Complete Graph Drawings up to Triangle Mutations

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Abstract

The main result of the paper can be stated in the following way: a complete graph drawing in the sphere, where two edges have at most one common point, which is either a crossing or a common endpoint, and no three edges share a crossing, is determined by the circular ordering of edges at each vertex, or equivalently by the set of pairs of edges that cross, up to homeomorphism and a sequence of triangle mutations. A triangle mutation (or switch, or flip) consists in passing an edge across the intersection of two other edges, assuming the three edges cross each other and the region delimited by the three edges has an empty intersection with the drawing. Equivalently, the result holds for a drawing in the plane assuming the drawing is given with a pair of edges indicating where the unbounded region is. The proof is constructive, based on an inductive algorithm that adds vertices and their incident edges one by one (actually yielding an extra property for the sequence of triangle mutations). This result generalizes Ringel's theorem on uniform pseudoline arrangements (or rank 3 uniform oriented matroids) to complete graph drawings. We also apply this result to plane projections (or visualizations) of a geometric spatial complete graph, in terms of the rank 4 uniform oriented matroid defined by its vertices. Independently, we prove that, for a complete graph drawing, the set of pairs of edges that cross determine, by first order logic formulas, the circular ordering of edges at each vertex, as well as further information.

1 Introduction

In the whole paper, a graph drawing is always understood in the sense of a topological graph drawing, that is a drawing whose edges are represented by Jordan curves (not necessarily straight). We consider *simple graph drawings*, where two edges have at most one common point, which is either a crossing or a common endpoint, and no three edges share a crossing (such drawings are also called *simple topological graphs* or *good drawings* in the literature).

We consider properties of such drawings of the complete graph in the sphere. We also consider drawings in the plane induced in the natural way from drawings in the sphere by the choice of an unbounded region. For our purpose, we assume that the unbounded region is defined by the choice of two given adjacent edges forming a so-called *corner*. Though, in general, a drawing in the plane does not necessarily have a corner (when the boundary of the unbounded region contains no vertex), the corner can be thought of as a starting point from which a drawing in the plane can be built in a unique way (up to homeomorphism) following the formal data structure that encodes the drawing in the sphere.

Those two viewpoints - drawings in the sphere and in the plane - are essentially equivalent. In the paper, the constructions and properties are mainly described in terms of drawings in the plane. This implies a few more technicalities than drawings in the sphere, but this allows for a detailed algorithmic construction.

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As observed in [12], a simple drawing of a connected graph with given corner in the plane is determined, up to an homeomorphism of the plane, by its *sketch*, that is: its underlying graph, the circular ordering of the edges at each vertex, the pairs of edges that cross, and the order of crossings on each edge. If the last data is removed, we get what we call the *subsketch* of the graph drawing. Hence the subsketch is intermediate between the sketch and the usual *map* of the drawing (which is defined by the circular ordering of the edges at each vertex, and determines the drawing if it is connected and planar, as well-known).

In the case of a complete graph drawing, it turns out that the subsketch is implied by the map, see [24, Section 5], and by the set of pairs of edges that cross as well, see Theorem 5.1. Both results are also proved in [18, Proposition 6].

A *triangle mutation* (named after the term *mutation* used in oriented matroid theory, also called *triangle-switch*, or *triangle-flip* in the literature) consists in passing an edge across the intersection of two other edges, assuming it crosses these two other edges (the three edges form a *triangle*) and the interior of the bounded region delimited by the three edges has an empty intersection with the drawing (the triangle is called *free*). Note that vertices of the graph are not involved in triangle mutations. Obviously a triangle mutation does not change the subsketch. This local transformation is shown on Figure 1.

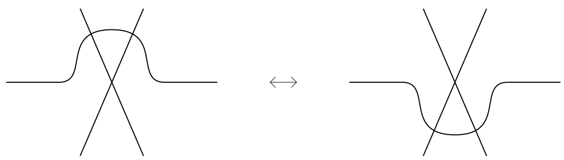


Figure 1: Triangle mutation.

The main result of the paper, namely Theorem 3.10, proves constructively that, for a complete graph drawing, the subsketch determines the drawing in the sphere up to a sequence of triangle mutations (and up to homeomorphism). The same result holds for drawings in the plane provided that a corner is given. In other words, if we consider that two complete graph drawings in the sphere are equivalent if they have the same subsketch, then the above result characterizes equivalence classes as sets of drawings obtained one from each other by a sequence of triangle mutations. For drawings in the plane, these equivalence classes are further subdivided so that they have a specific unbounded region defined by a corner. These equivalence classes are considered for instance for the sake of enumerating complete graph drawings in [18] where graphs having the same crossing pairs are called *weakly isomorphic*, and for the sake of testing whether a set of pairs of edges can be realized as the set of pairs of crossing edges in a complete graph drawing in [19].

Our proof is based on a few preliminary results and on an algorithm in which vertices and their incident edges are added one by one, maintaining a sequence of mutations with appropriate properties at each step. In this way, we actually get a stronger result than the existence of a sequence of triangle mutations, as we avoid mutations of triangles that are not contained in a triangle that necessarily has to be mutated at some step (called *permuted* triangle), see Section 3. We mention that it is not possible in general to use only mutations of permuted triangles (see below and Remark 3.14). As corollaries of the main result, we get that the subsketch and corner of a complete graph drawing determine most properties of the drawing.

If one considers a complete graph drawing with an even number of vertices, all of them being drawn on the same circle, such that edges between opposite vertices are inside the circle, then these edges define a pseudoline arrangement inside the circle, see Figure 2. Hence, the above result generalizes (and actually strengthens) Ringel's theorem on uniform pseudoline arrangements [25] (equivalent to rank 3 uniform oriented matroids [10]), see Section 4.1 and Theorem 4.1. Let us also point out Example 4.2, which shows that it is possible that some triangles must be mutated twice in a sequence from a drawing to another one.

An application of the above result is that two projections of complete spatial graphs, defined by finite sets of points in general position in $\mathbb{R}^3$ representing the same rank 4 uniform oriented matroid [10], are equivalent up to homeomorphism and a sequence of triangle mutations. Hence the combinatorial structure of the oriented matroid along with the combinatorial structure of the projected drawing form in some sense the two levels of a modelling for the visualization of a geometric spatial complete graph, see Section 4.2.

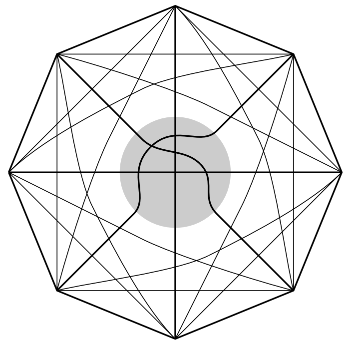


Figure 2: Complete graph drawing and pseudoline arrangement (in the grey circle).

In order to describe a graph drawing, amongst various equivalent possibilities, we choose to follow and build on the formalism from [12]. Let us mention that [12] addresses a logical viewpoint, where the question is to characterize graph drawing classes and properties by logic formulas, and where the aim is to use a constrained logic, namely monadic second-order logic, that is more flexible than first-order logic and more algorithmically usable than second-order logic. In this paper, we use this formalism because of its precision, but we do not address such logical questions, except in Section 5, which is independent of the rest of the paper and provides first-order logic formulas to relate various data in a complete graph drawing.

In a graph which is not complete, the subsketch is no more sufficient to determine the drawing up to triangle mutations. Figure 3 gives two examples of (pairs of) distinct graph drawings with the same crossings and the same circular orderings around each edge, but which cannot be transformed into each other with triangle mutations, since they simply have no triangle. It also gives an example showing that the result of the paper does not generalize to complete bipartite graphs. A possibly interesting setting for an extension to more general graphs of the question of determining a drawing up to a sequence of triangle mutations would be to consider drawings given with a planar frame, that is a planar 2-connected spanning subgraph, a structure considered in [12].

