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# Avoidability of palindrome patterns

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## Abstract

We characterize the formulas that are avoided by every  $\alpha$ -free word for some  $\alpha > 1$ . We show that the avoidable formulas whose fragments are of the form  $XY$  or  $XYX$  are 4-avoidable. The largest avoidability index of an avoidable palindrome pattern is known to be at least 4 and at most 16. We make progress toward the conjecture that every avoidable palindrome pattern is 4-avoidable.

**Mathematics Subject Classifications:** 68R15

## 1 Introduction

A *pattern*  $p$  is a non-empty finite word over an alphabet  $\Delta = \{A, B, C, \dots\}$  of capital letters called *variables*. An *occurrence* of  $p$  in a word  $w$  is a non-erasing morphism  $h : \Delta^* \rightarrow \Sigma^*$  such that  $h(p)$  is a factor of  $w$  (a morphism is *non-erasing* if the image of every letter is non-empty). The *avoidability index*  $\lambda(p)$  of a pattern  $p$  is the size of the smallest alphabet  $\Sigma$  such that there exists an infinite word over  $\Sigma$  containing no occurrence of  $p$ . Since there is no risk of confusion,  $\lambda(p)$  will be simply called the index of  $p$ .

A variable that appears only once in a pattern is said to be *isolated*. Following Cassaigne [5], we associate a pattern  $p$  with the *formula*  $f$  obtained by replacing every isolated variable in  $p$  by a dot. The factors between the dots are called *fragments*.

An *occurrence* of a formula  $f$  in a word  $w$  is a non-erasing morphism  $h : \Delta^* \rightarrow \Sigma^*$  such that the  $h$ -image of every fragment of  $f$  is a factor of  $w$ . As for patterns, the index  $\lambda(f)$  of a formula  $f$  is the size of the smallest alphabet allowing the existence of an infinite word containing no occurrence of  $f$ . Clearly, if a formula  $f$  is associated with a pattern  $p$ , every

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word avoiding  $f$  also avoids  $p$ , so  $\lambda(p) \leq \lambda(f)$ . Recall that an infinite word is *recurrent* if every finite factor appears infinitely many times and that any infinite factorial language contains a recurrent word [8, Proposition 5.1.13]. If there exists an infinite word over  $\Sigma$  avoiding  $p$ , then there exists an infinite recurrent word over  $\Sigma$  avoiding  $p$ . This recurrent word also avoids  $f$ , so that  $\lambda(p) = \lambda(f)$ . Without loss of generality, a formula is such that no variable is isolated and no fragment is a factor of another fragment.

Let us define the types of formulas we consider in this paper. A pattern is *doubled* if it contains every variable at least twice. Thus it is a formula with only one pattern. A formula  $f$  is *nice* if for every variable  $X$  of  $f$ , there exists a fragment of  $f$  that contains  $X$  at least twice. Notice that a doubled pattern is a nice pattern. A formula is an *xyx-formula* if every fragment is of the form  $XYX$ , i.e., the fragment has length 3 and the first and third variable are the same. A formula is *hybrid* if every fragment has length 2 or is of the form  $XYX$ . Thus, an *xyx-formula* is a hybrid formula.

In Section 3, we consider the avoidance of nice formulas. In Section 4, we find some formulas  $f$  such that every recurrent word avoiding  $f$  over  $\Sigma_{\lambda(f)}$  is equivalent to a well-known morphic word. In Section 5, we consider the avoidance of *xyx-formulas* and hybrid formulas. In Section 6, we consider the avoidance of patterns that are palindromes.

## 2 Preliminaries

Given a pattern  $p$ , the Zimin operator constructs the pattern  $Z(p) = pXp$  where  $X$  is a variable that is not contained in  $p$ . For every fixed  $t$ ,  $Z^t(p)$  denotes the pattern obtained by applying  $t$  times the Zimin operator to  $p$ . Notice that a recurrent word avoids  $Z^t(p)$  if and only if it avoids  $p$ .

We say that a formula  $f$  *divides* a formula  $f'$  if every recurrent word avoiding  $f$  also avoids  $f'$ . We denote by  $f \preceq f'$  the fact that  $f$  divides  $f'$ . By previous discussion,  $p \preceq Z^t(p)$  and  $Z^t(p) \preceq p$  for every pattern  $p$ . The basic case of divisibility is that  $f \preceq f'$  if  $f'$  contains an occurrence  $f$ , that is, if there exists a non-erasing morphism  $h$  such that the  $h$ -image of every fragment of  $f$  is a factor of a fragment of  $f'$ . Another case of divisibility obtained by transitivity: in order to obtain  $f \preceq p$ , it is sufficient to prove  $f \preceq Z^t(p)$ , since  $Z^t(p) \preceq p$ . We use this trick in the proof of Lemma 6 and Theorem 17. Of course, divisibility is related to avoidability: if  $f \preceq f'$ , then  $\lambda(f) \geq \lambda(f')$ .

Let  $\Sigma_k = \{0, 1, \dots, k-1\}$  denote the  $k$ -letter alphabet. We denote by  $\Sigma_k^n$  the  $k^n$  words of length  $n$  over  $\Sigma_k$ .

The operation of *splitting* a formula  $f$  on a fragment  $\phi$  consists in replacing  $\phi$  by two fragments, namely the prefix and the suffix of length  $|\phi| - 1$  of  $\phi$ . A formula  $f$  is *minimally avoidable* if splitting any fragment of  $f$  gives an unavoidable formula. The set of every minimally avoidable formula with at most  $n$  variables is called the  $n$ -avoidance basis.

The *adjacency graph*  $AG(f)$  of the formula  $f$  is the bipartite graph such that

- for every variable  $X$  of  $f$ ,  $AG(f)$  contains the two vertices  $X_L$  and  $X_R$ ,
- for every (possibly equal) variables  $X$  and  $Y$ , there is an edge between  $X_L$  and  $Y_R$  if and only if  $XY$  is a factor of  $f$ .

