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Homomorphisms of planar (m, n) -colored-mixed graphs to planar targets

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Abstract

An (m, n) -colored-mixed graph $G = (V, A_1, A_2, \dots, A_m, E_1, E_2, \dots, E_n)$ is a graph having m colors of arcs and n colors of edges. We do not allow two arcs or edges to have the same endpoints. A homomorphism from an (m, n) -colored-mixed graph G to another (m, n) -colored-mixed graph H is a morphism $\varphi : V(G) \rightarrow V(H)$ such that each edge (resp. arc) of G is mapped to an edge (resp. arc) of H of the same color (and orientation). An (m, n) -colored-mixed graph T is said to be $P_g^{(m,n)}$ -universal if every graph in $P_g^{(m,n)}$ (the planar (m, n) -colored-mixed graphs with girth at least g) admits a homomorphism to T .

We show that planar $P_g^{(m,n)}$ -universal graphs do not exist for $2m + n \geq 3$ (and any value of g) and find a minimal (in the number vertices) planar $P_g^{(m,n)}$ -universal graphs in the other cases.

1 Introduction

The concept of homomorphisms of (m, n) -colored-mixed graph was introduced by J. Nesěřil and A. Raspaud [1] in order to generalize homomorphisms of k -edge-colored graphs and oriented graphs.

An (m, n) -colored-mixed graph $G = (V, A_1, A_2, \dots, A_m, E_1, E_2, \dots, E_n)$ is a graph having m colors of arcs and n colors of edges. We do not allow two arcs or edges to have the same endpoints and we do not allow loops. The case $m = 0$ and $n = 1$ corresponds to simple graphs, $m = 1$ and $n = 0$ to oriented graphs and $m = 0$ and $n = k$ to k -edge-colored graphs. For the case $m = 0$ and $n = 2$ (2-edge-colored graphs) we refer to the two types of edges as *blue* and *red* edges.

A *homomorphism* from an (m, n) -colored-mixed graph G to another (m, n) -colored-mixed graph H is a mapping $\varphi : V(G) \rightarrow V(H)$ such that every edge (resp. arc) of G is mapped to an edge (resp. arc) of H of the same color (and orientation). If G admits a homomorphism to H , we say that G is *H-colorable* since this homomorphism can be seen as a coloring of the vertices of G using the vertices of H as colors. The edges and arcs of H (and their colors) give us the rules that this coloring must follow. Given a class of graphs \mathcal{C} , a graph is *\mathcal{C} -universal* if for every graph $G \in \mathcal{C}$ is H -colorable. The class $P_g^{(m,n)}$ contains every planar (m, n) -colored-mixed graph with girth at least g . Graph \vec{C}_6^2 is the graph with vertex set $\{0, 1, 2, 3, 4, 5\}$ such that uv is an arc if and only if $v - u \equiv 1 \pmod{6}$ or $v - u \equiv 2 \pmod{6}$.

In this paper, we consider some planar $P_g^{(m,n)}$ -universal graphs with few vertices. They are depicted in Figures 1 and 2. The known results about this topic are as follows.

Theorem 1.

1. K_4 is a planar $P_3^{(0,1)}$ -universal graph. This is the four color theorem.
2. K_3 is a planar $P_4^{(0,1)}$ -universal graph. This is Grötzsch's Theorem [2].
3. \vec{C}_6^2 is a planar $P_{16}^{(1,0)}$ -universal graph [3].

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Our first result shows that, in addition to the case of $(0,1)$ -graphs covered by Theorems 1.1 and 1.2, our topic is actually restricted to the cases of oriented graphs (i.e., $(m,n) = (1,0)$) and 2-edge-colored graphs (i.e., $(m,n) = (0,2)$).

Theorem 2. *For every $g \geq 3$, there exists no planar $P_g^{(m,n)}$ -universal graph if $2m + n \geq 3$.*

As Theorems 1.1 and 1.2 show for $(0,1)$ -graphs, there might exist a trade-off between minimizing the girth g and the number of vertices of the universal graph, for a fixed pair (m,n) . For oriented graphs, Theorem 1.3 tries to minimize the girth. For oriented graphs and 2-edge-colored graphs, we choose instead to minimize the number of vertices of the universal graph.

Theorem 3.

1. \vec{T}_5 is a planar $P_{28}^{(1,0)}$ -universal graph on 5 vertices.
2. T_6 is a planar $P_{22}^{(0,2)}$ -universal graph on 6 vertices.

The following results shows that Theorem 3 is optimal in terms of the number of vertices of the universal graph.

Theorem 4.

1. For every $g \geq 3$, there exists an oriented bipartite cactus graph (i.e., K_4^- minor-free graph) with girth at least g and oriented chromatic number at least 5.
2. For every $g \geq 3$, there exists a 2-edge-colored bipartite outerplanar graph (i.e., $(K_4^-, K_{2,3})$ minor-free graph) with girth at least g that does not map to a planar graph with at most 5 vertices.

Most probably, Theorem 3 is not optimal in terms of girth. The following constructions give lower bounds on the girth.

Theorem 5.

1. There exists an oriented bipartite 2-outerplanar graph with girth 14 that does not map to \vec{T}_5 .
2. There exists a 2-edge-colored planar graph with girth 11 that does not map to T_6 .
3. There exists a 2-edge-colored bipartite planar graph with girth 10 that does not map to T_6 .