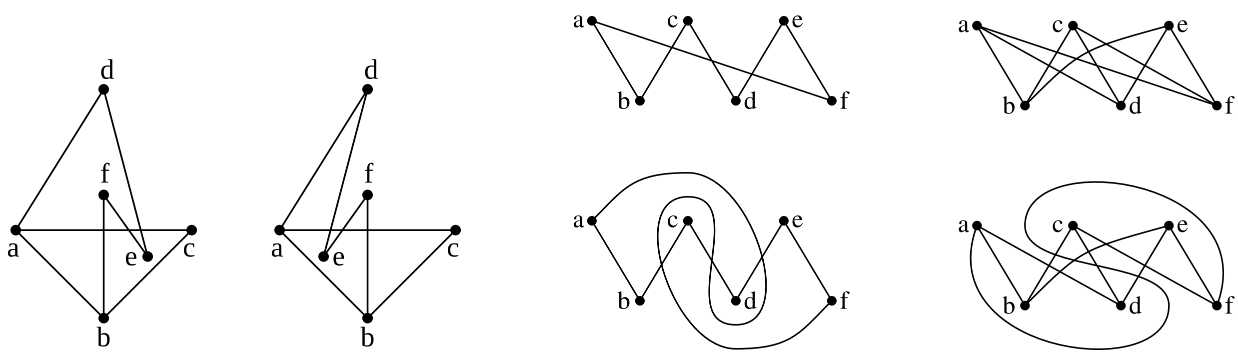


Figure 3: On the left and in the middle: examples of distinct 2-connected graph drawings with the same subsketch but no triangle. On the right: example of distinct drawings of a complete bipartite graph in the sphere with the same subsketch which are not equivalent up to triangle mutations (there is only one triangle).

The present paper is a reformed version with complete proofs of the preliminary conference paper [15] which gave the main result by means of an explicit algorithm and a guideline for the proof. This main result that the subsketch (or the set of pairs of edges that cross, or the map as well) determines a complete graph drawing up to a sequence of triangle mutations has been cited or used in several papers since then, e.g. [18, 19, 20, 1, 3]. A proof of this result using an alternative formalism was recently given in [5] (written in parallel with the present paper), also aiming at completing the 2005 paper [15], consistent with the guideline from [15], and initially motivated by crossing number questions; see Acknowledgments. Another proof can be found in [28], which appeared even more recently (while the present paper was being processed), repeating a proof proposed by the same author in [27, Section 4.3.2] and enhancing it with an extension of the result to some non-complete graphs (and with a characterization of pseudolinear drawings of complete graphs). A difference of approach between the proof in the present paper and the proofs in [5, 27, 28] is that these papers focus, in a more concise way, using topological formalisms for drawings in the sphere, on showing the existence of a sequence of mutations, whereas the present paper focuses, in a more detailed way, using a combinatorial/logical formalism for drawings in the plane, on algorithmically building such a sequence of mutations. Moreover, in this way, we actually state a stronger result (as we avoid mutations of triangles that are not contained in a permuted triangle); see how Theorems 3.10 and 4.1 are formulated.

Finally, let us also mention further references on close subjects. The problem of counting some special triplets of vertices in complete graph drawings is addressed in [16, 2, 7]. The problem of drawing complete graphs on general surfaces is addressed from a different viewpoint (though also related to oriented matroids) in [11]. Equivalence classes for general graph drawings with the same sets of pairs of edges that cross are addressed in [17, 24, 20]. The problem of counting edge crossings in monotone drawings of the complete graph is addressed [8]. Convex drawings of the complete graph are addressed in [6]. More specific references are also given along the paper.

2 Preliminaries

In this paper, a *graph* is always a finite, directed, loop-free, connected graph, with no multiple edges. The set of vertices of a graph G is denoted by V_G, or simply V, and its set of edges is denoted by $\overrightarrow{E}_G$, or simply $\overrightarrow{E}$. The underlying undirected set of edges is denoted by E_G, or simply E. In fact, the direction of an edge will be used only to distinguish the two ways of crossing between two edges, and to define an order of the points on a geometric representation of this edge. So, for $a, b \in V_G$ and $(a, b) \in \overrightarrow{E}_G$, we will denote $[a, b] = [b, a] \in E_G$.

Simple topological drawing. A *(simple topological) drawing* of a graph G (also called *simple topological graph*, or *simple drawing*, or *good drawing* in the literature) in the sphere, or in the real oriented affine plane, is a set of points representing V_G together with a set of drawn edges representing E_G satisfying the following properties:

D1: a drawn edge is a simple Jordan curve (i. e. homeomorphic to a closed segment) between the two endpoints representing the vertices of the edge; a drawn edge contains no other representation of a vertex of the graph than its endpoints.

D2: two edges having an endpoint in common meet only at this endpoint; when two edges with no common endpoint meet, they cross at this intersection point; two edges with no common endpoint cross at most once.

D3: no three edges meet at the same point, except if this point is an endpoint of the three edges.

Drawing in the sphere versus drawing in the plane. As well known, a drawing in the plane is equivalent to a drawing in the sphere along with the choice of a particular region playing the role of the unbounded region. The following pieces of information that we use are consistent for a drawing in the sphere or the plane as well. For our purpose, the choice of a particular unbounded region is made by the choice of two given adjacent edges forming a so-called *corner* (formally defined below). We will consider only drawings in the plane given with such a corner (in general, a drawing in the plane does not necessarily have a corner, when the boundary of the unbounded region contains no vertex). The corner can be thought of as a starting point from which a drawing in the plane can be built in a unique way (up to homeomorphism) following the formal data structure that encodes the drawing in the sphere. In the rest of the paper, we can sometimes switch between those two types of drawings, and all results can be expressed in both settings, even if it is not done explicitly.

In what follows, an edge of a drawing is represented by a *drawn edge* connecting the points corresponding to its endpoints. By *drawn vertex*, we also mean the topological representation of this vertex in the given drawing. For brevity, we sometimes omit the term *drawn* for edges and vertices represented in the drawing. By *segment*, we mean the portion of a drawn edge between two points of the drawn edge (containing these two points). The connected components of the complement of the union of all drawn edges in the plane are called *regions of the drawing*. Note that if Jordan arcs were replaced by straight line segments in axiom (D1), then we would have defined *geometric* graph drawings, for which various properties would become trivial, as noticed in [15].

We consider various pieces of information associated with a drawing D of the graph G, which encode the drawing at different levels of abstraction. To denote these pieces of information formally, we follow and build on the notations from [12]. They are illustrated in Figure 4.

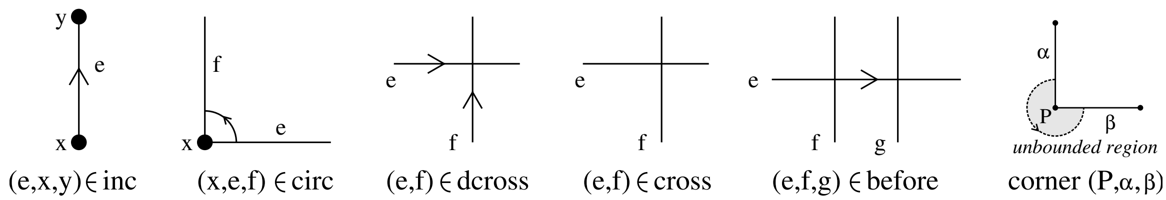


Figure 4: Relations forming the sketch of a drawing. A drawing is identified to its sketch plus a corner.

Incidence relations. This information is the combinatorial (directed) graph $G = (V_G, E_G)$ itself. Formally, it is encoded by the relation $inc_G \subseteq \overrightarrow{E}_G \times V_G \times V_G$ defined by: $(e, x, y) \in inc_G$ if and only if the edge e is directed from the vertex x to the vertex y.

