On the Approximation of Degree Constrained Spanning Problems in Graphs under Non Uniform Capacity Constraints

Miklós Molnár

On the Approximation of Degree Constrained Spanning Problems in Graphs under Non Uniform Capacity Constraints

Miklós Molnár

LIRMM, University of Montpellier, CNRS, Montpellier, France
`molnar@lirmm.fr`

Abstract. In this study, the constrained spanning problem supposing heterogeneous degree bounds on nodes capacities representing limited momentary capacities is analyzed. Given an undirected graph, we suppose different positive integer upper bounds associated with nodes to limit their degree for each visit. Finding the minimum cost connected spanning structure satisfying the degree constraints is the subject of our work. Usually, for budget constrained problems spanning trees are the solutions and the problem is NP hard which can not be approximated by a constant factor. Moreover, spanning the nodes with a tree respecting the degree constraints is not always possible. We demonstrate that the optimal solution to solve the capacity limited spanning problem with heterogeneous bounds can be different from a spanning tree, and an earlier proposed generalization of the tree concept, *i.e.* the hierarchy corresponds to the minimum cost solution. We investigate on the degree constrained minimum spanning hierarchy (DCMSH) under non-uniform constraints, on the conditions of its existence and on the possibility of its approximation being the problem NP hard. We prove necessary and sufficient conditions to find spanning hierarchies corresponding to the constraints.
Keywords: Graph theory, spanning problems, degree constrained minimum spanning tree, hierarchy, degree constrained minimum spanning hierarchy, inhomogeneous constraints, conditions for existence, approximation

1 Introduction

To optimally solve spanning problems in graphs (generally by minimizing the cost) is important in several domains, for instance in networks or for solving the routing in micro-circuits. In the simplest cases, spanning problems are formulated in graphs with positive costs associated to the edges, and the set of nodes or a given sub set of nodes should be spanned with minimum cost. It is well known the sub-graph which spans the set of nodes with minimum cost is a minimum spanning tree (MST) and several polynomial times algorithms are known to find it.

Some applications need to respect additional constraints. Various constrained spanning problems have been analyzed in graphs (cf. examples in [1,2,3]). Here we are interested in the degree constrained spanning problem. In this constrained spanning problem, each node $v \in V$ of the graph $G = (V, E)$ is assigned a positive integer value $d(v)$ which represents the maximum degree of the node in any spanning structure (for example in the spanning trees). This degree is potentially different from the degree of v in G indicated by $d_G(v)$. Note that only values $0 < d(v) \leq d_G(v)$ need to be considered for realistic cases. In the literature one can find several propositions to span the nodes of a graph respecting budget type degree bounds [4] [5]. In this cases, nodes are dotted by limited budgets and can not exceed the limit in the spanning trees. For instance, when the bounds are uniform and equal to 2, the problem corresponds to the Hamiltonian path problem which is one of the well known NP-hard problems and it is known the path does not always exist. Degree constrained spanning tree problems are hard to solve and unfortunately constant factor approximations do not exist for them [6]. Moreover, it is not always possible to span the nodes using trees with respect to the degree constraints [7].

If the degree bound does not correspond to a definitive budget but to a momentary limited capacity of the node, advantageous solutions, different from trees can be found. For example, in optical networks, the capacity of duplication of the switches may be limited for each incoming light, but a wavelength can be reused several times in a switch when it returns to that one (cf. [8]). As a result, the optical broadcast/multicast route may be different from a spanning tree. In [9], the authors propose a special walk containing returns to some nodes when the use of spanning trees or Hamiltonian paths is not possible due to the absence of nodes which may have a degree greater than two in the spanning structure (in their example, the under-layered optical network does not contain nodes which can duplicate the light, so spanning trees can not be used). Concretely, the walk proposed in [9] visits the nodes by using a Depth First Search algorithm on a spanning tree. Typically, the Depth First Search algorithm visits nodes at most twice.

In the cases where constraints correspond to momentary capacities but they are uniform in the node set, a hierarchy based solution has been proposed [10]. The analysis of this NP hard minimum spanning problem shows that a hierarchy based solution always exists and the minimum spanning hierarchy can be approximated. Here we propose the analysis of the cases where the momentary capacity constraints are not uniform but heterogeneous. We investigate on the following questions:

- In which cases is it possible to span the node set of the graph respecting heterogeneous degree constraints?
- Can the solutions be approximated?

We demonstrate that, similarly to the case with uniform capacity bounds, the optimal solution of the problem is a *hierarchy* and we formulate necessary and sufficient conditions to find it. The minimum cost spanning hierarchy problem, similarly to the minimum constrained spanning tree problem is NP-hard. In some cases the optimal solution can be approximated.

In the following sections, we propose a quick presentation of the well known and discussed degree constrained spanning tree problems followed by the definition of hierarchies and the analyzed capacity constrained spanning problems (cf. Section 2). The hardness of the problem is presented in Section 4, and Section 5 presents some approximations.

2 Degree Constrained Spanning Problems

The first version of the problems was formulated in [4] and extensively studied in [5]. The constraints on the degree of the nodes are budget-like constraints and the nodes can participate to the span until the exhaustion of their budget.