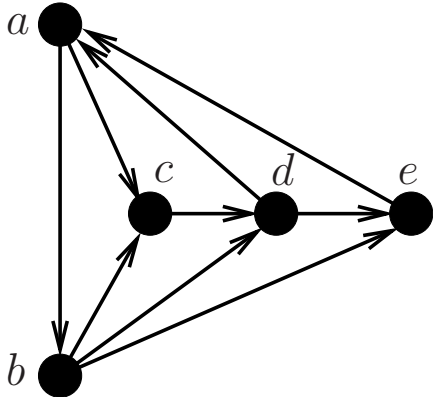


Figure 1: The $P_{28}^{(1,0)}$ -universal graph \vec{T}_5 .

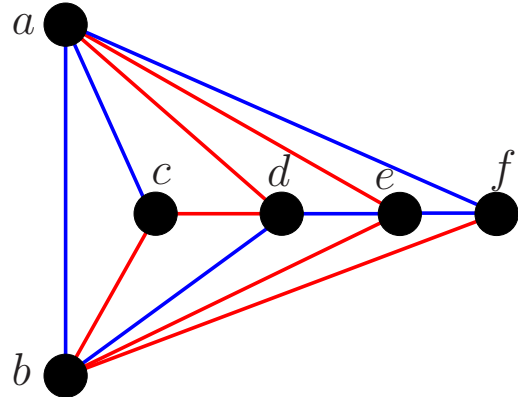


Figure 2: The $P_{22}^{(0,2)}$ -universal graph T_6 .

Next, we obtain the following complexity dichotomies:

Theorem 6.

1. For any fixed girth $g \geq 3$, either every graph in $P_g^{(1,0)}$ maps to \vec{T}_5 or it is NP-complete to decide whether a graph in $P_g^{(1,0)}$ maps to \vec{T}_5 . Either every bipartite graph in $P_g^{(1,0)}$ maps to \vec{T}_5 or it is NP-complete to decide whether a bipartite graph in $P_g^{(1,0)}$ maps to \vec{T}_5 .
2. Either every graph in $P_g^{(0,2)}$ maps to T_6 or it is NP-complete to decide whether a graph in $P_g^{(1,0)}$ maps to T_6 . Either every bipartite graph in $P_g^{(0,2)}$ maps to T_6 or it is NP-complete to decide whether a bipartite graph in $P_g^{(1,0)}$ maps to T_6 .

Finally, we can use Theorem 6 with the non-colorable graphs in Theorem 5.

Corollary 7.

1. Deciding whether a bipartite graph in $P_{14}^{(1,0)}$ maps to \vec{T}_5 is NP-complete.
2. Deciding whether a graph in $P_{11}^{(0,2)}$ maps to T_6 is NP-complete.
3. Deciding whether a bipartite graph in $P_{10}^{(0,2)}$ maps to T_6 is NP-complete.

A 2-edge-colored path or cycle is said to be *alternating* if any two adjacent edges have distinct colors.

Proposition 8 (folklore).

- Every planar simple graph on n vertices has at most $3n - 6$ edges.
- Every planar simple graph satisfies $(\text{mad}(G) - 2) \cdot (g(G) - 2) < 4$.

2 Proof of Theorem 3

We use the discharging method for both results in Theorem 3. The following lemma will handle the discharging part. We call a vertex of degree n an n -vertex and a vertex of degree at least n an n^+ -vertex. If there is a path made only of 2-vertices linking two vertices u and v , we say that v is a weak-neighbor of u . If v is a neighbor of u , we also say that v is a weak-neighbor of u . We call a (weak-)neighbor of degree n an n -(weak-)neighbor.

Lemma 9. *Let k be a non-negative integer. Let G be a graph with minimum degree 2 such that every 3-vertex has at most k 2-weak-neighbors and every path contains at most $\frac{k+1}{2}$ consecutive 2-vertices. Then $\text{mad}(G) \geq 2 + \frac{2}{k+2}$. In particular, G cannot be a planar graph with girth at least $2k + 6$.*

Proof. Let G be as stated. Every vertex has an initial charge equal to its degree. Every 3^+ -vertex gives $\frac{1}{k+2}$ to each of its 2-weak-neighbors. Let us check that the final charge $ch(v)$ of every vertex v is at least $2 + \frac{2}{k+2}$.

- If $d(v) = 2$, then v receives $\frac{1}{k+2}$ from each of its 3-weak-neighbors. Thus $ch(v) = 2 + \frac{2}{k+2}$.
- If $d(v) = 3$, then v gives $\frac{1}{k+2}$ to each of its 2-weak-neighbors. Thus $ch(v) \geq 3 - \frac{k}{k+2} = 2 + \frac{2}{k+2}$.
- If $d(v) = d \geq 4$, then v has at most $\frac{k+1}{2}$ 2-weak-neighbors in each of the d incident paths. Thus $ch(v) \geq d - d \left(\frac{k+1}{2}\right) \left(\frac{1}{k+2}\right) = \frac{d}{2} \left(1 + \frac{1}{k+2}\right) \geq 2 + \frac{2}{k+2}$.

This implies that $\text{mad}(G) \geq 2 + \frac{2}{k+2}$. Finally, if G is planar, then the girth of G cannot be at least $2k + 6$, since otherwise $(\text{mad}(G) - 2) \cdot (g(G) - 2) \geq \left(2 + \frac{2}{k+2} - 2\right) (2k + 6 - 2) = \left(\frac{2}{k+2}\right) (2k + 4) = 4$, which contradicts Proposition 8. \square

2.1 Proof of Theorem 3.1

We prove that the oriented planar graph \vec{T}_5 on 5 vertices from Figure 1 is $P_{28}^{(1,0)}$ -universal by contradiction. Assume that G is an oriented planar graph with girth at least 28 that does not admit a homomorphism to \vec{T}_5 and is minimal with respect to the number of vertices. By minimality, G cannot contain a vertex v with degree at most one since a \vec{T}_5 -coloring of $G - v$ can be extended to G . Similarly, G does not contain the following configurations.