Circular ordering relations (around vertices). This information gives the counterclockwise circular ordering of edges with a common endpoint around this endpoint. It is well defined by axiom (D2) in the definition of a drawing. Formally, it is encoded by the relation $circ_D \subseteq V_G \times E_G \times E_G$ defined by: $(x, e, f) \in circ_D$ if and only if x is an endpoint of e and f, and f is the next edge in the circular ordering around x in the counterclockwise sense of rotation.

Corner of the drawing. A *corner* of D is given by a vertex P and two edges α and β with endpoint P such that: β follows α in the counterclockwise ordering of edges around P; the drawn vertex P is in the topological boundary of the unbounded region of the plane delimited by D; and the intersection of this boundary with the union of α and β is a curve containing P. As mentioned above, the corner is not defined (and useless) for drawings in the sphere, and a drawing in the plane does not necessarily have a corner (when the boundary of the unbounded region contains no vertex). Formally, the corner is a particular element $(P, \alpha, \beta) \in circ_D$.

Map of the drawing. The incidence relations and the circular ordering relations (inc_G and $circ_D$) define the *map* associated with the drawing D of the graph G. It is well known (see for example [22]) that if D is a drawing in the plane with no edge crossing (except for common endpoints), and thus G is planar, then D is determined up to an orientation-preserving homeomorphism of the plane by its map and a corner. Obviously, the map alone, omitting the corner, determines the drawing in the sphere (up to homeomorphism).

Crossing relations. This information gives which pairs ef edges cross in the drawing. Formally, it is encoded by the relation $cross_D \subseteq E_G \times E_G$ defined by: $(e, f) \in cross_D$ if and only if the drawn edges e and f have no endpoint in common and the drawn edges e and f have one intersection point. Of course $(e, f) \in cross_D$ implies $(f, e) \in cross_D$. Then we say that $e \in E_G$ and $f \in E_G$ *cross* in D.

Subsketch of the drawing. The incidence relations, the circular ordering relations and the crossing relations (inc_G, $circ_D$, and $cross_D$) define the *subsketch* of the drawing D. In what follows, we mainly use the subsketch of a drawing, either alone for drawings in the sphere, or with a given corner for drawings in the plane. Notice that the subsketch may in general contain redundant information. It is noticed in [24, Section 5] that, in the case of complete graphs, the crossing relation is implied by the other ones. We prove in forthcoming Theorem 5.1 that, in the case of complete graphs, the crossing relation also implies the other ones. Both results are also proved in [18, Proposition 6].

Directed crossing relations. A refinement of the crossing relation takes into account edge directions. Formally, it is encoded by $dcross_D \subseteq \overrightarrow{E}_G \times \overrightarrow{E}_G$ such that: $(e, f) \in dcross_D$ if and only if the drawn edges e and f have no endpoint in common, the drawn edges e and f have one intersection point and f goes from the right of e to its left when e is directed from bottom to top. Of course $(e, f) \in dcross_D$ implies $(f, e) \notin dcross_D$. In this paper we will use the $cross_D$ relation, which makes no difference for complete graph drawings since the $dcross_D$ relation can be retrieved from the $cross_D$ relation and a corner, as shown in forthcoming Theorem 5.1.

Linear ordering relations (along directed edges). This information gives the ordering of the crossings along each directed edge. It is well defined by axiom (D2). Formally, we define the relation $before_D \subseteq \overrightarrow{E}_G \times E_G \times E_G$ by: $(e, f, g) \in before_D$ if and only if $f \neq g$, e and f cross in D, e and g cross in D, and the intersection point of e and f is before the intersection point of e and g on the directed drawn edge e. We can say that *f is before g along e*. Note that if e crosses f and g then either $(e, f, g) \in before_D$ or $(e, g, f) \in before_D$ but not both. By definition of a drawing, the relation $before_D$ induces, for any edge e, a linear ordering on the edges that cross e.

Sketch of the drawing. The incidence relations, the circular ordering relations, the directed crossing relations , and the linear ordering relations (inc_G, $circ_D$, $dcross_D$, and $before_D$) define the *sketch* associated with the drawing D, as introduced in [12]. A result of [12] is that a drawing D in the plane is determined up to an orientation-preserving homeomorphism of the plane by its sketch and a corner. This result is obtained by considering the planar graph drawing where intersections of edges in D are considered as new vertices, and then using the above result on planar graph drawings. Obviously, the sketch alone, omitting the corner, determines the drawing in the sphere (up to homeomorphism). In view of this result, in the rest of the paper, we can identify drawings (topological objects, always meant up to an orientation-preserving homeomorphism) and sketches (combinatorial objects, with a given corner for drawings in the plane), and the following definitions about drawings in the plane or sketches with a given corner can be made equivalently for one of these two objects. When the context is not ambiguous, we may omit the subscript $_D$ referring to the drawing.

Triangles. Let D be a drawing of a graph G. We define a *triangle* of D as a triple $(e, f, g) \in E_G \times E_G \times E_G$ such that e and f cross in D, e and g cross in D, and f and g cross in D. Then, e, f and g are called the *edges* of the triangle. The order of the elements in the triplet has no importance, and we denote the triangle by $[[e, f, g]]$. The *sides* of a triangle $[[e, f, g]]$ are the segments of the drawn edges e, f, or g which are delimited by the intersections with the two other edges of the triangle.

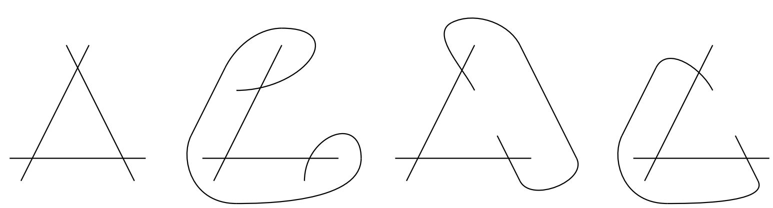


Figure 5: A tame triangle at first, and wild triangles next. Let us mention that every triangle in the plane is equivalent to one of these four triangles (up to homeomorphism). Also, the first and the second are equivalent as triangles in the sphere, as well as the third and the fourth (yielding two possible drawings in the sphere).

Let us consider a drawing in the plane. The *surface* of a triangle $[[e, f, g]]$ is the bounded region of the plane delimited by its sides and containing these sides. A triangle $[[e, f, g]]$ is *tame* if the intersection of the (topological) interior of its surface with the drawn edges e, f, g is empty (or, equivalently, if its surface does not contain a drawn endpoint of e, f, or g, as easily seen). Otherwise, the triangle is called *wild*. Tame and wild triangles are shown on Figure 5. A tame triangle is *contained* in another tame triangle if the two triangles are not equal, they have two common edges, and the surface of the first one is contained in the surface of the second one. We say that $h \in E_G$ *cuts the tame triangle* $[[e, f, g]]$ if the drawn edge h has a non-empty intersection with at least one side of $[[e, f, g]]$. We say that $h \in E_G$ *cuts the tame triangle* $[[e, f, g]]$ *twice* if the drawn edge h has a non-empty intersection with at least two side(s) of $[[e, f, g]]$. The following easy Lemma 2.1 is illustrated by Figure 6.

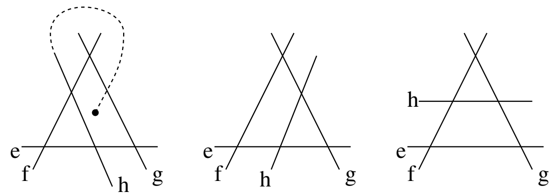


Figure 6: Tame triangle $[[e, f, g]]$ cut twice by h. The dashed curve illustrates the fact that the drawn edge h can have an endpoint in the surface of the triangle which it cuts twice.