We say that a set  $S$  of variables of  $f$  is *free* if for all  $X, Y \in S$ ,  $X_L$  and  $Y_R$  are in distinct connected components of  $AG(f)$ . A formula  $f$  is said to reduce to  $f'$  if it is obtained by deleting all the variables of a free set from  $f$ , discarding any empty word fragment. A formula is *reducible* if there is a sequence of reductions to the empty formula. Finally, a *locked* formula is a formula having no free set.

**Theorem 1** ([3]). *A formula is unavoidable if and only if it is reducible.*

Let us define here the following well-known pure morphic words. To specify a morphism  $m : \Sigma_s \rightarrow \Sigma_e$ , we use the notation  $m = m(0)/m(1)/\dots/m(s-1)$ . Assuming a morphism  $m : \Sigma_s \rightarrow \Sigma_s$  is such that  $m(0)$  starts with 0, the *fixed point* of  $m$  is the right infinite word  $m^\omega(0)$ .

- $b_2$  is the fixed point of 01/10.
- $b_3$  is the fixed point of 012/02/1.
- $b_4$  is the fixed point of 01/03/21/23.
- $b_5$  is the fixed point of 01/23/4/21/0

We also consider the morphic words  $v_3 = M_1(b_5)$  and  $w_3 = M_2(b_5)$ , where  $M_1 = 012/1/02/12/\varepsilon$  and  $M_2 = 02/1/0/12/\varepsilon$ . The languages of each of these words have been studied in the literature. Let us first recall the following characterization of  $b_3$ ,  $v_3$ , and  $w_3$ . We say that two infinite words are *equivalent* if they have the same set of factors.

**Theorem 2** ([1, 16]).

- *Every ternary square-free recurrent word avoiding 010 and 212 is equivalent to  $b_3$ .*
- *Every ternary square-free recurrent word avoiding 010 and 020 is equivalent to  $v_3$ .*
- *Every ternary square-free recurrent word avoiding 121 and 212 is equivalent to  $w_3$ .*

Interestingly, these three words can be characterized in terms of a forbidden distance between consecutive occurrences of one letter.

**Theorem 3.**

- *Every ternary square-free recurrent word such that the distance between consecutive occurrences of 1 is not 3 is equivalent to  $b_3$ .*
- *Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 2 is equivalent to  $v_3$ .*
- *Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 4 is equivalent to  $w_3$ .*

*Proof.*

- Another characterization for  $b_3$  is that every ternary square-free recurrent word avoiding 1021 and 1021 is equivalent to  $b_3$  [1]. This rules out the possibility that the distance between two occurrences of 1 is 3.
- Since  $v_3$  avoids 010 and 020, the distance between two occurrences of 0 is at least 3.
- Since  $w_3$  avoids 121 and 212, the distance between consecutive occurrences of 0 is at most 3.  $\square$

The word  $b_4$  is also known to avoid large families of formulas.

**Theorem 4** ([2]). *Every locked formula is avoided by  $b_4$ .*

**Theorem 5** ([5, Proposition 1.13]). *If every fragment of an avoidable formula  $f$  has length 2, then  $b_4$  avoids  $f$ .*

Theorem 5 will be extended to hybrid formulas, see Theorem 21 in Section 5.

Let us give here a result that will be needed in various parts of the paper.

**Lemma 6.**  $ABA.ACA.ABCA.ACBA.ABCBA \preceq AA$ .

*Proof.* Indeed,  $Z^2(AA) = AABAACAABAA$  contains the occurrence  $A \rightarrow A, B \rightarrow ABA, C \rightarrow ACA$  of  $ABA.ACA.ABCA.ACBA.ABCBA$ .  $\square$

Thus, if  $w$  is a recurrent word that avoids a formula dividing  $ABA.ACA.ABCA.ACBA.ABCBA$ , then  $w$  is square-free.

Recall that the repetition threshold  $RT(n)$  is the smallest real number  $\alpha$  such that there exists an infinite  $a^+$ -free word over  $\Sigma_n$ . The proof of Dejean's conjecture established that  $RT(2) = 2$ ,  $RT(3) = \frac{7}{5}$ ,  $RT(4) = \frac{7}{4}$ , and  $RT(n) = \frac{n}{n-1}$  for every  $n \geq 5$ . An infinite  $RT(n)^+$ -free word over  $\Sigma_n$  is called a Dejean word.

### 3 Nice formulas

All the nice formulas considered so far in the literature are also 3-avoidable. This includes doubled patterns [12], circular formulas [9], the nice formulas in the 3-avoidance basis [9], and the minimally nice ternary formulas in Table 1 [15].

**Theorem 7** ([9, 15]). *Every nice formula with at most 3 variables is 3-avoidable.*

We have a risky conjecture that would generalize both Theorem 7 and the 3-avoidability of doubled patterns.

**Conjecture 8.** Every nice formula is 3-avoidable.

Theorem 19 in Section 5 shows that there exist infinitely many nice formulas with index 3. It means that Conjecture 8 would be best possible and it contrasts with the case of doubled patterns, since we expect that there exist only finitely many doubled patterns with index 3 [12, 13]. In this section, we make progress toward Conjecture 8 by proving that every nice formula is avoidable and we explain how to get an upper bound on the index of a given nice formula.

### 3.1 The avoidability exponent

Let us consider a useful tool in pattern avoidance that has been defined in [12] and already used implicitly in [11]. The *avoidability exponent*  $AE(p)$  of a pattern  $p$  is the largest real  $\alpha$  such that every  $\alpha$ -free word avoids  $p$ . We extend this definition to formulas. The corresponding notion for the avoidance of patterns in the abelian setting has also been considered [7].