2.1 Minimum Spanning Trees under Degree Budget Constraints

Let us suppose that the edges and also the nodes in an undirected graph $G = (V, E)$ are assigned positive values. The positive integer value of a node limits the degree budget of the node in the spanning tree (it corresponds to a maximum budget which can be used to connect neighbors to the node) while the positive value of an edge corresponds to a cost or length function. Let $D(v)$ be the maximum possible number of neighbors (the budget constraint on the degree) of the node $v \in V$. If a degree bound $D(v)$ is equal to one, then v should only be a leaf in the span. (A special partial spanning problem with given leaf nodes has been formulated in [11].)

Property 1. If the objective is to span the node set respecting the budgets, the minimum cost solution is a spanning tree,

The proof is trivial, since cycles are useless. Let us suppose that the solution returns to a node several times and so there are cycles in the solution. Deleting one edge from each cycle, the node set remains covered but the cost of the solution is smaller. The connected solution if it exists, is a spanning tree.

The basic problem (DCMST) is known as follows.

Definition 1 (Degree Constrained Minimum Spanning Tree Problem) *Given an undirected graph $G =$ (V, E) with nonnegative costs $c(e)$ on the edges $e \in E$ and with positive integer degree bounds $D(v)$ on the nodes $v \in V$. The objective is finding a spanning tree in which the degree of any node v is at most $D(v)$ and the total cost is a minimum.*

The degree bounds can be uniform or not in the node set. In a homogeneous case, a bound B is given and the maximum degree of any node in the spanning tree is at most B. In the case of $B = 2$, no node can have a degree more than 2; and the solution is the minimum Hamiltonian path. The degree constrained spanning tree problem is known to be NP hard and the spanning tree (in the particular case the Hamiltonian path) does not always exist.

In [6], the authors present the problem as a hard network design problem. Their analysis fits in the frame of a generic bi-criteria optimization where the first objective corresponds to the respect of the budget (degree) constraints, the second is the minimization of the cost. The paper indicates the sub-graphs to solve the problem. The investigated classes are spanning trees, Steiner trees and generalized Steiner trees. The authors prove the hardness of the proposed optimizations. The DCMST problem is not in APX: there is no polynomial time ρ-approximation algorithm minimizing the cost when respecting the degree constraint even when the same upper bound is supposed for all of the nodes.

The case with non-uniform degree constraints was also shown to be NP hard using a reduction from the Traveling Salesman Problem (cf. [2]). It is known that it is not always possible to span the nodes using trees with respect of the degree constraints [7]. With reference to the observation in [2], a spanning tree solution of the degree bounded problem may exist if and only if

$$\sum_{v \in V} D(v) \geq 2n - 2$$

where $n = |V|$. Heuristic solutions were proposed in several works (cf. [5] [6] [12] [13] [14] [15] [16]). The partial spanning problem with non-uniform upper bounds cannot be approximated with $(2 - \epsilon, \rho)$ for any $\epsilon > 0$ and $\rho > 1$ [6]. For any $\epsilon > 0$ and $\rho > 1$, there is no polynomial time $(\tau - \epsilon, \rho)$-approximation algorithm for this problem, where τ is the lower bound on the performance ratio of any algorithm for finding minimum Steiner trees.

In unweighted graphs constrained by uniform degree bounds corresponding to the minimum possible maximum degree MB of possible spanning trees, Fürer and Raghavachari proposed an $(1, MB + 1)$-approximation algorithm for the degree constrained MST [17]. The paper [16] generalizes the result to weighted graphs.

When some nodes have a degree bound equal to 1, this sub-set of nodes is considered to be a leaf node set and these nodes must be leaves. The description of the Steiner problem is in [11] and a 2ρ-approximation algorithm for the partial spanning tree (leaf Steiner) problem can be found in [18].

Remember that the objective of the mentioned spanning problems is to cover a given set of the nodes using a connected structure minimizing the cost and satisfying the imposed degree constraints. *If the constraints are budget constraints, spanning trees are the cost optimal structures.* That is, a node belongs only once to the optimal solution (which is a sub-graph). In the case of momentary capacity limitation of the nodes, and supposing that this capacity can be renewed, the optimal solution can return several times in the nodes (the model has been presented in [10] for uniform degree bounds). The relaxation of the supposition that the solution corresponds to a tree is beneficial, the problem can be solved even if spanning trees satisfying the constraints do not exist. The solution always corresponds to a spanning hierarchy.

Before the reformulation of the degree constrained minimum cost spanning problem with non-uniform constraints, we propose the brief review of the *hierarchies* corresponding to the minimum cost solutions.

2.2 Hierarchies

The generalization of the tree concept was proposed in [19] using a simple definition. Graph homomorphism permits an accurate definition [20]. The homomorphic mapping between a tree T and a graph G can be used to define an eventually non-elementary tree in G called hierarchy. To simplify, let us suppose undirected graphs but the extension for digraphs is trivial.