- A path with 6 consecutive 2-vertices.
- A 3-vertex with at least 12 2-weak-neighbors.

Suppose that G contains a path $u_0u_1u_2u_3u_4u_5u_6u_7$ such that the degree of u_i is two for $1 \leq i \leq 6$. By minimality of G , $G - u_1, u_2, u_3, u_4, u_5, u_6$ admits a \vec{T}_5 -coloring φ . We checked on a computer that for any $\varphi(v_0)$ and $\varphi(v_6)$ in $V(\vec{T}_5)$ and every possible orientation of the 7 arcs u_iu_{i+1} , we can always extend φ into a \vec{T}_5 -coloring of G , a contradiction.

Suppose that G contains a 3-vertex v with at least 12 2-weak-neighbors. Let u_1, u_2, u_3 be the 3⁺-weak-neighbors of v and let l_i be the number of common 2-weak-neighbors of v and u_i , i.e., 2-vertices on the path between v and u_i . Without loss of generality and by the previous discussion, we have $5 \geq l_1 \geq l_2 \geq l_3$ and $l_1 + l_2 + l_3 \geq 12$. So we have to consider the following cases:

- **Case 1:** $l_1 = 5, l_2 = 5, l_3 = 2$.
- **Case 2:** $l_1 = 5, l_2 = 4, l_3 = 3$.
- **Case 3:** $l_1 = 4, l_2 = 4, l_3 = 4$.

By minimality, the graph G' obtained from G by removing v and its 2-weak-neighbors admits a \vec{T}_5 -coloring φ . Let us show that in all three cases, we can extend φ into a \vec{T}_5 -coloring of G to get a contradiction.

With an extensive search on a computer we found that if a vertex v is connected to a vertex u colored in $\varphi(u)$ by a path made of l 2-vertices ($0 \leq l \leq 5$) then v can be colored in:

- at least 1 color if $l = 0$,
- at least 2 colors if $l = 1$,
- at least 2 colors if $l = 2$ (the sets $\{c, d, e\}$ and $\{b, c, d\}$ are the only sets of size 3 that can be forbidden from v),
- at least 3 colors if $l = 3$,
- at least 4 colors if $l = 4$ and
- at least 4 colors if $l = 5$ (only the sets $\{b\}$, $\{c\}$, and $\{e\}$ can be forbidden from v).

In Case 1, u_3 forbids at most 3 colors from v since $l_3 = 2$. If it forbids less than 3 colors, we will be able to find a color for v since u_1 and u_2 forbid at most 1 color from v . The only sets of 3 colors that u_3 can forbid are $\{b, c, d\}$ and $\{c, d, e\}$. Since u_1 and u_2 can each only forbid b, c or e , we can always find a color for v .

In Case 2, u_1 and u_2 each forbid at most one color and u_3 forbids at most 2 colors so there remains at least one color for v .

In Case 3, u_1, u_2 , and u_3 each forbid at most one color, so there remains at least two colors for v .

We can always extend φ into a \vec{T}_5 -coloring of G , a contradiction.

So G contains at most 5 consecutive 2-vertices and every 3-vertex has at most 11 2-weak-neighbors. Using Lemma 9 with $k = 11$ contradicts the fact that the girth of G is at least 28.

2.2 Proof of Theorem 3.2

We prove that the 2-edge-colored planar graph T_6 on 6 vertices from Figure 2 is $P_{22}^{(0,2)}$ -universal by contradiction. Assume that G is a 2-edge-colored planar graphs with girth at least 22 that does not admit a homomorphism to T_6 and is minimal with respect to the number of vertices. By minimality, G cannot contain a vertex v with degree at most one since a T_6 -coloring of $G - v$ can be extended to G . Similarly, G does not contain the following configurations.

- A path with 5 consecutive 2-vertices.
- A 3-vertex with at least 9 2-weak-neighbors.

Suppose that G contains a path $u_0u_1u_2u_3u_4u_5u_6$ such that the degree of u_i is two for $1 \leq i \leq 5$. By minimality of G , $G - u_1, u_2, u_3, u_4, u_5$ admits a T_6 -coloring φ . We checked on a computer that for any $\varphi(v_0)$ and $\varphi(v_6)$ in $V(T)$ and every possible colors of the 6 edges u_iu_{i+1} , we can always extend φ into a T_6 -coloring of G , a contradiction.

Suppose that G contains a 3-vertex v with at least 9 2-weak-neighbors. Let u_1, u_2, u_3 be the 3⁺-weak-neighbors of v and let l_i be the number of common 2-weak-neighbors of v and u_i , i.e., 2-vertices on the path between v and u_i . Without loss of generality and by the previous discussion, we have $4 \geq l_1 \geq l_2 \geq l_3$ and $l_1 + l_2 + l_3 \geq 9$. So we have to consider the following cases:

- **Case 1:** $l_1 = 3, l_2 = 3, l_3 = 3$.
- **Case 2:** $l_1 = 4, l_2 = 3, l_3 = 2$.
- **Case 3:** $l_1 = 4, l_2 = 4, l_3 = 1$.

By minimality of G , the graph G' obtained from G by removing v and its 2-weak-neighbors admits a T_6 -coloring φ . Let us show that in all three cases, we can extend φ into a T_6 -coloring of G to get a contradiction.