Let us show that  $AE(ABCBA.CBABC) = \frac{4}{3}$ . Suppose for contradiction that a  $\frac{4}{3}$ -free word contains an occurrence  $h$  of  $ABCBA.CBABC$ . We write  $y = |h(Y)|$  for every variable  $Y$ . The factor  $h(ABCBA)$  is a repetition with period  $|h(ABCBA)|$ . So we have  $\frac{a+b+c+b+a}{a+b+c+b} < \frac{4}{3}$ . This simplifies to  $2a < 2b + c$ . Similarly,  $CBABC$  gives  $2c < a + 2b$ ,  $BAB$  gives  $2b < a$ , and  $BCB$  gives  $2b < c$ . Summing up these four inequalities gives  $2a + 4b + 2c < 2a + 4b + 2c$ , which is a contradiction. On the other hand, the word 01234201567865876834201234 is  $(\frac{4}{3})$ -free and contains the occurrence  $A \rightarrow 01$ ,  $B \rightarrow 2$ ,  $C \rightarrow 34$  of  $ABCBA.CBABC$ .

As a second example, we obtain that  $AE(ABCDACBD) = 1.246266172\dots$ . When we consider a repetition  $uvu$  in an  $\alpha$ -free word, we derive that  $\frac{|uvu|}{|uv|} < \alpha$ , which gives  $\beta|u| < |v|$  with  $\alpha = 1 + \frac{1}{\beta+1}$ . We consider an occurrence  $h$  of the pattern. The maximal repetitions in  $ABCDACBD$  are  $ABCDAB$ ,  $BCDB$ ,  $BACB$ ,  $CDBAC$ , and  $DBACBD$ . They imply the following inequalities.

$$\begin{cases} \beta a \leq 2b + c + d \\ \beta b \leq c + d \\ \beta b \leq a + c \\ \beta c \leq a + b + d \\ \beta d \leq a + 2b + c \end{cases}$$

We look for the smallest  $\beta$  such that this system has no solution. Notice that  $a$  and  $d$  play symmetric roles. Thus, we can set  $a = d$  and simplify the system.

$$\begin{cases} \beta a \leq a + 2b + c \\ \beta b \leq a + c \\ \beta c \leq 2a + b \end{cases}$$

Then  $\beta$  is the largest eigenvalue of the matrix  $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$  that corresponds to the latter system. So  $\beta = 3.060647027\dots$  is the largest root of the characteristic polynomial  $x^3 - x^2 - 5x - 4$ . Then  $\alpha = 1 + \frac{1}{\beta+1} = 1.246266172\dots$

This matrix approach is a convenient trick to use when possible. It was used in particular for some doubled patterns such that every variable occurs exactly twice [12]. It may fail if the number of inequalities is strictly greater than the number of variables or if the formula contains a repetition  $uvu$  such that  $|u| \geq 2$ . In any case, we can fix a rational value to  $\beta$  and ask a computer algebra system whether the system of inequalities is solvable. Then we can get arbitrarily good approximations of  $\beta$  (and thus  $\alpha$ ) by a dichotomy method.

Of course, the avoidability exponent is related to divisibility.

**Lemma 9.** *If  $f \preceq g$ , then  $AE(f) \leq AE(g)$ .*

The avoidability exponent depends on the repetitions induced by  $f$ . We have  $AE(f) = 1$  for formulas such as  $f = AB.BA.AC.CA.BC$  or  $f = AB.BA.AC.BC.CDA.DCD$  that do not have enough repetitions. That is, for every  $\varepsilon > 0$ , there exists a  $(1 + \varepsilon)$ -free word that contains an occurrence of  $f$ .

Let us investigate formulas with non-trivial avoidability exponent, that is,  $AE(f) > 1$ . To show that a nice formula has a non-trivial avoidability exponent (see Lemma 10), we first introduce a notion of minimality for nice formulas similar to the notion of minimally avoidable for general formulas. A nice formula  $f$  is *minimally nice* if there exists no nice formula  $g$  such that  $v(g) \leq v(f)$  and  $g \prec f$ . Alternatively, splitting a minimally nice formula on any of its fragments leads to a non-nice formula. The following property of every minimally nice formula is easy to derive. If a variable  $V$  appears as a prefix of a fragment  $\phi$ , then

- $V$  is also a suffix of  $\phi$  (since otherwise we can split on  $\phi$  and obtain a nice formula),
- $\phi$  contains exactly two occurrences of  $V$  (since otherwise we can remove the prefix letter  $V$  from  $\phi$  and obtain a nice formula),
- $V$  is neither a prefix nor a suffix of any fragment other than  $\phi$  (since otherwise we can remove this prefix/suffix letter  $V$  from the other fragment and obtain a nice formula),
- Every fragment other than  $\phi$  contains at most one occurrence of  $V$  (since otherwise we can remove the prefix letter  $V$  from  $\phi$  and obtain a nice formula).

**Lemma 10.** *If  $f$  is a nice formula with  $v(f) \geq 3$ , then  $AE(f) \geq 1 + \frac{1}{2v(f)-3}$ .*

*Proof.* First remark that if a word  $uvu$  is  $\left(1 + \frac{1}{2v(f)-3}\right)$ -free then  $2|u| + |v| < (|u| + |v|) \left(1 + \frac{1}{2v(f)-3}\right)$  which implies  $(2v(f) - 4)|u| < |v|$ .

Suppose that  $f$  contradicts the lemma. Then there exists a  $\left(1 + \frac{1}{2v(f)-3}\right)$ -free word  $w$  containing an occurrence  $h$  of  $f$ . Let  $X$  be a variable of  $f$  such that  $|h(X)| \geq |h(Y)|$  for every variable  $Y$ . Since  $f$  is nice,  $f$  contains a factor of the form  $XPX$  where  $P$  is a sequence of variables that does not contain  $X$ . Remark that  $v(P) \leq v(f) - 1$ .