Definition 2 (Hierarchy) *Let $G = (V, E)$ be an arbitrary graph and $T = (W, F)$ a tree. Let $h : W \to V$ be a homomorphic function which associates a node $v \in V$ to each node $w \in W$. The application (T, h, G) defines a hierarchy in G.*

Since a node $v \in V$ can be associated with several nodes in W, a hierarchy can "return" several times to nodes and pass several times edges in G (as it may be the case in walks): it corresponds to a "non-elementary tree" in the graph G. In hierarchies some nodes are eventually branching nodes (they are the node occurrences corresponding to the branching nodes of the tree T). Fig. 1 illustrates a hierarchy in an undirected graph. Some nodes of the graph (namely the nodes c and d) participate on different levels of the hierarchy shown in the labeled tree in Fig. 1/b). Notice that a hierarchy can also be given by two multisets: $H = (U, D)$ where U is the multi-set of the concerned nodes and D is the multi-set of edges in H using the labels from G. More details can be found in [20].

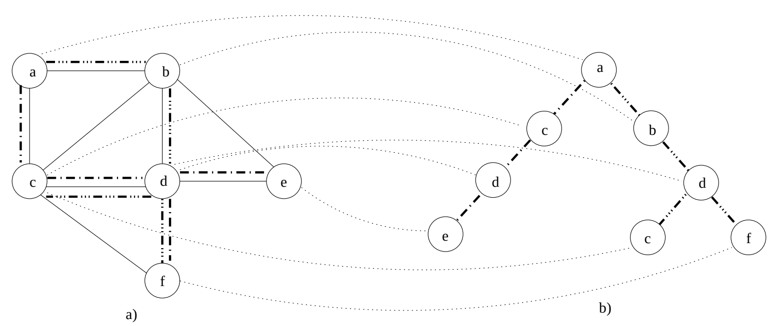

Fig. 1. Example of a hierarchy in a undirected graph

Hierarchies generalize trees. Trees are special hierarchies without repetition of nodes (applying an injective mapping h) and consequently inherit the properties of hierarchies. Let us notice that the sub-graph of G generated by a hierarchy (the projection) can contain cycles in G but the expanded hierarchy itself is a tree. In the following, we use the term *hierarchy* to reference the defined tree-like structure and we use the term *image of the hierarchy* for the sub-graph implicated in the original graph. Hierarchies allow the exact definition of some constrained spanning problems. In this paper, we analyze the degree constrained spanning problem with inhomogeneous constraints when the constraints limit the degree of nodes for a given visit. Some previous results are available concerning cases with uniform degree bounds.

2.3 Degree Capacity Constrained Spanning Problems

The minimum cost connected sub-graph spanning the node set corresponds to a minimum spanning tree [21] and, as we have already seen, the solution is a tree even if there are budget like degree constraints. If the constraints are due to limited instantaneous capacities of the nodes, the minimum cost spanning structure is always a hierarchy [20]. Fig. 2 illustrates the interest of the hierarchies in the degree capacity constrained spanning problems. Let us suppose unity cost edges and a uniform upper bound on the node degrees which is equal to three. The minimum cost connected spanning of this graph is required such that the degree of the nodes for each visit is limited to three (any node occurrence can not have a degree 4 or greater in the spanning structure). Trivially, there is no spanning tree but there is a spanning hierarchy satisfying the constraint. It uses the node b twice, but each occurrence of this node respects the degree constraint as it is shown in Fig. 2/b).

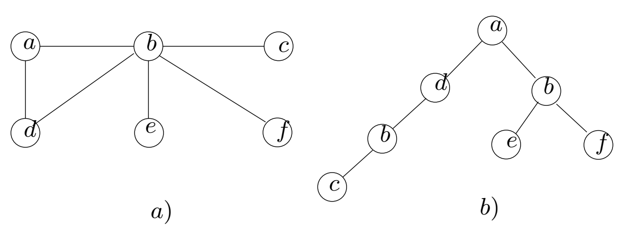

Fig. 2. A minimum cost spanning hierarchy

In [10] it has been demonstrated that the cost optimal solution of the degree constrained spanning problem with uniform capacity bounds corresponds to a spanning hierarchy. Here, we will show that the optimal solution is a hierarchy even if the degree bounds are non-uniform, and that such a solution exists.

Definition 3 (Node Capacity Constrained Minimum Spanning Problem) *Let $G = (V, E)$ be an arbitrary connected graph with positive costs $c(e)$ associated with edges $e \in E$. Here the positive integer degree bound $D(v)$ represents the instantaneous maximum capacity of the node $v \in V$. The problem is in finding the minimum cost hierarchy spanning the node set V s.t. the degree constraints are respected for each visit of the nodes.*

In the case of uniform capacity bounds, the problem is NP-hard but the solution always exists [22]. The following section discusses the conditions of existence for the solution in the cases of non-uniform bounds.

3 Necessary and Sufficient Conditions for Spanning Hierarchies

We show that the degree constrained spanning hierarchy does not always exist in the cases where degree bounds are heterogeneous. The following lemmas indicate trivial conditions for the existence of a spanning hierarchy satisfying non-uniform degree constraints.

Remember, a hierarchy in G is given by a triplet (T, h, G), where T is a tree. We use the notation $V_d \subseteq V$ to indicate the sub-set of nodes with degree bound d.