With an extensive search on a computer we found that if a vertex v is connected to a vertex u colored in $\varphi(u)$ by a path P made of l 2-vertices ($0 \leq l \leq 4$) then v can be colored in:

- at least 1 color if $l = 0$ (the sets a, c, d, e, f and b, c, d, e, f of colors are the only sets of size 5 that can be forbidden from v for some $\varphi(u) \in T$ and edge-colors on P),
- at least 2 colors if $l = 1$ (the sets a, b, c, f and b, c, e, f are the only sets of size 4 that can be forbidden from v),
- at least 3 colors if $l = 2$ (the sets b, c, f, c, e, f and d, e, f are the only sets of size 3 that can be forbidden from v),
- at least 4 colors if $l = 3$ (the set c, b is the only set of size 2 that can be forbidden from v), and
- at least 5 colors if $l = 4$ (the sets c and f are the only sets of size 1 that can be forbidden from v).

Suppose that we are in Case 1. Vertices u_1, u_2 , and u_3 each forbid at most 2 colors from v since $l_1 = l_2 = l_3 = 3$. Suppose that u_1 forbids 2 colors. It has to forbid colors c and f (since it is the only pair of colors that can be forbidden by a path made of 3 2-vertices). If u_2 or u_3 also forbids 2 colors, they will forbid the exact same pair of colors. We can therefore assume that they each forbid 1 color from v . There are 6 available colors in T_6 , so we can always find a color for v and extend φ to a T_6 -coloring of G , a contradiction. We proceed similarly for the other two cases.

So G contains at most 4 consecutive 2-vertices and every 3-vertex has at most 8 2-weak-neighbors. Then Lemma 9 with $k = 8$ contradicts the fact that the girth of G is at least 22.

3 Proof of Theorem 4.1

We construct an oriented bipartite cactus graph with girth at least g and oriented chromatic number at least 5. Let g' be such that $g' \geq g$ and $g' \equiv 4 \pmod{6}$. Consider a circuit $v_1, \dots, v_{g'}$. Clearly, the oriented chromatic number of this circuit is 4 and the only tournament on 4 vertices it can map to is the tournament \vec{T}_4 induced by the vertices a, b, c , and d in \vec{T}_5 . Now we consider the cycle $C = w_1, \dots, w_{g'}$ containing the arcs $w_{2i-1}w_{2i}$ with $1 \leq i \leq g'/2$, $w_{2i+1}w_{2i}$ with $1 \leq i \leq g'/2 - 1$, and $w_{g'}w_1$.

Suppose for contradiction that C admits a homomorphism φ such that $\varphi(w_1) = d$. This implies that $\varphi(w_2) = a$, $\varphi(w_3) = d$, $\varphi(w_4) = a$, and so on until $\varphi(w_{g'}) = a$. Since $\varphi(w_{g'}) = a$ and $\varphi(w_1) = d$, $w_{g'}w_1$ should map to ad , which is not an arc of \vec{T}_4 , a contradiction.

Our cactus graph is then obtained from the circuit $v_1, \dots, v_{g'}$ and g' copies of C by identifying every vertex v_i with the vertex w_1 of a copy of C . This cactus graph does not map to \vec{T}_4 since one of the v_i would have to map to d and then the copy of C attached to v_i would not be \vec{T}_4 -colorable.

4 Proof of Theorem 4.2

We construct a 2-edge-colored bipartite outerplanar graph with girth at least g that does not map to a 2-edge-colored planar graph with at most 5 vertices. Let g' be such that $g' \geq g$ and $g' \equiv 2 \pmod{4}$. Consider an alternating cycle $C = v_0, \dots, v_{g'-1}$. For every $0 \leq i \leq g' - 3$, we add $g' - 2$ 2-vertices $w_{i,1}, \dots, w_{i,g'-2}$ that form the path $P_i = v_i w_{i,1} \dots w_{i,g'-2} v_{i+1}$ such that the edges of P_i get the color distinct from the color of the edge $v_i v_{i+1}$. Let G be the obtained graph. The 2-edge-colored chromatic number of C is 5. So without loss of generality, we assume for contradiction that G admits a homomorphism φ to a 2-edge-colored planar graph H on 5 vertices. Let us define $\mathcal{E} = \bigcup_{i \text{ even}} \varphi(v_i)$ and $\mathcal{O} = \bigcup_{i \text{ odd}} \varphi(v_i)$. Since C is alternating, $\varphi(v_i) \neq \varphi(v_{i+2})$ (indices are modulo g'). Since $g' \equiv 2 \pmod{4}$, there is an odd number of v_i with an even (resp. odd) index. Thus, $|\mathcal{E}| \geq 3$ and $|\mathcal{O}| \geq 3$. Therefore we must have $\mathcal{E} \cap \mathcal{O} \neq \emptyset$.

Notice that every two vertices v_i and v_j in G are joined by a blue path and a red path such that the lengths of these paths have the same parity as $i - j$. Thus, the blue (resp. red) edges of H must induce a connected spanning subgraph of H . Since $|V(H)| = 5$, H contains at least 4 blue (resp. red) edges. Since red and blue edges play symmetric roles in G and since $|E(H)| \leq 9$ by Proposition 8, we assume without loss of generality that H contains exactly 4 blue edges. Moreover, these 4 blue edges induce a tree. In particular, the blue edges induce a bipartite graph which partitions $V(H)$ into 2 parts. Thus, every v_i with even index is mapped into one part of $V(H)$ and every v_i with odd index is mapped into the other part of $V(H)$. So $\mathcal{E} \cap \mathcal{O} = \emptyset$, which is a contradiction.

5 Proof of Theorem 2

Let T be a $P_g^{(m,n)}$ -universal planar graph for some g that is minimal with respect to the subgraph order.