For any variable  $Z$ , let  $|P|_Z$  be the number of occurrences of  $Z$  in  $P$ . Let  $Y$  be the variable that maximizes  $|h(Y)| \times |P|_Y$ , that is,  $|h(W)| \times |P|_W \leq |h(Y)| \times |P|_Y$  for every variable  $W$  in  $P$ . We have

$$|h(P)| = \sum_{W \in Var(P)} |h(W)| \times |P|_W \leq (v(f) - 1)|h(Y)| \times |P|_Y \leq (v(f) - 1)|h(X)| \times |P|_Y.$$

If  $|P|_Y = 1$ , then  $|h(P)| \leq (v(f) - 1)|h(X)|$  and the exponent of  $|h(XPX)|$  is at least  $\frac{(v(f)+1)|h(X)|}{v(f)|h(X)|} = 1 + \frac{1}{v(f)}$ , which is a contradiction.

If  $|P|_Y \geq 2$ , then the number of letters of  $h(P)$  that do not belong to an occurrence of  $h(Y)$  is at most

$$\sum_{W \in \text{Var}(P) \setminus \{Y\}} |h(W)| \times |P|_W \leq (v(f) - 2)|h(Y)| \times |P|_Y.$$

Thus there exist two occurrences of  $h(Y)$  in  $h(P)$  that are separated by at most  $\frac{(v(f)-2)|h(Y)| \times |P|_Y}{|P|_Y - 1}$  letters. Since  $h(P)$  is  $\left(1 + \frac{1}{2v(f)-3}\right)$ -free, we obtain

$$(2v(f) - 4)|h(Y)| < \frac{(v(f) - 2)|h(Y)| \times |P|_Y}{|P|_Y - 1}.$$

This can be simplified to

$$(2v(f) - 4)(|P|_Y - 1) < (v(f) - 2) \times |P|_Y$$

and finally

$$|P|_Y < \frac{2v(f) - 4}{v(f) - 2} = 2,$$

which is a contradiction.  $\square$

The circular formulas studied in [9] show that  $AE(f)$  can be as low as  $1 + (v(f))^{-1}$ . Moreover, our example  $AE(ABCDBACBD) = 1.246266172\dots$  shows that lower avoidability exponents exist among nice formulas with at least 4 variables.

We will describe below a method to construct infinite words avoiding a formula. This method can be applied if and only if the formula  $f$  satisfies  $AE(f) > 1$ . So we are interested in characterizing the formulas  $f$  such that  $AE(f) > 1$ . By Theorems 9 and 10, if  $f$  is a formula such that there exists a nice formula  $g$  satisfying  $g \preceq f$ , then  $AE(f) > 1$ . Now we prove that the converse also holds, which gives the following characterization.

**Theorem 11.** *A formula  $f$  satisfies  $AE(f) > 1$  if and only if there exists a nice formula  $g$  such that  $g \preceq f$ .*

*Proof.* What remains to prove is that for every formula  $f$  that is not divisible by a nice formula and for every  $\varepsilon > 0$ , there exists an infinite  $(1 + \varepsilon)$ -free word  $w$  containing an occurrence of  $f$ , such that the size of the alphabet of  $w$  only depends on  $f$  and  $\varepsilon$ .

First, we consider the equivalent pattern  $p$  obtained from  $f$  by replacing every dot by a distinct variable that does not appear in  $f$ . We will actually construct an occurrence of  $p$ . Then we construct a family  $f_i$  of pseudo-formulas as follows. We start with  $f_0 = p$ . To obtain  $f_{i+1}$  from  $f_i$ , we choose a variable that appears at most once in every fragment of  $f_i$ . This variable is given the alias name  $V_i$  and every occurrence of  $V_i$  is replaced by a dot. We say that  $f_i$  is a pseudo-formula since we do not try to normalize  $f_i$ , that is,  $f_i$  can contain consecutive dots and  $f_i$  can contain fragments that are factors of other fragments. However, we still have a notion of fragment for a pseudo-formula. Since  $f$  is not divisible by a nice formula, this process ends with the pseudo-formula  $f_{v(p)}$  with no variable and



$|p|$  consecutive dots. The goal of this process is to obtain the ordering  $V_0, V_1, \dots, V_{v(p)-1}$  on the variables of  $p$ .

The image of every  $V_i$  is a finite factor  $w_i$  of a Dejean word over an alphabet of  $\lfloor \varepsilon^{-1} \rfloor + 2$  letters, so that  $w_i$  is  $(1 + \varepsilon)$ -free. The alphabets are disjoint: if  $i \neq j$ , then  $w_i$  and  $w_j$  have no common letter. Finally, we define the length of  $w_i$  as follows:  $|w_{v(p)-1}| = 1$  and  $|w_i| = \lfloor \varepsilon^{-1} \rfloor \times |p| \times |w_{i+1}|$  for every  $i$  such that  $0 \leq i \leq v(p) - 2$ . Let us show by contradiction that the constructed occurrence  $h$  of  $p$  is  $(1 + \varepsilon)$ -free. Consider a repetition  $xyx$  of exponent at least  $1 + \varepsilon$  that is maximal, that is, which cannot be extended to a repetition with the same period and larger exponent. Since every  $w_i$  is  $(1 + \varepsilon)$ -free and since two matching letters must come from distinct occurrences of the same variable, then  $x = h(x')$  and  $y = h(y')$  where  $x'$  and  $y'$  are factors of  $p$ . Our ordering of the variables of  $p$  implies that  $y'$  contains a variable  $V_i$  such that  $i < j$  for every variable  $V_j$  in  $x'$ . Thus,  $|y| \geq |w_i| = \lfloor \varepsilon^{-1} \rfloor \times |p| \times |w_{i+1}| \geq \lfloor \varepsilon^{-1} \rfloor \times |x|$ , which contradicts the fact that the exponent of  $xyx$  is at least  $1 + \varepsilon$ .