Lemma 1. *If V_1 contains a separator*[1]*, there is no hierarchy spanning V and satisfying the degree constraints.*

Proof. A separator S divides the node set into (at least) two sub-sets A and B. Any path from a node in A to an arbitrary node in B should traverse S but the nodes in S can only be leaves in the connection. Trivially, there are no possible connections between nodes in A and B satisfying the degree constraints imposed by V_1. ∎

Lemma 2. *If $V_d = \emptyset, \forall d > 2$ (there is no node with degree bound $D(v) > 2$) and $|V_1| > 2$ (there are more than two nodes with degree bound 1), there is no hierarchy spanning V and satisfying the constraints.*

Proof. To satisfy the degree constraint, a node $v \in V_1$ must be a leaf in the spanning hierarchy (and trivially in T). Since $|V_1| > 2$, the tree T and the hierarchy must have more than 2 leaves. Because $V_d = \emptyset, \forall d > 2$, the degree of internal nodes in T is equal to 2 for all nodes. T must be a path and the hierarchy corresponds to a walk. A path (and an opened walk) has exactly two extremities. The contradiction is trivial. ∎

Lemma 3. *If $|V \setminus V_1| = 1$, and the only one potential non-leaf branching node is $v_b \in V_d, d \geq 2$ and $|V_1| > d$, there is no hierarchy spanning V and respecting the constraints.*

Proof. Occurrences of v_b can be internal (and eventually branching) nodes in the spanning hierarchy. One occurrence of $v_b \in V_d$ can have at most d neighbors forming a star. The leaves V_L of this star are in V_1 and consequently they must

[1] Remember that a set of nodes separating V in two independent non empty sub-sets is a separator

be leaves in the spanning hierarchy. Since $|V_1| > d$, the remaining nodes $V_1 \setminus V_L$ can not be connected to the star and can not be covered by any hierarhy respecting the degree constraints. ■

These lemmas give conditions for the nonexistence of spanning hierarchies. The following theorem formulates necessary and sufficient conditions for their existence.

Theorem 1. *A hierarchy spanning the whole node set of a connected graph and respecting heterogeneous degree constraints can be found, iff*
a) there is no separator in V_1 (separator nodes having a degree bound 1) and
b) $|V_1| \leq 2$ (there are at most two nodes with degree bound 1) or
c) $|V_1| > 2$ and these nodes are neighbor nodes of a node $v \in V_m, m \geq |V_1|$ or
d) $|V \setminus V_1| \geq 2$ (there are at least two nodes with degree bound greater than 1) and $|V \setminus V_2 \setminus V_1| \geq 1$ (there is at least one node with degree bound greater than 2).

Proof. Following Lemma 1, condition a) is necessary. Moreover, following Lemmas 2 and 3 one of conditions b) or c) or d) is also necessary. These conditions are sufficient as it is proved in the following.

b) can be proved as follows. Let us suppose that there are two nodes a and b with degree bound 1 and they do not compose a separator. In this case, a Hamiltonian walk[2] in which the extremities correspond to a and b exists due to the following. Since these nodes do not form a separator, there is at least a path between any node pair in the graph. A complete graph (for example the metrical closure) can be constructed representing the shortest paths by edges. In this complete graph, Hamiltonian paths exist in which the extremities coincide with a and b. Each Hamiltonian path corresponds to a Hamiltonian walk in the original graph covering the node set and respecting the degree constraints. The same demonstration is trivially true with only one node with degree bound 1 or without this kind of nodes. In these cases, in the metrical closure, one or both ends of the Hamiltonian path can be freely chosen.

c) This condition is trivially sufficient. If all of the nodes in V_1 are connected to a node $v \in V_m, m \geq |V_1|$, the graph is a star and it corresponds to the degree constraints.

d) If there is at least one node with a degree bound greater than 2, let say the node c, another with a degree bound at least 2, let say the node b, and there is no separator composed from nodes having a degree bound 1, a degree bounded hierarchy spanning the node set can be found as follows. Because there is no separator composed from nodes in V_1, each node in $V \setminus b \setminus c$ can be connected to c with a path and each path respects the degree constraints. Then groups with these paths can be created s.t. there are $D(c) - 2$ paths in every group. Each group corresponds to a spider respecting the degree constraints and having c as

[2] A Hamiltonian walk is not obligatory an elementary Hamiltonian path and it can return several times to a node

central node. The different occurrences of c can be chained by a walk visiting the node b (having a degree bound at least 2) s.t. the degree of the central nodes is at most $D(c)$. The obtained structure is a hierarchy spanning the node set and respecting the constraints. ∎

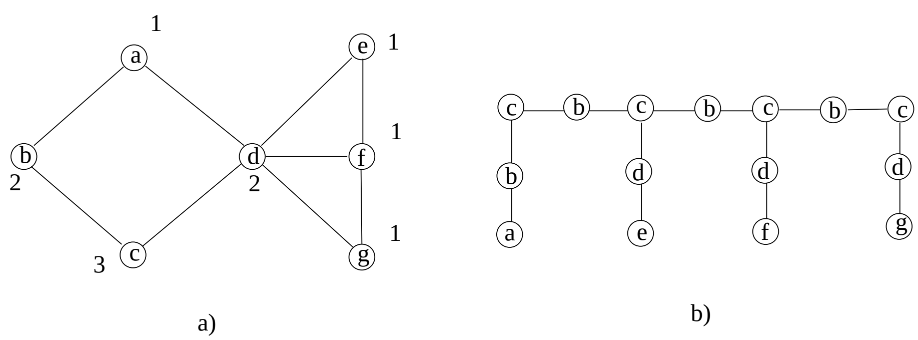

Fig. 3. Illustration of Theorem 1

Figure 3 illustrates a spanning hierarchy following the construction described in Theorem 1. The occurrences of the node c (having a degree bound 3) are the central nodes of spiders which are connected by paths passing via the node b. Trivially it is not the "best" spanning hierarchy. In some cases, one can easily find hierarchies containing less edges.

