** The strong geodetic number of  $G$

```

1  $n \leftarrow \text{sum}(L)$ 
2  $k \leftarrow \text{len}(L)$ 
3  $nc \leftarrow 0$ 
4 for  $jin(0..L[0])$  do
5    $d[0][j][L[0] - j] \leftarrow j * (j - 1)/2$ 
6 end
7 for  $iin(0..k - 2)$  do
8    $nc \leftarrow nc + L[i]$ 
9   for  $jin(0..nc)$  do
10    for  $rin(0..nc)$  do
11      if  $d[i][j][r] == -1$  then
12        continue
13      end
14       $dc \leftarrow d[i][j][r]$ 
15      for  $sin(0..L[i + 1])$  do
16        for  $spin(0..s * (s - 1)/2)$  do
17          for  $dpin(0..dc)$  do
18             $rp \leftarrow \text{max}(0, r - sp) + \text{max}(0, L[i + 1] - s - dp)$ 
19             $dd \leftarrow s * (s - 1)/2 - sp + \text{max}(0, -(r - sp)) +$ 
20               $\text{max}(0, -(L[i + 1] - s - dp)) + dc - dp$ 
21            if  $dd > d[i + 1][j + s][rp]$  then
22               $d[i + 1][j + s][rp] = dd$ 
23            end
24          end
25        end
26      end
27    end
28  end
29   $j \leftarrow 0$  ;
30  while  $d[k - 1][j][0] < 0$  do
31     $j \leftarrow j + 1$ 
32  end
33  return  $j$ ;

```

---