To obtain the infinite word  $w$ , we can insert our occurrence of  $p$  into a bi-infinite  $(1 + \varepsilon)$ -free word over an alphabet of  $\lfloor \varepsilon^{-1} \rfloor + 2$  new letters. So  $w$  is an infinite  $(1 + \varepsilon)$ -free word over an alphabet of  $v(p)(\lfloor \varepsilon^{-1} \rfloor + 2) + 1$  letters which contains an occurrence of  $f$ .  $\square$

By Lemma 10, every nice formula is avoidable since it is avoided by a Dejean word over a sufficiently large alphabet. Thus, if a formula is nice and minimally avoidable, then it is minimally nice. This is the case for every formula in the 3-avoidance basis, except  $AB.AC.BA.CA.CB$ . However, a minimally nice formula is not necessarily minimally avoidable. Indeed, we have shown [15] that the set of minimally nice ternary formulas consists of the nice formulas in the 3-avoidance basis, together with the minimally nice formulas in Table 1 that can be split to  $AB.AC.BA.CA.CB$ .

- $ABA.BCB.CAC$
- $ABCA.BCAB.CBAC$  and its reverse
- $ABCA.BAB.CAC$
- $ABCA.BAB.CBC$  and its reverse
- $ABCA.BAB.CBAC$  and its reverse
- $ABCBA.CABC$  and its reverse
- $ABCBA.CAC$

Table 1: The minimally nice ternary formulas that are not minimally avoidable.

### 3.2 Avoiding a nice formula

Recall that a nice formula  $f$  is such that  $AE(f) > 1$ . We consider the smallest integer  $s$  such that  $RT(s) < AE(f)$ . Thus, every Dejean word over  $\Sigma_s$  avoids  $f$ , which already gives  $\lambda(f) \leq s$ . Recall that a morphism is  $q$ -uniform if the image of every letter has length  $q$ . Also, a uniform morphism  $h : \Sigma_s^* \rightarrow \Sigma_e^*$  is *synchronizing* if for any  $a, b, c \in \Sigma_s$  and  $v, w \in \Sigma_e^*$ , if  $h(ab) = vh(c)w$ , then either  $v = \varepsilon$  and  $a = c$  or  $w = \varepsilon$  and  $b = c$ . For increasing values of  $q$ , we look for a  $q$ -uniform morphism  $h : \Sigma_s^* \rightarrow \Sigma_e^*$  such that  $h(w)$  avoids  $f$  for every  $RT(s)^+$ -free word  $w \in \Sigma_s^\ell$ , where  $\ell$  is given by Lemma 12 below. Recall that a word is  $(\beta^+, n)$ -free if it contains no repetition with exponent strictly greater than  $\beta$  and period at least  $n$ .

**Lemma 12.** [11] *Let  $\alpha, \beta \in \mathbb{Q}$ ,  $1 < \alpha < \beta < 2$  and  $n \in \mathbb{N}^*$ . Let  $h : \Sigma_s^* \rightarrow \Sigma_e^*$  be a synchronizing  $q$ -uniform morphism (with  $q \geq 1$ ). If  $h(w)$  is  $(\beta^+, n)$ -free for every  $\alpha^+$ -free word  $w$  such that  $|w| < \max\left(\frac{2\beta}{\beta-\alpha}, \frac{2(q-1)(2\beta-1)}{q(\beta-1)}\right)$ , then  $h(w)$  is  $(\beta^+, n)$ -free for every (finite or infinite)  $\alpha^+$ -free word  $w$ .*

Given such a candidate morphism  $h$ , we use Lemma 12 to show that for every  $RT(s)^+$ -free word  $w \in \Sigma_s^*$ , the image  $h(w)$  is  $(\beta^+, n)$ -free. The pair  $(\beta, n)$  is chosen such that  $RT(s) < \beta < AE(f)$  and  $n$  is the smallest possible for the corresponding  $\beta$ . If  $\beta < AE(f)$ , then every occurrence  $h$  of  $f$  in a  $(\beta^+, t)$ -free word is such that the length of the  $h$ -image of every variable of  $f$  is upper bounded by a function of  $n$  and  $f$  only. Thus, the  $h$ -image of every fragment of  $f$  has bounded length and we can check that  $f$  is avoided by inspecting a finite set of factors of words of the form  $h(w)$ .

### 3.3 The number of fragments of a minimally avoidable formula

Interestingly, the notion of (minimally) nice formula is helpful in proving the following.

**Theorem 13.** *The only minimally avoidable formula with exactly one fragment is  $AA$ .*

*Proof.* A formula with one fragment is a doubled pattern. Since it is minimally avoidable, it is a minimally nice formula. By the properties of minimally nice formulas discussed above, the unique fragment of the formula is either  $AA$  or is of the form  $ApA$  such that  $p$  does not contain the variable  $A$ . Thus,  $p$  is a doubled pattern such that  $p \prec ApA$ , which contradicts that  $ApA$  is minimally avoidable.  $\square$

By contrast, the family of *two-birds* formulas, which consists of  $ABA.BAB$ ,  $ABCBA.CBABC$ ,  $ABCD.CBADC$ , and so on, shows that there exist infinitely many minimally avoidable formulas with exactly two fragments. Every two-birds formula is nice. Let us check that every two-birds formula  $AB \cdots X \cdots BA.X \cdots A \cdots X$  is minimally avoidable. Since the two fragments play symmetric roles, it is sufficient to split on the first fragment. We obtain the formula  $AB \cdots X \cdots B.B \cdots X \cdots BA.X \cdots A \cdots X$  which divides the pattern  $B \cdots X \cdots BAB \cdots X \cdots B = Z(B \cdots X \cdots B)$ . This pattern is equivalent to  $B \cdots X \cdots B$ , which is unavoidable. Thus, every two-birds formula is indeed minimally avoidable.

Concerning the index of two-birds formulas, we have seen that  $\lambda(ABA.BAB) = 3$  and  $\lambda(ABCBA.CBABC) = 2$  [9]. Computer experiments suggest that larger two-birds formulas are easier to avoid.