Lemma 2.1 (Cutting)**.** *If* $[[e, f, g]]$ *is a tame triangle cut twice by* h*, and* h *does not have its two endpoints in the surface of* $[[e, f, g]]$*, then one and only one triplet in* $\{ \{e, f, h\}, \{e, g, h\}, \{f, g, h\} \}$ *defines a tame triangle contained in* $[[e, f, g]]$. □

Mutations. Let D and D' be two drawings of the graph G with the same subsketch. As D and D' have the same crossing relation, they have the same triangles. We say that a triangle $[[e, f, g]]$ is *permuted between* D *and* D' if the ordering of crossings between its edges along each of its three edges is different in the two drawings, that is, formally: $\mathit{before}_D(e, f, g) = \neg \mathit{before}_{D'}(e, f, g)$, $\mathit{before}_D(f, e, g) = \neg \mathit{before}_{D'}(f, e, g)$, and $\mathit{before}_D(g, e, f) = \neg \mathit{before}_{D'}(g, e, f)$, where $\mathit{before}_D(x, y, z)$ means $(x, y, z) \in \mathit{before}_D$.

For a drawing in the plane, we call a triangle *free* if the (topological) interior of the surface of the triangle has an empty intersection with the drawing. Equivalently, a triangle is *free* if it is tame and it is not cut by any element (indeed a tame triangle whose surface contains a drawn vertex has to be cut since the graph is connected).

Given a drawing D of a graph G in the plane and a free triangle $[[e, f, g]]$ of D, the *(triangle) mutation of* $[[e, f, g]]$ *from* D is the drawing D' of G for which all relations are the same as in D, except that the triangle $[[e, f, g]]$ is permuted between D and D'. We denote this situation by $D \to D'$, and call $[[e, f, g]]$ the *mutated* triangle from D to D'. See Figure 1. Observe that, of course, a tame triangle may be permuted between two drawings D and D', without being free in D or in D'.

Obviously, the definition of a *mutation* can be extended to drawings in the sphere in the natural way (even if the notion of "(topological) interior" of a region is not defined in the sphere, a triangle mutation involves a triangle such that one region delimited by the sides of the triangle contains all the endpoints of the edges of the triangle, and the other region has an empty intersection with the drawing).

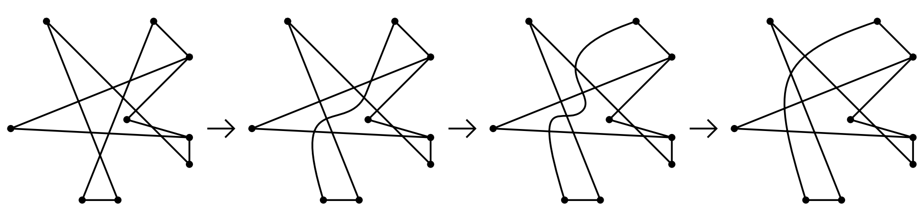
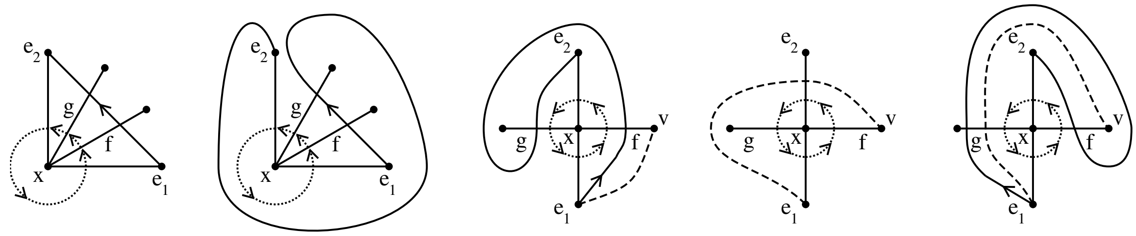


Figure 7: A sequence of mutations.

A *sequence of mutations* from the drawing D is a sequence of drawings, each one being the mutation of a free triangle from the previous one. On the example of Figure 7, the triangle containing a vertex cannot be mutated, but the three other triangles can be mutated triangles in a sequence of mutations. Obviously, a mutation does not change the subsketch. Observe that a sequence of mutations from a drawing D to a drawing D' involves an odd number of times each permuted triangle between the D and D', and possibly an even number of times some non-permuted triangles between D and D' (it is not possible in general to use only permuted triangles, see Section 4.1).

3 Building a sequence of mutations between two complete graph drawings with the same subsketch

The aim of this section is to prove that two complete graph drawings in the sphere, or two complete graph drawings in the plane with the same corner, have the same subsketch if and only if they can be transformed into each other by a sequence of mutations. We essentially deal with drawings in the plane, as the result for drawings in the sphere follows directly. The "if" way is obvious since a mutation does not change the subsketch, the "only if" way is obtained constructively. The algorithm, given in Theorem 3.10, consists in adding vertices and their incident edges one by one. At each new edge insertion, the previous sequence of mutations can be updated as it involves triangles whose surfaces do not contain vertices of the graph, and, then, completed by a sweeping of the new edge. For a partial illustration, the algorithm is applied on a simple example of a pseudoline arrangement in Section 4.1. As corollaries, we get that the subsketch (and corner) of a complete graph drawing determine most properties of the drawing, except those involving orderings of crossings of edges of tame triangles whose surface contain no vertex. First, we need to state a few preliminary results in order to ensure that the algorithm is well defined.



(i)-a (i)-b (ii) (iii) (iv)

Figure 8: Proof of Lemma 3.1.

Lemma 3.1. *The subsketch of a complete graph drawing (in the plane or the sphere) determines the ordering of crossings along an edge for the edges crossing this edge and all sharing the same endpoint. Formally: if the edges f and g share an endpoint and cross the edge e, then the subsketch determines if (e, f, g) belongs to $before_D$ or not.*

Proof. The proof is illustrated in Figure 8. Assume e is directed from the vertex e_1 to the vertex e_2. Recall that, by axiom (D2), two drawn edges with a common endpoint intersect only at this endpoint.

In case (i), we assume that the circular ordering of edges around the vertex x is the following: $[x, e_1]$, f, g, $[x, e_2]$. It is easy to see that e cannot cross g first, otherwise e could not cross f, or it would have to cross g or $[x, e_1]$ twice. Then e has to cross f first and g next. This case is illustrated by pictures (i)-a and (i)-b. Observe that these two pictures are equivalent as drawings in the sphere. We will not illustrate such variants in the other cases. The case where the ordering around x is $[x, e_1]$, g, f, $[x, e_2]$ is analogous.

Let us now assume, without loss of generality, that the ordering around x is $[x, e_1]$, f, $[x, e_2]$, g. Let us call v the endpoint of f distinct from x. Let us prove that, in every possible case, the ordering of crossings of f and g along e is determined by the crossing relation.

In case (ii), we assume that $[e_1, v]$ does not cross $[x, e_2]$ nor g. It is easy to see that e cannot cross f after g, otherwise e would have to cross f or $[e_1, v]$ or $[x, e_1]$ twice. Then e has to cross f first and g next, as illustrated by picture (ii).

In case (iii), we assume that $[e_1, v]$ crosses $[x, e_2]$, as illustrated by picture (iii). It is easy to see that e cannot cross f at all, otherwise e would have to cross f or $[e_1, v]$ or $[x, e_2]$ twice. Hence this case is impossible.

In case (iv), we assume that $[e_1, v]$ does not cross $[x, e_2]$ and crosses g. It is easy to see that e cannot cross f first, otherwise e would have to cross g or $[e_1, v]$ or $[x, e_1]$ twice. Then e has to cross g first and f next, as illustrated by picture (iv). □

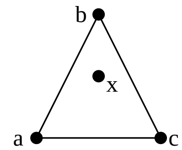


Figure 9: Inside relation.