4 Hardness and Some Ideas for Computation

Theorem 2. *The computation of the minimum cost hierarchy spanning the node set of a connected graph and respecting non-uniform degree constraints is an NP-hard problem if the solution exists.*

Proof. The proof is similar to the proof in [22] for the case of homogeneous degree bounds.

Let $G = (V, E)$ be a connected graph and let $D(v)$ be the positive integer bound associated to node $v \in V$. Let $G' = (V', E')$ be the graph obtained from G by connecting $D(v) - 2$ new leaves to each vertex $v \in V$ s.t. $D(v) > 2$ as it is illustrated in Figure 4. Let L be the newly added set of leaves. Following the construction: $V'_1 = V_1 \bigcup L$ and $V'_i = V_i, \forall i > 1$.

Let us suppose that G' corresponds to the conditions of Theorem 1 and admits a solution. For this the following conditions should be held: a) There is no separator in V'_1. It is the case when there is no separator in V_1 (since the new leaves do not form a separator).

b) The first part of this condition corresponds to $|V_1'| \leq 2$. It is satisfied in the cases where $|V_1| + |V_2| \leq 2$. Following the second (alternative) part of the condition: $|V' \setminus V_1'| \geq 2$ and $|V' \setminus V_2' \setminus V_1'| \geq 1$. Trivially $V' \setminus V_1' = V \setminus V_1$ and $V_2' = V_2$. Consequently this part is true if $|V \setminus V_1| \geq 2$ and $|V \setminus V_2 \setminus V_1| \geq 1$.

The degree constrained minimum spanning hierarchy of G' contains all edges leaving to the new leaves. The degree constraints in the attachment vertices are respected, iff G is covered by a Hamiltonian walk, in which the degree of nodes is at most 2. The computation of this latter is NP-hard [23]. ■

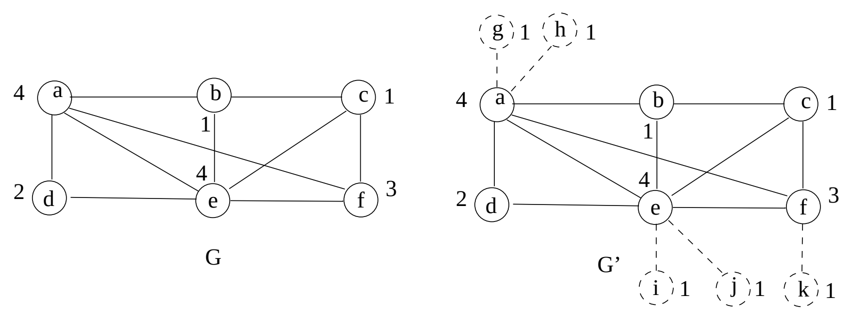

Fig. 4. The initial graph G and the completed graph G'

Several algorithms can be used to compute the optimal solution. For instance with a small modification of the ILP found in [22], an exact mathematical program for the computation can be obtained. Branch and bound, branch and price like algorithms are also candidates for the computation, but they are out of scope of the recent study.

5 Approximation

The approximation with a constant ratio of the minimum spanning hierarchy under uniform capacity degree bound has been presented in [10]. In the recent study the possibility of guaranteed approximations of the cases with non-uniform degree bounds is analyzed. At first, we propose a particular spanning tree (*cf.* sub section 5.1), where an eventual leaf node set is given (the given nodes must be leaves in the spanning tree). The analysis of the conditions in the previous section indicates that the set of leaves V_1 strongly influences the solution. Sub section 5.2 examines the cases where the leaf node set is not fixed: $V_1 = \emptyset$. The problem is more complex in sub sections 5.3 and 5.4 when this leaf node set V_1 is not empty.

First of all, a potential lower bound of costs is proposed.

5.1 A Lower Bound

Let us examine the construction of a particular spanning tree when a set V_1 of nodes is given s.t. these nodes should be leaves in the tree. Notice that the nodes in the set V_1 should be leaves in the solution, but some another nodes in $V \setminus V_1$ can also be leaves.

Definition 4 (Minimum Spanning Tree with Fixed Leaves - MSTFL) *In the connected graph $G = (V, E)$ with $D(v) > 0$ strictly positive degree bounds on the nodes, let $V_1 \subset V$ be a non empty sub-set of desired leaves. Let us suppose that V_1 does not contain any separator. The degree of the other nodes is not limited in the covering: for example $D(v) = d_G(v), \forall v \in V \setminus V_1$. The problem consists in finding a minimum cost spanning tree covering the node set s.t. the nodes in V_1 are leaves in the tree.*

Note: V_1 should be free of any separator. If V_1 is not a separator but a separator sub-set of V_1 exists, this later isolates another sub-set of leaves in V_1 which are unreachable from the remaining graph. Consequently, a spanning tree (and a spanning hierarchy) satisfying the degree constraints does not exist. Figure 5 illustrates this case. Here, $V_1 = \{b, c, d\}$ is not a separator but the set $\{b, d\}$ is.