**Conjecture 14.** Every two-birds formula with at least 3 variables is 2-avoidable.

## 4 Characterization of some famous morphic words

Our next result gives characterizations of  $w_3$ , up to renaming, that use just one formula. Then we give similar characterizations of  $b_3$  and  $b_2$ . Let  $\sigma = 1/2/0$  be the morphism that cyclically permutes  $\Sigma_3$ .

**Theorem 15.** Let  $f_h = ABA.BCB.ACA$ ,  $f_e = ABA.ABCBA.ACA.ACB.BCA$ , and let  $f$  be such that  $f_h \preceq f \preceq f_e$ . Every ternary recurrent word avoiding  $f$  is equivalent to  $w_3$ ,  $\sigma(w_3)$ , or  $\sigma^2(w_3)$ .

*Proof.* Using Cassaigne's algorithm [4], we have checked that  $w_3$  avoids  $f_h$ . By divisibility,  $w_3$  avoids  $f$ .

Let  $w$  be a ternary recurrent word avoiding  $f$ . By Lemma 6,  $w$  is square-free.

Let  $v = 210201202101201021$ . A computer check shows that no infinite ternary word avoids  $f_e$ , squares,  $v$ ,  $\sigma(v)$ , and  $\sigma^2(v)$ . So, without loss of generality,  $w$  contains  $v$ . If  $w$  contains 121, then  $w$  contains the occurrence  $A \rightarrow 1, B \rightarrow 2, C \rightarrow 0$  of  $f_e$ . Similarly, if  $w$  contains 212, then  $w$  contains the occurrence  $A \rightarrow 2, B \rightarrow 1, C \rightarrow 0$  of  $f_e$ . Thus,  $w$  avoids squares, 121, and 212. By Theorem 2,  $w$  is equivalent to  $w_3$ .

By symmetry, every ternary recurrent word avoiding  $f$  is equivalent to  $w_3$ ,  $\sigma(w_3)$ , or  $\sigma^2(w_3)$ .  $\square$

**Theorem 16.** Let  $f$  be such that

- $ABCA.ABA.ACA \preceq f \preceq ABCA.ABA.ACA.ACB.CBA$ ,
- $ABCA.ABA.BCB.AC \preceq f \preceq ABCA.ABA.ABCBA.ACB$ , or
- $ABCA.ABA.BCB.CBA \preceq f \preceq ABCA.ABA.ABCBA.ACB$ .

Every ternary recurrent word avoiding  $f$  is equivalent to  $b_3$ ,  $\sigma(b_3)$ , or  $\sigma^2(b_3)$ .

*Proof.* Using Cassaigne's algorithm [4], we have checked that  $b_3$  avoids  $ABCA.ABA.ACA$ ,  $ABCA.ABA.BCB.AC$ , and  $ABCA.ABA.BCB.CBA$ . By divisibility,  $b_3$  avoids  $f$ . Let  $w$  be a ternary recurrent word avoiding  $f$ . By Lemma 6,  $w$  is square-free.

Let  $v = 20210121020120$ . A computer check shows that no infinite ternary word avoids  $ABCA.ABA.ACA.ACB.CBA$  (resp.  $ABCA.ABA.ABCBA.ACB$ ), squares,  $v$ ,  $\sigma(v)$ , and  $\sigma^2(v)$ .

So, without loss of generality,  $w$  contains  $v$ . If  $w$  contains 010, then  $w$  contains the occurrence  $A \rightarrow 0, B \rightarrow 1, C \rightarrow 2$  of  $ABCA.ACA.ABCA.ACBA.ABCBA$ . Similarly, if  $w$  contains 212, then  $w$  contains the occurrence  $A \rightarrow 2, B \rightarrow 1, C \rightarrow 0$  of

$ABA.ACA.ABCA.ACBA.ABCBA$ . Thus,  $w$  avoids squares, 010, and 212. By Theorem 2,  $w$  is equivalent to  $b_3$ .

By symmetry, every ternary recurrent word avoiding  $f$  is equivalent to  $b_3$ ,  $\sigma(b_3)$ , or  $\sigma^2(b_3)$ .  $\square$

Notice that Theorem 16 is a complement to [15, Theorem 2] in which we gave a disjoint set of formulas with the same property. The difference between Theorem 16 and [15, Theorem 2] is that a different occurrence of  $f$  shows that  $f$  divides  $Z^n(AA)$ .

**Theorem 17.** *Let  $f_h = AABC AA.BCB$ ,  $f_e = AABCAAB.AABCAB.AABCB$ , and let  $f$  be such that  $f_h \preceq f \preceq f_e$ . Every binary recurrent word avoiding  $f$  is equivalent to  $b_2$ .*

*Proof.* Using Cassaigne's algorithm [4], we have checked that  $b_2$  avoids  $f_h$ . First,  $f_e \preceq AAA$  because  $Z(AAA) = AAABAAA$  contains the occurrence  $A \rightarrow A$ ,  $B \rightarrow A$ ,  $C \rightarrow B$  of  $f_e$ . Second,  $f_e \preceq ABABA$  because  $Z(ABABA) = ABABACABABA$  contains the occurrence  $A \rightarrow AB$ ,  $B \rightarrow A$ ,  $C \rightarrow C$  of  $f_e$ .

Thus, every recurrent word avoiding  $f_e$  also avoids  $AAA$  and  $ABABA$ , which means that it is overlap-free. Finally, it is well-known that every binary recurrent word that is overlap-free is equivalent to  $b_2$ .  $\square$

## 5 $xyx$ -formulas

Recall that every fragment of an  $xyx$ -formula is of the form  $XYX$ . We associate to an  $xyx$ -formula  $F$  the directed graph  $\vec{G}$  such that every variable corresponds to a vertex and  $\vec{G}$  contains the arc  $\overrightarrow{XY}$  if and only if  $F$  contains the fragment  $XYX$ . We will also denote by  $G$  the underlying simple graph of  $\vec{G}$ .