Inside relations. Let us consider a drawing of the graph G in the plane. For three vertices $a, b, c \in V_G$, we denote by $[a, b, c]$ the bounded region of the plane delimited by the drawn edges $[a, b]$, $[b, c]$ and $[a, c]$, and containing these drawn edges. Note that, by definition of a topological drawing, such a region is equivalent to a closed disc up to homeomorphism. We say that $x \in V_G$ is *inside* $[a, b, c]$ if $x \notin \{a, b, c\}$ and the drawn vertex x is inside the region $[a, b, c]$. Formally, we define the relation $inside_D \subseteq V_G \times V_G \times V_G \times V_G$ by: $(x, a, b, c) \in inside_D$ if and only if x is inside $[a, b, c]$. See Figure 9.

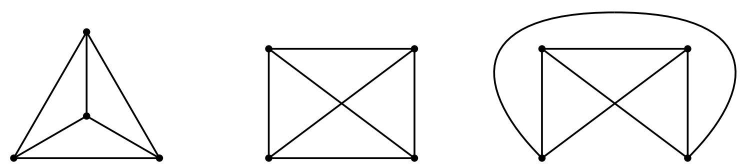


Figure 10: Illustration for Lemma 3.2, with the three possible types of drawings of K_4 in the plane, where, respectively, one, zero or two vertices have the property of being inside the region formed by the three other vertices.

Lemma 3.2. *The corner and subsketch of a complete graph drawing in the plane determine its inside relations.*

Proof. This result is easy to prove directly by considering the different possible drawings of K_4 and noticing that a region $[a, b, c]$ does not contain the vertex P of the corner when $P \notin \{a, b, c\}$.

Briefly, Figure 10 shows the three possible types of drawings of K_4 in the plane. Observe that the drawings in the middle and on the right have the same map and crossing relations. The three vertices involving the corner are not represented, but the position of the corner with respect to the drawing of K_4 indicates which vertices of K_4 are in the boundary of the region delimited by this drawing. In this way, if there is no crossing between the edges (drawing on the left), then the vertex inside the region formed by the other ones is determined; and if there is one crossing between the edges, then the fact that zero (drawing in the middle) or two vertices (drawing on the right) are inside the region formed by the other ones is determined, as well as which vertices are concerned.

In details, this result is contained in Theorem 5.1 below. A precise construction and a case by case analysis is given in the first part of the proof of Theorem 5.1 (cases (i) to (v), proved independently of the rest of the paper, and using only the corner and crossing relations in first order logic). □

Proposition 3.3. *Let D and D' be two complete graph drawings in the plane with the same corner and subsketch. Let T be a tame triangle permuted between the two drawings. Then, in each of the two drawings, the surface of T contains no drawn vertex of the graph.*

Proof. The proof of this key result is rather technical and done in three parts: in part (a), we reduce the problem to one situation and one assumption for a contradiction; in part (b), we show that there are two main cases to handle; and in part (c), we analyse these cases. Assume the triangle $T = [[e, f, g]]$ is permuted between D and D'. Let e_1 and e_2 be the endpoints of e, and f_1 and f_2 be the endpoints of f.

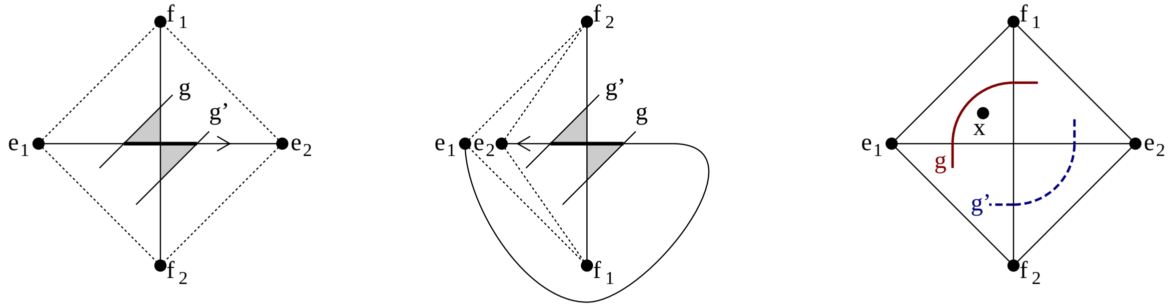


Figure 11: Proof of Proposition 3.3 part (a). On the left and in the middle: two situations for a drawing in the plane, which are equivalent for a drawing in the sphere. On the right: assumption for a contradiction. The curves named g and g' represent a portion of the drawn edge g in D and D', respectively.

• *Part (a), illustrated in Figure 11.*

As shown in Figure 10, since e and f cross, there are two possible types of drawings in the plane for the graph K_4 defined by the endpoints of e and f. In one situation, no endpoint of e or f is inside the region delimited by K_4. In the other situation, one endpoint of e and f is inside the triangle formed by the three other endpoints. However, the two situations are equivalent for a drawing in the sphere. The two situations are shown on the first two pictures of Figure 11, where e_2 is either not inside $[e_1, f_1, f_2]$ or inside $[e_1, f_1, f_2]$. By Lemma 3.2, the corner and subsketch determine if a vertex is inside a triple of vertices or not. This property will be the same in both D and D' since they have the same corner and subsketch. We claim that we only have to prove the proposition in the first situation.

Indeed, the result of the proposition in this first situation directly implies, for drawings in the sphere, the following claim. Assume that a triangle $T = [[e, f, g]]$ is permuted between two complete graph drawings D and D' in the sphere. Call R the region of the sphere delimited by the graph K_4 formed by the endpoints of e and f, such that R contains the drawn edges e and f. The sides of T delimit two regions of the sphere, among which one is contained in R, call it R_T. Then, by the above result, we directly have: *in D and D' as well, there is no drawn vertex in R_T.* Now, the claim in the plane in the second situation alluded to above directly follows from this claim in the sphere, as the two situations are equivalent in the sphere (this reasoning technically consists in changing the unbounded region via the sphere, even if changing the unbounded region can change the inside relation and the fact that a triangle is tame or not).

Hence, in what follows, we assume, without loss of generality, that e_2 is not inside $[e_1, f_1, f_2]$ (in both drawings), and we also assume that e is directed from e_1 to e_2. By axiom (D2), and since e crosses f, the drawn edge e is formed by a segment with endpoint e_1 contained in $[e_1, f_1, f_2]$, and a segment with endpoint e_2 contained in the complement of $[e_1, f_1, f_2]$. So, since e is directed from e_1 to e_2, we have the following equivalence: the portion of the drawn edge e between the intersections of e with f and g is contained in $[e_1, f_1, f_2]$ if and only if g is before f along e. The surface of $[[e, f, g]]$ is delimited by the three sides contained in the drawn edges e, f, g between the intersections of these edges. Since the triangle T is tame, axiom (D2) implies that the surface of $[[e, f, g]]$ is either contained in the region $[e_1, f_1, f_2]$, or contained in the complement of this region. Hence, the surface of $[[e, f, g]]$ is contained in $[e_1, f_1, f_2]$ if and only if g is before f along e. Since $[[e, f, g]]$ is permuted between D and D', we then have that the surface of $[[e, f, g]]$ is contained in $[e_1, f_1, f_2]$ either in D, or in D', but not both. The property that a vertex x is inside $[e_1, f_1, f_2]$ or not is determined by the corner and subsketch, by Lemma 3.2, hence this property of x is the same in D and D'. So, assume a vertex x is inside the surface of T in D, then it is not inside the surface of T in D'.

So, from now on, we assume that a vertex x is inside the surface of T in D but not in D', and that the surface of T is contained in $[e_1, f_1, f_2]$ in D, but not in D'. Up to symmetries, we assume that x is inside $[e_1, f_1, e_2] \cap [e_1, f_1, f_2]$, and that the four regions $[e_1, f_1, e_2]$, $[e_1, f_1, f_2]$, $[e_1, e_2, f_2]$, $[f_1, e_2, f_2]$ do not contain a vertex in $\{e_1, e_2, f_1, f_2\}$. This assumption is shown on the third picture of Figure 11. We now look for a contradiction. For brevity, as shown in Figure 12, we will denote $Q(e_1, f_1)$ for $[e_1, f_1, e_2] \cap [e_1, f_1, f_2]$, $Q(e_2, f_1)$ for $[e_2, f_1, e_1] \cap [e_2, f_1, f_2]$, $Q(e_1, f_2)$ for $[e_1, f_2, e_2] \cap [e_1, f_2, f_1]$, and $Q(e_2, f_2)$ for $[e_2, f_2, e_1] \cap [e_2, f_2, f_1]$.