By minimality of T , there exists a graph $G \in P_g^{(m,n)}$ such that every color in T has to be used at least once to color G . Without loss of generality, G is connected, since otherwise we can replace G by the connected graph obtained from G by choosing a vertex in each component of G and identifying them. We obtain a graph G' from G as follows:

For each edge or arc uv in G , we keep uv in G' and we add $4m + n$ paths starting at u and ending at v made of vertices of degree 2:

- For each type of edge, we add a path made of $g - 1$ edges of this type.
- For each type of arc, we add two paths made of $g - 1$ arcs of this type such that the paths alternate between forward and backward arcs. We make the paths such that u is the tail of the first arc of one path and the head of the first arc of the other path.

- Similarly, for each type of arc we add two paths made of g arcs of this type such that the paths alternate between forward and backward arcs. We make the paths such that u is the tail of the first arc of one path and the head of the first arc of the other path.

Notice that G' is in $P_g^{(m,n)}$ and thus admits a homomorphism φ to T . Since G is a connected subgraph of G' and every color in T has to be used at least once to color G , we can find for each pair of vertices (c_1, c_2) in T and each type of edge a path (v_1, v_2, \dots, v_l) in G' made only of edges of this type such that $\varphi(v_1) = c_1$ and $\varphi(v_l) = c_2$.

This implies that for every pair of vertices (c_1, c_2) in T and each type of edge, there exists a walk from c_1 to c_2 made of edges of this type. Therefore, for $1 \leq j \leq n$, the subgraph induced by $E_j(T)$ is connected and contains all the vertices of T . So $E_j(T)$ contains a spanning tree of T . Thus T contains at least $|V(T)| - 1$ edges of each type.

Similarly, we can find for each pair of vertices (c_1, c_2) in T and each type of arc a path of even length $(v_1, v_2, \dots, v_{2l-1})$ in G' made only of arcs of this type, starting with a forward arc and alternating between forward and backward arcs such that $\varphi(v_1) = c_1$ and $\varphi(v_l) = c_2$. We can also find a path of the same kind with odd length.

This implies that for every pair of vertices (c_1, c_2) in T and each type of arc there exist a walk of odd length and a walk of even length from c_1 to c_2 made of arcs of this type, starting with a forward arc and alternating between forward and backward arcs. Let p be the maximum of the length of all these paths. Given one of these walks of length l , we can also find a walk of length $l + 2$ that satisfies the same constraints by going through the last arc of the walk twice more. Therefore, for every $l \geq p$, every pair of vertices (c_1, c_2) in T , and every type of arc, it is possible to find a homomorphism from the path P of length l made of arcs of this type, starting with a forward arc and alternating between forward and backward arcs to T such that the first vertex is colored in c_1 and the last vertex is colored in c_2 .

We now show that this implies that $|A_j(T)| \geq 2|V(T)| - 1$ for $1 \leq j \leq m$. Let P be a path $(v_1, v_2, \dots, v_p, v_{p+1})$ of length p starting with a forward arc and alternating between forward and backward arcs of the same type. We color v_1 in some vertex c of T . Let C_i be the set of colors in which vertex v_i could be colored. We know that $C_1 = c$ and C_2 is the set of direct successors of c . Set C_3 is the set of direct predecessors of vertices in C_2 so $C_1 \subseteq C_3$ and, more generally, $C_i \subseteq C_{i+2}$. Let uv be an arc in T . If $u \in C_i$ with i odd, then $v \in C_{i+1}$. If $v \in C_i$ with i even then $u \in C_{i+1}$. We can see that uv is capable of adding at most one vertex to a C_i (and every C_j with $j \equiv i \pmod{2}$ and $i \leq j$). We know that $C_{p+1} = V(T)$ hence T contains at least $2|V(T)| - 1$ arcs of each type.

Therefore, the underlying graph of T contains at least $m(2|V(T)| - 1) + n(|V(T)| - 1) = (2m + n)|V(T)| - m - n$ edges, which contradicts Proposition 8 for $2m + n \geq 3$.

6 Proof of Theorem 5.1

We construct an oriented bipartite 2-outerplanar graph with girth 14 that does not map to \vec{T}_5 .

The oriented graph X is a cycle on 14 vertices v_0, \dots, v_{13} such that the tail of every arc is the vertex with even index, except for the arc $v_{13}v_0$. Suppose for contradiction that X has a \vec{T}_5 -coloring h such that no vertex with even index maps to b . The directed path $v_{12}v_{13}v_0$ implies that $h(v_{12}) \neq h(v_0)$. If $h(v_0) = a$, then $h(v_1) \in \{b, c\}$ and $h(v_2) = a$ since $h(v_2) \neq b$. By contagion, $h(v_0) = h(v_2) = \dots = h(v_{12}) = a$, which is a contradiction. Thus $h(v_0) \neq a$. If $h(v_0) = c$, then $h(v_1) = d$ and $h(v_2) = c$ since $h(v_2) \neq b$. By contagion, $h(v_0) = h(v_2) = \dots = h(v_{12}) = c$, which is a contradiction. Thus $h(v_0) \neq c$. So $h(v_0) \notin \{a, b, c\}$, that is, $h(v_0) \in \{d, e\}$. Similarly, $h(v_{12}) \in \{d, e\}$. Notice that \vec{T}_5 does not contain a directed path xyz such that x and z belong to

$\{d, e\}$. So the path $v_{12}v_{13}v_0$ cannot be mapped to \vec{T}_5 . Thus X does not have a \vec{T}_5 -coloring h such that no vertex with even index maps to b .

Consider now the path P on 7 vertices p_0, \dots, p_6 with the arcs $\overrightarrow{p_1p_0}, \overrightarrow{p_1p_2}, \overrightarrow{p_3p_2}, \overrightarrow{p_4p_3}, \overrightarrow{p_5p_4}, \overrightarrow{p_5p_6}$. It is easy to check that there exists no \vec{T}_5 -coloring h of P such that $h(p_0) = h(p_6) = b$.