**Lemma 18.** *Let  $F_1$  and  $F_2$  be  $xyx$ -formulas associated to  $\vec{G}_1$  and  $\vec{G}_2$ . If there exists a homomorphism  $\vec{G}_1 \rightarrow \vec{G}_2$ , then  $F_1 \preceq F_2$ .*

*Proof.* Since both digraph homomorphism and formula divisibility are transitive relations, we only need to consider the following two cases. If  $G_1$  is a subgraph of  $G_2$ , then  $F_1$  is obtained from  $F_2$  by removing some fragments. So every occurrence of  $F_2$  is also an occurrence of  $F_1$  and thus  $F_1 \preceq F_2$ . If  $G_2$  is obtained from  $G_1$  by identifying the vertices  $u$  and  $v$ , then  $F_2$  is obtained from  $F_1$  by identifying the variables  $U$  and  $V$ . So every occurrence of  $F_2$  is also an occurrence of  $F_1$  and thus  $F_1 \preceq F_2$ .  $\square$

For every  $i$ , let  $T_i$  be the  $xyx$ -formula corresponding to the directed circuit  $\vec{C}_i$  of length  $i$ , that is,  $T_1 = AAA$ ,  $T_2 = ABA.BAB$ ,  $T_3 = ABA.BCB.CAC$ ,  $T_4 = ABA.BCB.CDC.DAD$ , and so on. More formally,  $T_i$  is the formula with  $i$  variables  $A_0, \dots, A_{i-1}$  which contains the  $i$  fragments of length three of the form  $A_j A_{j+1} A_j$  such that the indices are taken modulo  $i$ . Notice that  $T_i$  is a nice formula.

**Theorem 19.** *For every  $i \geq 2$ ,  $\lambda(T_i) = 3$ .*

*Proof.* We use Lemma 12 to show that the image of every  $(7/4^+)$ -free word over  $\Sigma_4$  by the following 58-uniform morphism is  $(3/2, 3)$ -free.

$0 \rightarrow 0012211002201021120022100112201002112001022011002211201022$   
 $1 \rightarrow 0012210022010211220010221120011022010021122011002211201022$   
 $2 \rightarrow 0011221002201021122001102201002112001022110012200211201022$   
 $3 \rightarrow 0011221002201021120011022010021122001022110012200211201022$

In these words, the factor 010 is the only occurrence  $m$  of  $ABA$  such that  $|m(A)| \geq |m(B)|$ . This implies that these ternary words avoid  $T_i$  for every  $i \geq 1$ , so that  $\lambda(T_i) \leq 3$ .

To show that  $\lambda(T_i) \geq 3$ , we consider the  $xyx$ -formula  $H = ABA.BAB.ACA.CBC$  associated to the directed graph  $\vec{D}_3$  on 3 vertices and 4 arcs that contains a circuit of length 2 and a circuit of length 3. Standard backtracking shows that  $\lambda(H) > 2$ , and even the stronger result that  $\lambda(ABAB.ACA.CAC.BCB.CBC) > 2$ .

For every  $i \geq 2$ , the circuit  $\vec{C}_i$  admits a homomorphism to  $\vec{D}_3$ . By Lemma 18, this means that  $T_i \preceq H$ , which implies that  $\lambda(T_i) \geq \lambda(H) \geq 3$ .  $\square$

**Theorem 20.** *For every  $i \geq 1$ ,  $b_4$  avoids  $T_i$ .*

*Proof.* Suppose for contradiction that there exist  $i$  and  $n$  such that  $m^n(0)$  contains an occurrence  $h$  of  $T_i$ . Further assume that  $n$  is minimal. Notice that in  $b_4$ , every even (resp. odd) letter appears only at even (resp. odd) positions. Thus, for every fragment  $XYX$  of  $T_i$ , the period  $|h(XY)|$  of the repetition  $h(XYX)$  must be even. This implies that  $|h(X)|$  and  $|h(Y)|$  have the same parity. By contagion, the lengths of the images of all the variables of  $T_i$  have the same parity. Now we proceed to a case analysis.

- Every  $|h(X)|$  is even.
  - Every  $h(X)$  starts with 0 or 2. By taking the pre-image by  $m$  of every  $h(X)$ , we obtain an occurrence of  $T_i$  that is contained in  $m^{n-1}(0)$ . This contradicts the minimality of  $n$ .
  - Every  $h(X)$  starts with 1 or 3. Notice that in  $b_4$ , the letter 1 (resp. 3) is in position 1 (mod 4) (resp. 3 (mod 4)).  $m^n(0)$  contains the occurrence  $h'$  of  $T_i$  such that  $h'(X)$  is obtained from  $h(X)$  by adding to the right the letter 1 or 3 depending on its position modulo 4 and by removing the first letter. Since  $h'$  is also contained in  $m^n(0)$  and every  $h'(X)$  starts with 0 or 2,  $h'$  satisfies the previous subcase.
- Every  $|h(X)|$  is odd. It is not hard to check that every factor  $uvu$  in  $b_4$  with  $|v| = 1$  satisfies  $v \in \{1, 3\}$  and  $u \in \{0, 2\}$ . So  $|h(X)| \geq 3$  for every variable  $X$  of  $T_i$ . Let  $X_1, \dots, X_i$  be the variables of  $T_i$ . Up to a shift of indices, we can assume that  $j$  and the first and last letters of  $h(X_j)$  have the same parity. We construct the occurrence  $h'$  of  $T_i$  as follows. If  $j$  is odd, then  $h'(X_j)$  is obtained by removing the first letter of  $h(X_j)$ . If  $j$  is even, then  $h'(X_j)$  is obtained by adding to the right the letter 1 or 3 depending on its position modulo 4. Since  $h'$  is also contained in  $m^n(0)$  and every  $|h'(X)|$  is even,  $h'$  satisfies the previous case.  $\square$

Our next result generalizes Theorems 5 and 20. Recall that every fragment of a hybrid formula has length 2 or is of the form  $XYX$ .