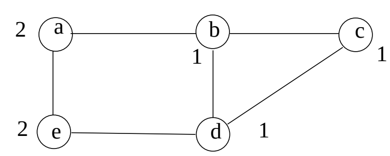

Fig. 5. V_1 is not a separator, but contains separators

Lemma 4. *To find the MSTFL in which the nodes in V_1 are leaves polynomial time algorithms can be used.*

Proof. The following greedy algorithm computes the solution.

Let $G' = (V \setminus V_1, E')$ the graph obtained by deleting the desired leaves from G. G' is connected, since V_1 does not contain any separator.

Let T' be the MST in G'. T' can be computed in polynomial time.

Let E_v be the set of adjacent edges of the node $v \in V_1$ s.t. the other extremity of the edge is not in V_1. Let e_v be the edge with minimum cost in E_v. Connecting each node $v \in V_1$ to T' by the edge e_v produces a tree T''.

T'' is a tree which spans the whole node set and in which the nodes in V_1 are leaves. Moreover it is with the minimum cost as it is proved in the following.

Let us suppose that a tree T_m exists with less cost and satisfying the constraints on the leaves. Let T'_m be the tree after the deletion of leaves in V_1 from T_m. In order to connect the leaves in V_1 to T'_m, trivially the less cost edges are used. Consequently, the cost of T'_m must be less than the cost of T' which is contrary the fact that T' is an MST. ■

Trivially, if V_1 does not contain any separator, the MSTFL exists.

In the following, we focus on the approximation of degree bounded minimum spanning hierarchies.

A polynomial time calculable lower bound for the cost of the minimum spanning hierarchy respecting the degree constraints can be found.

Lemma 5. *Let us suppose that spanning hierarchies corresponding to the given degree constraints exist, satisfying the conditions in Theorem 1. Then the MSTFL with leaf nodes fixed by V_1 exists and its cost is a lower bound for the spanning hierarchies, and in this manner for the minimum spanning hierarchy H, corresponding to the given degree constraints.*

Proof. In the minimum spanning hierarchy H respecting the constraints, each node in V_1 is a unique leaf occurrence. By deleting these nodes a hierarchy H' is obtained. The deletion involves the deletion of a set E_1 of edges leading the the deleted leaves (these edges are also present only once in the hierarchy). The image I' of H' in G may eventually contain cycles. From each cycle in I', an edge can be deleted to obtain a tree T' in G covering the node set $V \setminus V_1$. (It is possible that T' does not respect the degree constraints.) By adding the deleted nodes in V_1 using the edges in E_1 to T' a spanning tree T_1 corresponding to the fixed leaf nodes in V_1 is created. Since at least one spanning tree (T_1) with fixed leaf nodes given by V_1 exists, the MSTLF (indicated here by T) exists. Moreover $c(T) \leq c(T_1) \leq c(H)$. ■

The eventual approximations depend strongly on the number and the position of the desired leaf nodes. If there are a few numbers of desired leaves in V_1, guaranteed ratios can be given.

5.2 Case of $V_1 = \emptyset$

If $V_1 = \emptyset$, trivially the MSTFL corresponds to the MST.

Lemma 6. *Let us suppose that there is no node with degree bound* 1 *($V_1 = \emptyset$). The problem can be approximated from the MST and a trivial ratio corresponds to $\frac{D}{D-1}$ where $D = \min_{v \in V} D(v)$. In the worst case, the minimum of degree bounds is equal to* 2 *and the ratio is also* 2.

Proof. Similarly to the case with uniform degree bounds, an approximation scheme can be proposed starting from the MST T^* of G [10]. T^* can be covered by a set $S = \{S_i, i = 1, ...k\}$ of stars. In each star S_i, a spanning hierarchy respecting the degree constraint of the (eventually multiplied) central node v_i^c can be built. The cost of the hierarchy spanning the star S_i is limited by $\frac{d(v_i^c)}{d(v_i^c)-1} c(S_i)$

where $c(S_i)$ is the cost of the star. These hierarchies spanning the stars can be connected as described in [10], and a hierarchy H spanning the node set V and respecting the degree constraints is obtained. Moreover, since $D = \min_{v \in V} D(v)$:

$$\frac{d(v)}{d(v)-1} \leq \frac{D}{D-1}, \forall v \in V$$

If there are k stars in the decomposition:

$$c(H) = \sum_{i=1}^{k} c(H_{S_i}) \leq \sum_{i=1}^{k} \frac{d(v_i^c)}{d(v_i^c)-1} c(S_i) \leq \frac{D}{D-1} c(T^*)$$

■

Figure 6 illustrates the proposed decomposition. The degree bound $d(v)$ is indicated for each node $v \in V$. Since this value can be greater than the degree of the node in the MST, spanning hierarchies respecting the degree bounds are computed in the different stars and reconnected to form the final hierarchy spanning the MST as it is indicated by Figure 7.