We construct the graph Y as follows: we take 8 copies of X called $X_{\text{main}}, X_0, X_2, X_4, \dots, X_{12}$. For every couple $(i, j) \in \{0, 2, 4, 6, 8, 10, 12\}^2$, we take a copy $P_{i,j}$ of P , we identify the vertex p_0 of $P_{i,j}$ with the vertex v_i of X_{main} and we identify the vertex p_6 of $P_{i,j}$ with the vertex v_j of H_i .

So Y is our oriented bipartite 2-outerplanar graph with girth 14. Suppose for contradiction that Y has a \vec{T}_5 -coloring h . By previous discussion, there exists $i \in \{0, 2, 4, 6, 8, 10, 12\}$ such that the vertex v_i of X_{main} maps to b . Also, there exists $j \in \{0, 2, 4, 6, 8, 10, 12\}$ such that the vertex v_j of X_i maps to b . So the corresponding path $P_{i,j}$ is such that $h(p_0) = h(p_6) = b$, a contradiction. Thus Y does not map to \vec{T}_5 .

7 Proof of Theorem 5.2

We construct a 2-edge-colored 2-outerplanar graph with girth 11 that does not map to T_6 . We take 12 copies X_0, \dots, X_{11} of a cycle of length 11 such that every edge is red. Let $v_{i,j}$ denote the j^{th} vertex of X_i . For every $0 \leq i \leq 10$ and $0 \leq j \leq 10$, we add a path consisting of 5 blue edges between $v_{i,11}$ and $v_{j,i}$.

Notice that in any T_6 -coloring of a red odd cycle, one vertex must map to c . So we suppose without loss of generality that $v_{0,11}$ maps to c . We also suppose without loss of generality that $v_{0,0}$ maps to c . The blue path between $v_{0,11}$ and $v_{0,0}$ should map to a blue walk of length 5 from c to c in T_6 . Since T_6 contains no such walk, our graph does not map to T_6 .

8 Proof of Theorem 5.3

We construct a 2-edge-colored bipartite 2-outerplanar graph with girth 10 that does not map to T_6 . By Theorem 4.2, there exists a bipartite outerplanar graph M with girth at least 10 such that for every T_6 -coloring h of M , there exists a vertex v in M such that $h(v) = c$.

Let X be the graph obtained as follows. Take a main copy Y of M . For every vertex v of Y , take a copy Y_v of M . Since Y_v is bipartite, let A and B the two independent sets of Y_v . For every vertex w of A , we add a path consisting of 5 blue edges between v and w . For every vertex w of B , we add a path consisting of 4 edges colored (blue, blue, red, blue) between v and w .

Notice that X is indeed a bipartite 2-outerplanar graph with girth 10. We have seen in the previous proof that T_6 contains no blue walk of length 5 from c to c . We also check that T_6 contains no walk of length 4 colored (blue, blue, red, blue) from c to c . By the property of M , for every T_6 -coloring h of X , there exist a vertex v in Y and a vertex w in Y_v such that $h(v) = h(w) = c$. Then h cannot be extended to the path of length 4 or 5 between v and w . So X does not map to T_6 .

9 Proof of Theorem 6.1

Let g be the largest integer such that there exists a graph in $P_g^{(1,0)}$ that does not map to \vec{T}_5 . Let $G \in P_g^{(1,0)}$ be a graph that does not map to \vec{T}_5 and such that the underlying graph of G is minimal with respect to the homomorphism order.

Let G' be obtained from G by removing an arbitrary arc v_0v_3 and adding two vertices v_1 and v_2 and the arcs v_0v_1, v_2v_1, v_2v_3 . By minimality, G' admits a homomorphism φ to \vec{T}_5 . Suppose for contradiction that $\varphi(v_2) = c$. This implies that $\varphi(v_1) = \varphi(v_3) = d$. Thus φ provides a \vec{T}_5 -coloring of G , a contradiction. So $\varphi(v_2) \neq c$ and, similarly, $\varphi(v_2) \neq e$.

Given a set S of vertices of \vec{T}_5 , we say that we force S if we specify a graph H and a vertex $v \in V(H)$ such that for every vertex $x \in V(\vec{T}_5)$, we have $x \in S$ if and only if there exists a

\vec{T}_5 -coloring φ of H such that $\varphi(v) = x$. Thus, with the graph G' and the vertex v_2 , we force a non-empty set $\mathcal{S} \subset V(\vec{T}_5) \setminus \{c, e\} = \{a, b, d\}$.

We use a series of constructions in order to eventually force the set $\{a, b, c, d\}$ starting from \mathcal{S} . Recall that $\{a, b, c, d\}$ induces the tournament \vec{T}_4 . We thus reduce \vec{T}_5 -coloring to \vec{T}_4 -coloring, which is NP-complete for subcubic bipartite planar graphs with any given girth [4].

These constructions are summarized in the tree depicted in Figure 3. The vertices of this forest contain the non-empty subsets of $\{a, b, d\}$ and a few other sets. In this tree, an arc from S_1 to S_2 means that if we can force S_1 , then we can force S_2 . Every arc has a label indicating the construction that is performed. In every case, we suppose that S_1 is forced on the vertex v of a graph H_1 and we construct a graph H_2 that forces S_2 on the vertex w .