**Theorem 21.** *Every avoidable hybrid formula is avoided by  $b_4$ .*

*Proof.* Let  $f$  be a hybrid formula. If  $f$  contains a locked formula or a formula  $T_i$ , then  $b_4$  avoids  $f$  by Theorems 4 and 20. If  $f$  contains neither a locked formula nor a formula  $T_i$ , then we show that  $f$  is unavoidable. By induction and by theorem 1 it is sufficient to show that  $f$  is reducible to a hybrid formula containing neither a locked formula nor a formula  $T_i$ . Since  $f$  is not locked,  $f$  contains a free set of variables and thus  $f$  has a free singleton  $\{X\}$ . If  $f$  contains a fragment  $YXY$ , then  $\{Y\}$  is also a free singleton of  $f$ . Using this argument iteratively, we end up with a free singleton  $\{Z\}$  such that  $f$  contains no fragment  $TZT$ , since  $f$  contains no formula  $T_i$ .

So we can assume that  $f$  contains a free singleton  $\{Z\}$  and no fragment  $TZT$ . Thus, deleting every occurrence of  $Z$  from  $f$  gives an hybrid sub-formula containing neither a locked formula nor a formula  $T_i$ . By induction,  $f$  is unavoidable.  $\square$

So the index of an avoidable  $xyx$ -formula is at most 4 and we have seen examples of  $xyx$ -formulas with index 3 in Theorems 15 and 19. The next results give an  $xyx$ -formula with index 4 and an  $xyx$ -formula with index 2 that is not divisible by  $AAA$ .

**Theorem 22.**  $\lambda(ABA.BCB.DCD.DED.AEA) = 4$ .

*Proof.* By Theorem 21,  $ABA.BCB.DCD.DED.AEA$  is 4-avoidable.

Notice that  $ABA.BCB.DCD.DED.AEA \preceq ABA.BCB.ACA$  via the homomorphism  $A \rightarrow A, B \rightarrow B, C \rightarrow C, D \rightarrow B, E \rightarrow C$ . Moreover,  $w_3$  contains the occurrence  $A \rightarrow 0, B \rightarrow 1, C \rightarrow 02, D \rightarrow 01, E \rightarrow 2$  of  $ABA.BCB.DCD.DED.AEA$ . By Theorem 15, the formula is not 3-avoidable.  $\square$

**Theorem 23.** *The fixed point of  $001/011$  avoids the  $xyx$ -formula associated to the directed graph on 4 vertices with all the 12 arcs.*

*Proof.* We use again Cassaigne's algorithm.  $\square$

## 6 Palindrome patterns

Mikhailova [10] has considered the index of an avoidable pattern that is a palindrome and proved that it is at most 16. She actually constructed a morphic word over  $\Sigma_{16}$  that avoids every avoidable palindrome pattern.

We make a distinction between the largest index  $\mathcal{P}_w$  of an avoidable palindrome pattern and the smallest alphabet size  $\mathcal{P}_s$  allowing an infinite word avoiding every avoidable palindrome pattern. We obtained [15] the lower bound

$$\lambda(ABCADACBA) = \lambda(ABCA.ACBA) = 4,$$

so that  $4 \leq \mathcal{P}_w \leq \mathcal{P}_s \leq 16$ .

The following result is a slight improvement to  $\lambda(ABCA.ACBA) = 4$  that is not related to palindromes.

**Theorem 24.**  $\lambda(ABCA.ACBA.ABCBA) = 4$ .

*Proof.* By Lemma 6, every recurrent word avoiding  $ABCA.ACBA.ABCBA$  is square-free. A computer check shows that no infinite ternary square-free word avoids the occurrences  $h$  of  $ABCA.ACBA.ABCBA$  such that  $|h(A)| = 1$ ,  $|h(B)| \leq 2$ , and  $|h(C)| \leq 3$ .  $\square$

Let us give necessary conditions on a palindrome pattern  $P$  so that  $5 \leq \lambda(P) \leq 16$ .

1. The length of  $P$  is odd and the central variable of  $P$  is isolated. Indeed, otherwise  $P$  would be a doubled pattern and thus 3-avoidable [12].
2. No variable of  $P$  appears both at an even and an odd position. Indeed, if  $P$  had a variable that appears both at an even and an odd position, then  $P$  would be divisible by a formula in the family  $AA$ ,  $ABCA.ACBA$ ,  $ABCDEA.AEDCBA$ ,  $ABCDEFGA.AGFEDCBA$ ,  $\dots$ . Such formulas (with an odd number of variables) are locked and thus are avoided by  $b_4$  by Theorem 4. So  $P$  would be 4-avoidable.

We have found three patterns/formulas satisfying these conditions (see Theorem 25), but they seem to be 2-avoidable. We use again Cassaigne's algorithm with simple pure morphic words to ensure that they are 4-avoidable. Let  $z_3$  be the fixed point of  $01/2/20$ .

**Theorem 25.**

1.  $ADBDCDAD.DADCDBDA$  is avoided by  $b_4$ .
2.  $ABCDADC.CDADCBA$  is avoided by  $z_3$ .
3.  $ABACDBAC.CABDCABA$  is avoided by  $z_3$  and  $b_4$ .

## 7 Discussion

Let us briefly mention the things that we have attempted to do in this paper, without success.

- Find a result similar to Theorems 15 and 16 for  $v_3$ , the morphic word avoiding squares, 010, and 020.
- Improve Theorem 23 by showing that some  $xyx$ -formula on 4 variables and fewer fragments is 2-avoidable.
- Show that the  $xyx$ -formula associated to the transitive tournament on 5 vertices is 2-avoidable.

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