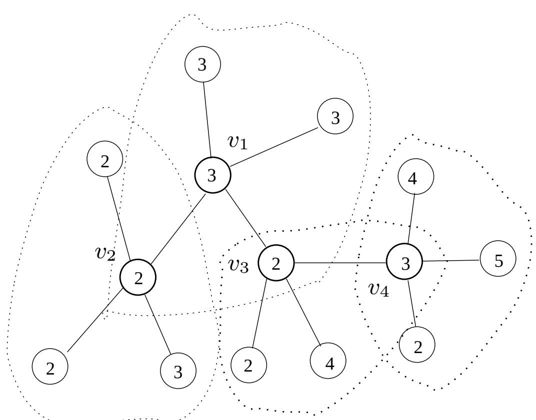

Fig. 6. An MST and its decomposition into a set of stars

5.3 Case of $0 < |V_1| \leq 2$

In this case, we suppose that V_1 is not empty but it contains at most two nodes which are not separators.

At first we propose to analyze the case of $|V_1| = 2$. Let a and b be the two nodes in V_1. a and b must be leaves in any solution but other leaves may also exist. Following Theorem 5, the MSTFL having nodes a and b as fixed leaves gives a lower bound for the cost of the optimum.

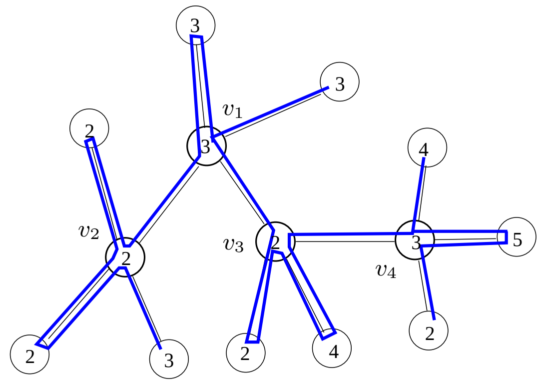

Fig. 7. The connected final hierarchy spanning the MST and respecting the degree constraints

Lemma 7. *The minimum spanning hierarchy in the case of non-uniform degree bounds and at most two fixed leaves can be approximated by a factor* 2*.*

Proof. If $V_1 \neq \emptyset$, following Theorem 1 the optimal hierarchy H exist. Then, following Lemma 4 the MSTFL also exists. Let T_2 be this minimum spanning tree of the graph having nodes a and b as leaves. $c(T_2) \leq c(H)$ Trivially, following T_2, a trail can be computed starting from a and ending at b. This trail t contains the edges of T_2 at most twice and corresponds to a spanning hierarchy satisfying the degree constraints (the internal nodes in t have only degree 2). $c(t) \leq 2 \cdot c(T_2) \leq 2 \cdot c(H)$. ■

5.4 Case of $|V_1| > 2$

As Theorem 1 indicates, in some cases there is no solution. In this section we suppose that conditions of existence of the solution are satisfied and we focus on the approximation.

Unfortunately, in arbitrary graphs with arbitrary positive degree bounds, an approximation with constant factor can not be guaranteed from the MSTFL.

Lemma 8. *Even if the solution exists (the conditions of its existence are satisfied), the non-uniform degree constrained minimum spanning hierarchy can not be approximated by a constant factor from the corresponding MSTFL when* $|V_1| > 2$*.*

Proof. Let us construct a graph (corresponding to a star) as follows. Let a be a node with degree bound 3 and b the central node with degree bound 2. Nodes with degree bound 1 are connected to these nodes without creating any separator. The solution exists. Let us suppose that k nodes with bound 1 are connected to the node b. Consider the cost of the edges leading to leaves being ϵ negligible to the cost of edge $\{b, c\}$. In this graph (cf. Figure 8), in the DCMSH the costly edge $\{b, c\}$ should be repeated $3(k-2)$-times if $k > 3$ and this edge should be repeated k times if $k \leq 3$. Edges leaving to nodes in V_1 can be covered only once. In any case:

$$c(DCMSH) \geq k(1+\epsilon)$$

The MSTFL having nodes in V_1 as leaves is the graph itself. The ratio between the costs is bounded.

$$\frac{c(DCMSH)}{c(MSTFL)} \geq \frac{k(1+\epsilon)}{1+k\epsilon}$$

When ϵ tends to zero, the lower bound tends to k. when k tends to infinity, the lower bound tends to infinity (to $\frac{1+\epsilon}{\epsilon}$) and can not be limited by a constant valid for all stars.

■

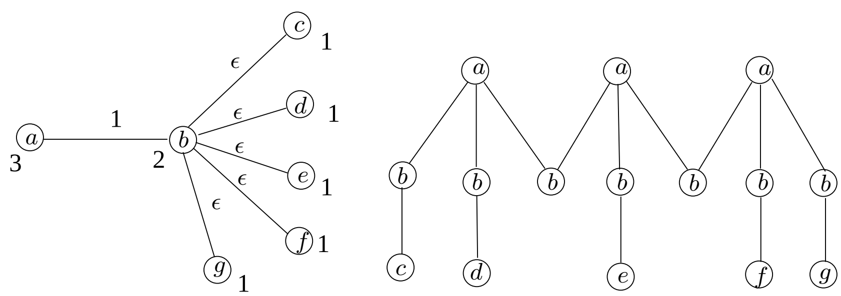

Fig. 8. A particular graph and its coverage by the optimal hierarchy

Notice that the example used in Lemma 8 shows an example where the MSTFL is the same that the MST. Consequently, the constant approximation is not possible either using the MST.