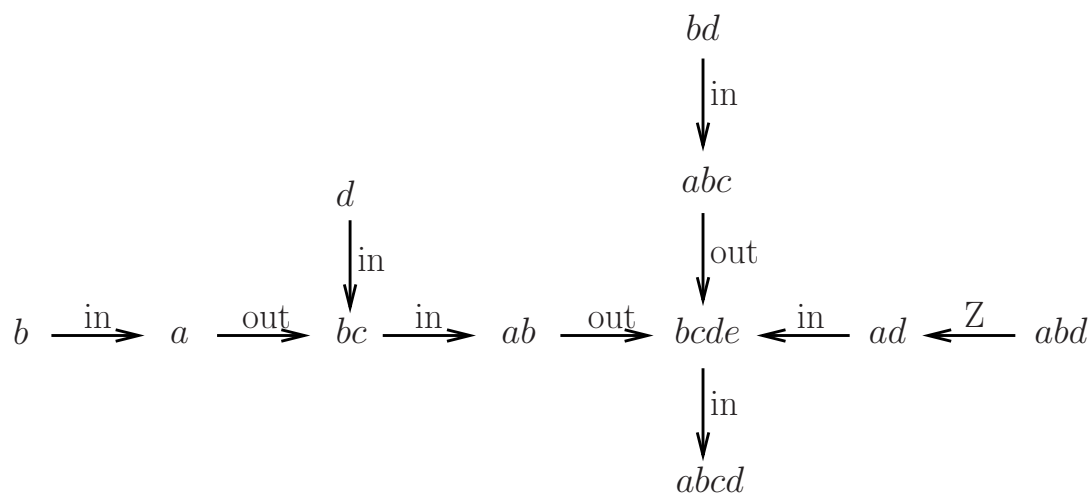


Figure 3: Forcing the set $\{a, b, c, d\}$.

- Arcs labelled "out": The set S_2 is the out-neighborhood of S_1 in \vec{T}_5 . We construct H_2 from H_1 by adding a vertex w and the arc vw . Thus, S_2 is indeed forced on the vertex w of H_2 .
- Arcs labelled "in": The set S_2 is the in-neighborhood of S_1 in \vec{T}_5 . We construct H_2 from H_1 by adding a vertex w and the arc wv . Thus, S_2 is indeed forced on the vertex w of H_2 .
- Arc labelled "Z": Let g' be the smallest integer such that $g' \geq g$ and $g' \equiv 4 \pmod{6}$. We consider a circuit $v_1, \dots, v_{g'}$. For $2 \leq i \leq g'$, we take a copy of H_1 and we identify its vertex v with v_i . We thus obtain the graph H_2 and we set $w = v_2$. Let φ be any T_6 -coloring of H_2 . By construction, $\{\varphi(v_2), \dots, \varphi(v_{g'})\} \subset S_1 = \{a, b, d\}$. A circuit of length $\not\equiv 0 \pmod{3}$ cannot map to the 3-circuit induced by $\{a, b, d\}$, so $\varphi(v_1) \in \{c, e\}$. If $\varphi(v_1) = c$ then $\varphi(v_2) = d$ and if $\varphi(v_1) = e$ then $\varphi(v_2) = a$. Thus $S_2 = \{ad\}$.

10 Proof of Theorem 6.2

Let g be the largest integer such that there exists a graph in $P_g^{(0,2)}$ that does not map to T_6 . Let $G \in P_g^{(0,2)}$ be a graph that does not map to T_6 and such that the underlying graph of G is minimal with respect to the homomorphism order.

Let G' be obtained from G by subdividing an arbitrary edge v_0v_3 twice to create the path $v_0v_1v_2v_3$ such that the edges v_0v_1 and v_1v_2 are red and the edge v_2v_3 gets the color of the original edge v_0v_3 . By minimality, G' admits a homomorphism φ to T_6 . Suppose for contradiction