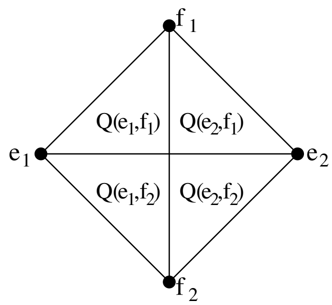


Figure 12: Proof of Proposition 3.3. Notation for the four quarters of the region delimited by e_1, f_1, e_2, f_2.

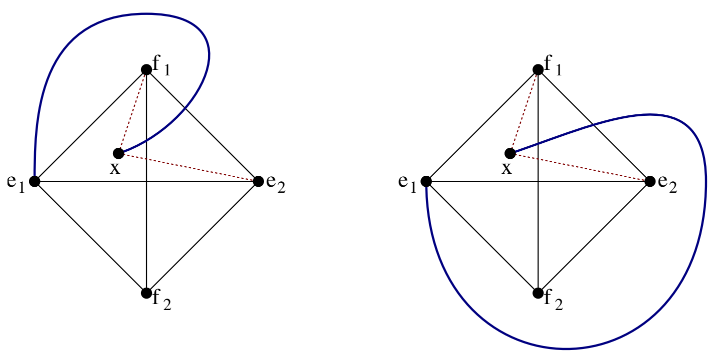


Figure 13: Proof of Proposition 3.3 part (b). The curved edge $[x, e_1]$ implies the dashed edges $[x, e_2]$ and $[x, f_1]$.

• *Part (b), illustrated in Figure 13.*

What follows is available in D and D' as well. For two vertices v and w, let us denote by $]v, w[$ the drawn edge $[v, w]$ minus its endpoints v and w.

Let us assume that $]x, e_1[$ is not contained in $Q(e_1, f_1)$ and see what is implied by this assumption. Then, by axiom (D2), this drawn edge has to cross $[f_1, e_2]$. Note that $[x, e_1]$ cannot pass through $[e_1, e_2, f_2]$, otherwise it would have to cross $[f_1, f_2]$ twice. So, there are two cases illustrated in Figure 13. By axiom (D2), $]x, f_1[$ cannot cross the drawn edges $[x, e_1]$, $[e_1, f_1]$ or $[f_1, f_2]$. So, in each case, $]x, f_1[$ has to be contained in $Q(e_1, f_1)$. Moreover, similarly, by axiom (D2), $]x, e_2[$ cannot cross the drawn edges $[x, e_1]$, $[e_1, e_2]$ or $[f_1, e_2]$. So, in each case, it is direct to check that $]x, e_2[$ has to be contained in $[e_1, e_2, f_1]$. We have proven: if $]x, e_1[$ is not contained in $Q(e_1, f_1)$, then $]x, f_1[$ is contained in $Q(e_1, f_1)$ and $]x, e_2[$ is contained in $[e_1, e_2, f_1]$.

Now, assume that $]x, f_1[$ is not contained in $Q(e_1, f_1)$. Then, by the same reasoning as above applied to $]x, f_1[$ instead of $]x, e_1[$, we have that $]x, e_1[$ is contained in $Q(e_1, f_1)$. So, we have proven that either $]x, f_1[$ or $]x, e_1[$ is contained in $Q(e_1, f_1)$.

So, finally, there are two cases to handle: either both $]x, f_1[$ and $]x, e_1[$ are contained in $Q(e_1, f_1)$ (case 1), or $]x, f_1[$ is contained in $Q(e_1, f_1)$ and $]x, e_2[$ is contained in $[e_1, e_2, f_1]$ (case 2).

• *Part (c), illustrated in Figure 14.*

In case 1 (first picture of Figure 14), we assume that both $]x, f_1[$ and $]x, e_1[$ are contained in $Q(e_1, f_1)$. Then, by assumption on the position of x (made at the end of part (a) of the proof), the edge g has to cross

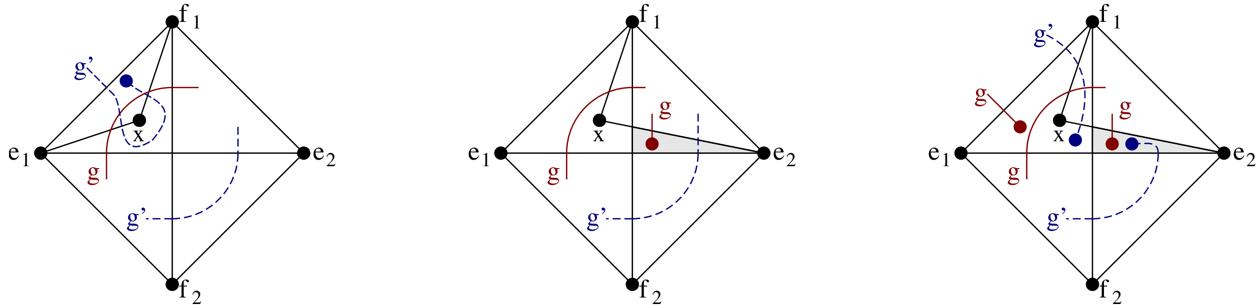


Figure 14: Proof of Proposition 3.3 part (c). Three cases yielding a contradiction.

$[x, e_1]$ and $[x, f_1]$ in D. Since D and D' have the same subsketch, then g has to cross those two edges also in D'. So, g has to cross first $[e_1, f_1]$ in D', because of the portion of g in D' contained by assumption in $Q(e_2, f_2)$ and to cross next $[x, e_1]$ and $[x, f_1]$ (in any order). So, an endpoint of g in D', say g_1, has to be in the region $[x, e_1, f_1]$, as shown on the picture. Moreover, the edge $[x, g_1]$ has to be contained in the same region $[x, e_1, f_1]$ in D', otherwise it would meet g twice, which is forbidden by axiom (D2). By Lemma 3.2, the endpoint g_1 of g has to be in the same region $[x, e_1, f_1]$ in D, and the edge $[x, g_1]$ must not cross e or f in D as it does not in D'. If g_1 is in the region delimited by g, $[x, e_1]$ and $[x, f_1]$ in D, then g has to meet itself or to meet twice $[x, e_1]$ or $[x, f_1]$ in D, which is forbidden by axiom (D2). If g_1 is not in the region delimited by g, $[x, e_1]$ and $[x, f_1]$ in D, then the edge $[x, g_1]$ has to meet g twice in D. A contradiction.

In case 2, we assume that $]x, f_1[$ is contained in $Q(e_1, f_1)$ and $]x, e_2[$ is contained in $[e_1, e_2, f_1]$. See the second and third pictures of Figure 14. Let us denote by R the region intersection of $Q(e_2, f_1)$ and the complement of $[x, f_1, e_2]$ (grey region delimited by $[x, e_2]$, $[e_1, e_2]$ and $[f_1, f_2]$ on the pictures).

In a first case (second picture of Figure 14), we assume that the portion of g in D' contained in $Q(e_2, f_2)$ continues with crossing $[x, e_2]$. Since D and D' have the same subsketch, this implies that g crosses $[x, e_2]$ also in D. Since g cannot meet $[x, e_2]$, $[e_1, e_2]$ and $[f_1, f_2]$ more than once, we get that an endpoint of g, say g_1 belongs to the region R in D. By Lemma 3.2, g_1 belongs to the same region R in D', which is impossible since g cannot meet $[x, e_2]$, $[e_1, e_2]$ or $[f_1, f_2]$ more than once. A contradiction.