It is an open question whether an approximation can be found using another sub-graph, tree or polynomial time computable spanning hierarchy as reference or not.

6 Perspectives

An important perspective is the analysis of partial spanning problems like degree bounded Steiner problems. We suppose that a good part of the recent results can be applied in partial spanning cases.

Important research work should investigate the fast computation of advantageous spanning hierarchies for constrained spanning problems and for various related applications. These spanning problems have applications, for instance, in optical multicast routing. In this kind of applications, additional constraints may exist and the different constraints should altogether be satisfied. The analysis of these problems promises further interesting challenges.

References

1. Papadimitriou, C.H., Yannakakis, M.: The Complexity of Restricted Minimum Spanning Tree Problems (Extended Abstract). In Maurer, H.A., ed.: ICALP. Volume 71 of Lecture Notes in Computer Science., Springer (1979) 460–470
2. Cieslik, D.: The vertex degrees of minimum spanning trees. European Journal of Operational Research **125** (2000) 278–282
3. Ruzika, S., Hamacher, H.W.: A Survey on Multiple Objective Minimum Spanning Tree Problems. In Lerner, J., Wagner, D., Zweig, K., eds.: Algorithmics of Large and Complex Networks: Design, Analysis, and Simulation, LNCS 5515. Springer-Verlag, Berlin, Heidelberg (2009) 104–116
4. Deo, N., Hakimi, S.: The shortest generalized Hamiltonian tree. In: Sixth Annual Allerton Conference. (1968) 879–888
5. Boldon, B., Deo, N., Kumar, N.: Minimum-Weight Degree-Constrained Spanning Tree Problem: Heuristics and Implementation on an SIMD Parallel Machine. Parallel Computing **22** (1996) 369–382
6. Ravi, R., Marathe, M.V., Ravi, S.S., Rosenkrantz, D.J., Iii, H.B.H.: Approximation algorithms for degree-constrained minimum-cost network-design problems. Algorithmica **31** (2001) 58–78
7. Bauer, F., Varma, A.: Degree-constrained multicasting in point-to-point networks. In: INFOCOM '95: Proceedings of the Fourteenth Annual Joint Conference of the IEEE Computer and Communication Societies (Vol. 1)-Volume, Washington, DC, USA, IEEE Computer Society (1995) 369
8. Mukherjee, B.: Optical WDM Networks (Optical Networks). Springer-Verlag, Berlin, Heidelberg (2006)
9. Ali, M., Deogun, J.: Cost-effective implementation of multicasting in wavelength-routed networks. IEEE J. Lightwave Technol., Special Issue on Optical Networks **18** (2000) 1628–1638
10. Molnár, M., Durand, S., Merabet, M.: Approximation of the Degree-Constrained Minimum Spanning Hierarchies. In: SIROCCO. (2014) 96–107
11. Lin, G., Xue, G.: On the terminal Steiner tree problem. Inf. Process. Lett. **84** (2002) 103–107
12. Könemann, J., Ravi, R.: A Matter of Degree: Improved Approximation Algorithms for Degree-Bounded Minimum Spanning Trees. SIAM J. Comput. **31** (2002) 1783–1793
13. Könemann, J., Ravi, R.: Primal-Dual Meets Local Search: Approximating MSTs With Nonuniform Degree Bounds. SIAM J. Comput. **34** (2005) 763–773
14. Ravi, R.: Matching Based Augmentations for Approximating Connectivity Problems. In: Proc. of the 7th Latin American Symposium on Theoretical Informatics (LATIN'06), Valdivia, Chile (2006) 13–24
15. Ravi, R., Singh, M.: Delegate and Conquer: An LP-Based Approximation Algorithm for Minimum Degree MSTs. In: ICALP (1). (2006) 169–180
16. Singh, M., Lau, L.C.: Approximating minimum bounded degree spanning trees to within one of optimal. In: STOC '07: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing, New York, NY, USA, ACM (2007) 661–670

17. Fürer, M., Raghavachari, B.: Approximating the minimum-degree steiner tree to within one of optimal. Journal of Algorithms **17** (1994) 409–423
18. Fuchs, B.: A note on the terminal Steiner tree problem. Inf. Process. Lett. **87** (2003) 219–220
19. Molnár, M.: Optimisation des communications multicast sous contraintes. Mémoire of habilitation to advise doctoral theses (in French), University Rennes 1 (2008)
20. Molnár, M.: Hierarchies to Solve Constrained Connected Spanning Problems. Technical Report 11029, LIRMM (2011)
21. Kruskal, J.B.: On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem. Proceedings of the American Mathematical Society **7** (1956) 48–50
22. Merabet, M., Molnár, M., Durand, S.: ILP formulation of the degree-constrained minimum spanning hierarchy problem. Journal of Combinatorial Optimization **36** (2018) 789–811
23. Nishizeki, T., Asano, T., Watanabe, T.: An approximation algorithm for the Hamiltonian walk problem on maximal planar graphs. Discrete Applied Mathematics **5** (1983) 211 – 222