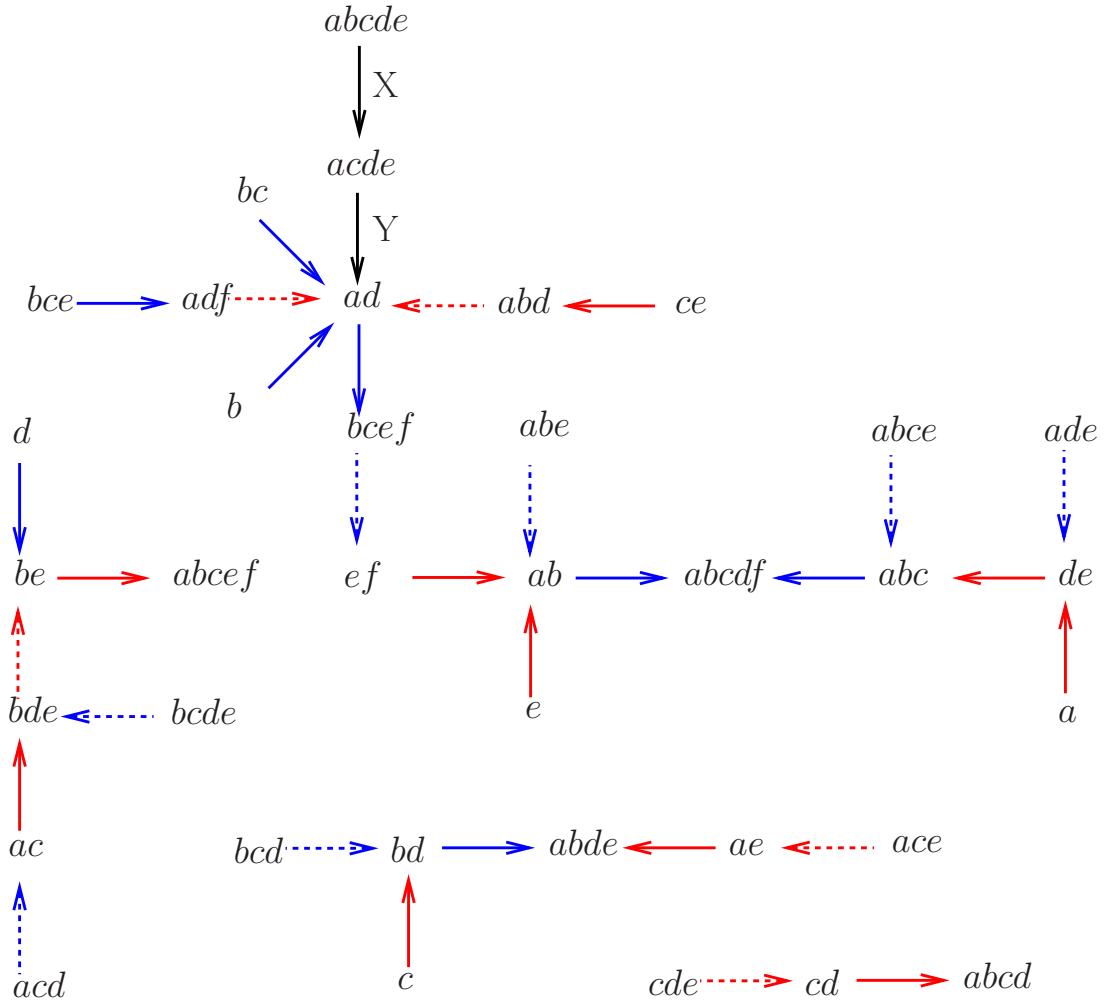


Figure 4: Forcing a good set.

that $\varphi(v_1) = f$. This implies that $\varphi(v_0) = \varphi(v_2) = b$. Thus φ provides a T_6 -coloring of G , a contradiction.

Given a set S of vertices of T_6 , we say that we force S if we specify a graph H and a vertex $v \in V(H)$ such that for every vertex $x \in V(T_6)$, we have $x \in S$ if and only if there exists T_6 -coloring φ of H such that $\varphi(v) = x$. Thus, with the graph G' and the vertex v_1 , we force a non-empty set $\mathcal{S} \subset V(T_6) \setminus \{f\} = \{a, b, c, d, e\}$.

Recall that the core of a graph is the smallest subgraph which is also a homomorphic image. We say that a subset S of $V(T_6)$ is *good* if the core of the subgraph induced by S is isomorphic to the graph T_4 which is a a clique on 4 vertices such that both the red and the blue edges induce a path of length 3. We use a series of constructions in order to eventually force a good set starting from \mathcal{S} . We thus reduce T_6 -coloring to T_4 -coloring, which is NP-complete for subcubic bipartite planar graphs with any given girth [5].

These constructions are summarized in the forest depicted in Figure 4. The vertices of this forest are the non-empty subsets of $\{a, b, c, d, e\}$ together with a few auxiliary sets of vertices containing f . In this forest, an arc from S_1 to S_2 means that if we can force S_1 , then we can force S_2 . Every set with no outgoing arc is good. We detail below the construction that is performed for each arc. In every case, we suppose that S_1 is forced on the vertex v of a graph H_1 and we construct a graph H_2 that forces S_2 on the vertex w .

- Blue arcs: The set S_2 is the blue neighborhood of S_1 in T_6 . We construct H_2 from H_1 by adding a vertex w adjacent to v such that vw is blue. Thus, S_2 is indeed forced on the vertex w of H_2 .
- Red arcs: The set S_2 is the red neighborhood of S_1 in T_6 . The construction is as above except that the edge vw is red.
- Dashed blue arcs: The set S_2 is the set of vertices incident to a blue edge contained in the subgraph induced by S_1 in T_6 . We construct H_2 from two copies of H_1 by adding a blue edge between the vertex v of one copy and the vertex v of the other copy. Then w is one of the vertices v .
- Dashed red arcs: The set S_2 is the set of vertices incident to a red edge contained in the subgraph induced by S_1 in T_6 . The construction is as above except that the added edge is red.
- Arc labelled "X": Let $g' = 2 \lceil g/2 \rceil$. We consider an even cycle $v_1, \dots, v_{g'}$ such that $v_1v_{g'}$ is red and the other edges are blue. For every vertex v_i , we take a copy of H_1 and we identify its vertex v with v_i . We thus obtain the graph H_2 and we set $w = v_1$. Let φ be any T_6 -coloring of H_2 . In any T_6 -coloring of H_2 , the cycle $v_1, \dots, v_{g'}$ maps to a 4-cycle with exactly one red edge contained in the subgraph of T_6 induced by $S_1 = \{a, b, c, d, e\}$. These 4-cycles are $aedb$ with red edge ae and $cdba$ with red edge cd . Since w is incident to the red edge in the cycle $v_1, \dots, v_{g'}$, w can be mapped to a, e, c , or d but not to b . Thus $S_2 = \{a, c, d, e\}$.
- Arc labelled "Y": We consider an alternating cycle v_0, \dots, v_{8g-1} . For every vertex v_i , we take a copy of H_1 and we identify its vertex v with v_i . We obtain the graph H_2 by adding the vertex x adjacent to v_0 and v_{4g+2} such that xv_0 and xv_{4g+2} are blue. We set $w = v_0$. In any T_6 -coloring φ of H_2 , the cycle $v_1, \dots, v_{g'}$ maps to the alternating 4-cycle $acde$ contained in $S_1 = \{a, c, d, e\}$ such that $\varphi(v_i) = \varphi(v_{i+4 \pmod{8g}})$. So, a priori, either $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{a, d\}$ or $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{c, e\}$. In the former case, we can extend φ to H_2 by setting $\varphi(x) = b$. In the latter case, we cannot color x since c and e have no common blue neighbor in T_6 . Thus, $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{a, d\}$ and $S_2 = \{a, d\}$.

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