In a second case (third picture of Figure 14), we assume that the portion of g in D' contained in $Q(e_2, f_2)$ ends with an endpoint, say g_1, in the region R. As above, by Lemma 3.2, this implies that g_1 belongs to the same region in D. This implies that g crosses $[x, e_2]$ in D, and hence in D' too. Independently, g crosses $[x, f_1]$ in D, which implies that g crosses $[x, f_1]$ in D' too. Because of the portion of g in D' contained in $Q(e_2, f_2)$, we must have that g first crosses $[e_1, f_1]$ in D', and next crosses $[x, f_1]$ and $[x, e_2]$ (in any order a priori). This implies also that g crosses $[e_1, f_1]$ in D too, and hence that the endpoint g_2 of g in D belongs to the region $Q(e_1, f_1)$. This implies that $[x, g_2]$ crosses $[e_1, e_2]$ in D (otherwise $[x, g_2]$ would meet either g or $[x, e_2]$ twice), and hence in D' too. If, in D', g crosses $[e_1, f_1]$, and next $[x, e_2]$, and next $[x, f_1]$ (this case is not represented on the picture), then $[x, g_2]$ cannot cross $[e_1, e_2]$ in D' (otherwise $[x, g_2]$ would have to meet twice either $[e_1, f_1]$ or $[x, e_2]$). So, in D', g crosses $[e_1, f_1]$, and next $[x, f_1]$, and next $[x, e_2]$ (this case is represented on the picture). Finally, assuming that g is directed from g_1 to g_2, we have that, in D, g crosses the edge $[x, e_2]$ first and the edge $[x, f_1]$ next, whereas, in D', g crosses the edge $[x, f_1]$ first and the edge $[x, e_2]$ next. Since the edges $[x, f_1]$ and $[x, e_2]$ share the vertex x, this is a contradiction with Lemma 3.1. □

The first part of Lemma 3.4 below, from which the second part follows easily, is equivalent to the first part of [8, Lemma 4.7]. We shall not prove it again. Another proof was recently presented in [9].

Lemma 3.4. *Every region of a complete graph drawing in the plane, except the unbounded one, is contained in a region $[x, y, z]$ of the plane for some vertices x, y, z of the graph. Moreover, for every couple of regions R and R' of the drawing, there exists a region $[x, y, z]$ of the plane, for some vertices x, y, z, that contains either R or R' but not both.*

Proof. The first claim is equivalent to the first claim of [8, Lemma 4.7]. Let us prove the second claim.

The first claim can be directly applied to a drawing in the sphere with the following statement: choose a particular region R' of the drawing in the sphere and consider it as the unbounded region for a drawing in the plane, then for every other region R of the drawing in the sphere, there exist three vertices x, y, z such that the curve formed by the union of $[x, y]$, $[x, y]$, and $[x, y]$ (homeomorphic to a circle), separates the two regions R and R'. Hence, back to a drawing in the plane, we get that: for every couple of regions R and R' of the drawing in the plane (bounded or not), there exist three vertices x, y, z such that the region $[x, y, z]$ of the plane contains either R or R' but not both. □

Lemma 3.5. *Regions of a complete graph drawing in the plane are exactly non-empty intersections, for all triplets $\{x, y, z\}$ of vertices, of regions $R_{x,y,z}$ of the plane where $R_{x,y,z}$ is either the (topological) interior of $[x, y, z]$ or the complement of $[x, y, z]$.*

Proof. Consider an intersection of regions of the plane of type $R_{x,y,z}$ for all triplets $\{x, y, z\}$. This intersection is either empty, or it is a union of regions of the drawing (since its intersection with any drawn edge is empty, since $R_{x,y,z}$ has an empty intersection with the edges joining x, y and z). A region R of the drawing is contained in the intersection of the regions $R_{x,y,z}$ for all triplets $\{x, y, z\}$, where $R_{x,y,z}$ is the region containing R among the interior of $[x, y, z]$ and the complement of $[x, y, z]$. It remains to prove that this intersection does not contain another region R' of the drawing. This is given by the second claim of Lemma 3.4: for every region R' distinct from R, there exists $R_{x,y,z}$ that contains R but not R'. □

Lemma 3.6. *Let D and D' be two complete graph drawings of K_n in the plane with the same corner and subsketch. Let a_n be a vertex of K_n. Assume that the two drawings of $K_{n-1} = K_n - \{a_n\}$ induced by D and D' are equal. Then the drawn vertex a_n belongs to the same region of this subdrawing in both D and D'.*

Proof. By Lemma 3.5, the region of the drawing D'' of K_{n-1} containing a_n in D is the intersection of suitable regions $R_{x,y,z}$ for $x, y, z \in V \setminus \{a_n\}$. By Lemma 3.2, the inside relation for D is determined by its subsketch, which is the same for D and D'. Hence a_n belongs to the same regions of type $[x, y, z]$ for $x, y, z \in V \setminus \{a_n\}$ both for D and D'. Hence a_n belongs to the same intersection of regions $R_{x,y,z}$ for $x, y, z \in V \setminus \{a_n\}$ in D' as in D, hence to the same region of D'' in D' as in D. □

Notation 3.7. For a drawing D of a graph G, and a drawn edge e of D, we denote by $D - e$ the drawing obtained by removing the drawn edge e except the intersection points with other edges. Note that if an endpoint a of e has degree one in G, then $G - e$ is not connected and a is isolated in $G - e$. By definition, the vertex a is not represented in $D - e$.

Let G be a complete graph with n vertices $\{a_1, \dots, a_n\}$, and let D be a drawing of G. For $1 \leq i < j \leq n$, the (undirected) drawn edges of G are denoted by $e_{i,j} = [a_i, a_j]$. We denote $D_n = D$ and, for $1 \leq i < n$, $D_i = D - \{e_{i,n}, e_{i+1,n}, \dots, e_{n-1,n}\}$ (vertices of the graph are not deleted, except a_n which is deleted in D_1). In particular, D_1 is a drawing of the complete graph on $n - 1$ vertices $a_1, \dots, a_{n-1}$. When D is given with a corner (P, α, β), we choose to numerate vertices so that $P = a_1$, $\alpha = e_{1,2}$ and $\beta = e_{1,3}$, so that it remains a corner of the considered subdrawings.

Proposition 3.8 (Sweeping)**.** *Let D and D' be two complete graph drawings in the plane, with the same corner and subsketch. Using Notation 3.7, assume there exists i, $1 \leq i < n$, such that $D_i = D'_i$ and $D_{i+1} \neq D'_{i+1}$ (thus, these drawings differ only in the drawn edge $e_{i,n}$). Then, following an orientation-preserving homeomorphism of the plane, there exists a sequence of mutations from D_{i+1} to D'_{i+1}, using only permuted triangles between D_{i+1} and D'_{i+1} (thus, these triangles all contain the edge $e_{i,n}$), and using each of these triangles exactly once.*

Remark 3.9. The result of Proposition 3.8 above does not generalize to general graph drawings. For example, the middle part of Figure 3 shows two distinct 2-connected graph drawings, with the same subskecth and corner, such that the drawings are the same when the edge $[a, f]$ is removed, but there is no triangle.

Proof. The integer i is fixed for the whole proof. The proof intuitively consists in a “sweeping” of $e_{i,n}$ from D to D', which is described in part (c) of the proof. At first, we need to transform the drawing D by means of an homeomorphism of the plane, preserving its sketch and corner, which is done in the preliminary step below and in parts (a) and (b) of the proof, so that $e_{i,n}$ is suitably drawn and this sweeping notion works properly. It ensures that, from the algorithmic viewpoint, one can deal with combinatorial sketches only, and one just needs the construction in part (c). The proof is illustrated by, and can be understood from, Figures 15, 16, 17, 18, 19.

First of all, let us observe that, up to an orientation-preserving homeomorphism of the plane, a_n is represented by the same drawn vertex in D_{i+1} and D'_{i+1}. Indeed, by hypothesis, if $1 < i < n$, then a_n is represented by the same drawn vertex in D_{i+1} and D'_{i+1} as the topological representation of D_i and D'_i is the same and involves the vertex a_n. Now, assume that $i = 1$ (recall that a_n is not represented in D_1). By Lemma 3.6, the drawn vertices representing a_n in $D_{i+1} = D_2$ and $D'_{i+1} = D'_2$ are in the same region of the plane delimited by $D_i = D'_i = D_1 = D'_1$, which is a drawing of K_{n-1}. So, up to an orientation-preserving homeomorphism that affects only this region, a_n is also represented by the same drawn vertex in $D_{i+1} = D_2$ and $D'_{i+1} = D'_2$, and this does not affect $D_1 = D'_1$ nor the intersections of $e_{1,n}$ with $D_1 = D'_1$. For the rest of the proof, we make the assumption that a_n is represented by the same drawn vertex in D_{i+1} and D'_{i+1}.

The edge $e_{i,n}$ is represented by a curve c in D, and by a curve c' in D', with both the same endpoints a_i and a_n which are drawn vertices of the graph. The intersection of the curves c and c' consists of a disjoint union of closed segments (by segments, we always mean curve segments, which possibly contain only one point). We can assume that this set of segments is finite because all drawings of a given graph can be realized in piecewise-linear topology using the same finite set of straight segments (up to homeomorphism). We denote by R_∞ the unbounded region of the plane delimited by $c \cup c'$, that is, the unbounded connected component of the plane minus $c \cup c'$. A point (or a set of points) in $c \cup c'$ is called *exterior* if it is contained in the boundary of R_∞. The bounded connected components of the plane minus $c \cup c'$ are called *components*. We denote by R the union of all components, that is, R equals the plane minus $R_\infty \cup c \cup c'$. We call *proper homeomorphism* an orientation-preserving homeomorphism of the plane that, when applied to D_{i+1}, does not affect $D_i = D'_i$, nor a_n, nor the intersections of the curve representing $e_{i,n}$ with $D_1 = D'_1$, nor the sketch of D_{i+1}, nor the property of being a corner. Finally, recall that $D_1 = D'_1$ is the drawing of the complete graph K_{n-1} with vertices $\{a_1, \dots, a_{n-1}\}$ contained in $D_i = D'_i$.

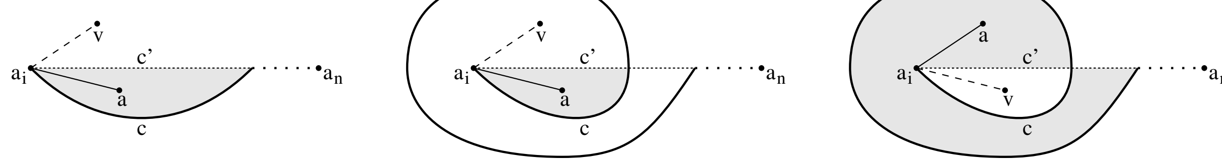


Figure 15: Proof of Proposition 3.8, part (a), first step. If a drawn vertex is in R, then all drawn vertices are in the same component, shown as a grey region (because of the same circular ordering around a_i).

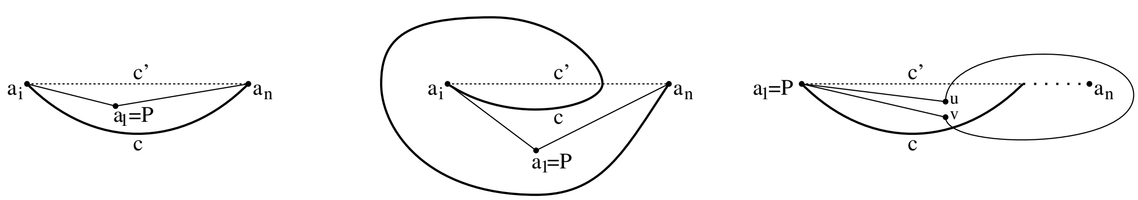


Figure 16: Proof of Proposition 3.8, part (a), second step. The pictures show impossible situations. If a drawn vertex is in R, then $i = 1$ and a_n is in the unbounded region of the common drawing $D_1 = D'_1$ of K_{n-1} (because of the same corner).

• *Part (a), illustrated in Figures 15, 16 and 17.*

In this part of the proof, we prove the following claim: *up to a proper homeomorphism transforming D, we can assume there is no drawn vertex inside R.*

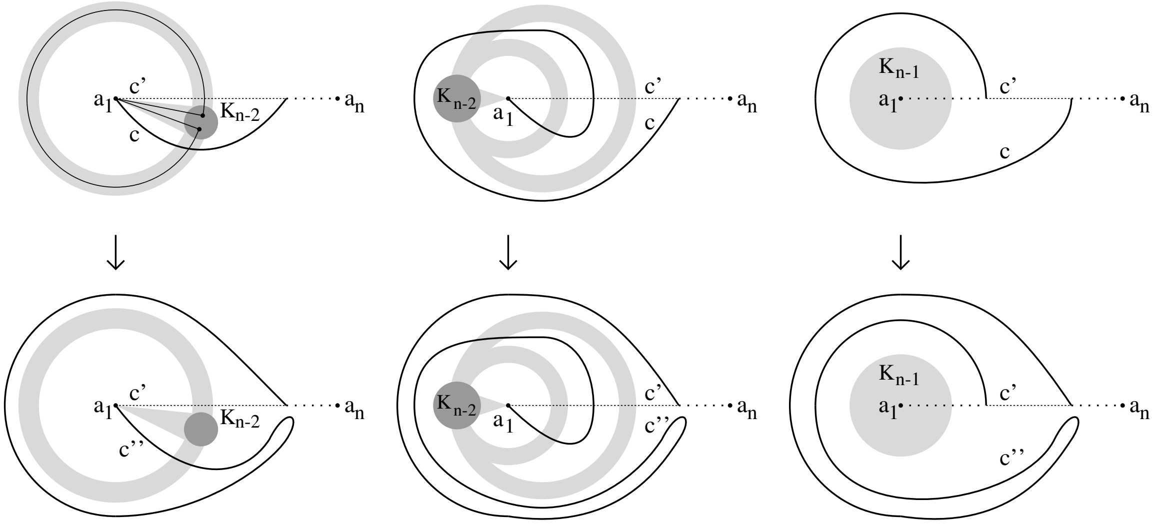


Figure 17: Proof of Proposition 3.8, part (a), third step. If a drawn vertex is in R, then, by a proper homeomorphism, the curve c representing the edge $e_{1,n}$ can be transformed into a curve c'', so that all vertices (except a_1 and a_n) are in the unbounded region R_∞ defined with respect to the curves c' and c''. The grey regions show where edges of the drawing $D_1 = D'_1$ of K_{n-1} are drawn, and the dark grey regions, denoted by K_{n-2}, show where the vertices $\{a_2, \ldots, a_{n-1}\}$ are drawn.

Let us first informally explain this claim. It can happen, for instance, that the two initial curves c and c' delimit a disc in which the two drawings are contained (thus this disc is the closure of R). In this case, one would perform first a proper homeomorphism on D to move c onto c' (without affecting the rest of the drawing, nor the sketch of the drawing, nor the corner), and then consider R as defined for the resulting drawings instead (thus R is now empty). This claim handles when such preliminary topological transformations are made (not affecting the combinatorial sketchs).

Let us assume that there is a drawn vertex a inside R (hence $a \in V \setminus \{a_i, a_n\}$), and let us denote by R_a the component containing a. Since the graph is complete, then there is an edge $[a_i, a]$. By axiom (D2), this edge does not meet the drawn edge $e_{i,n}$ except at a_i, so the edge $[a_i, a]$ is also contained in R_a, except its endpoint a_i (which is in the closure of R_a).

First, let us prove that all the vertices, except a_i and a_n, are drawn inside R_a. Assume there is a drawn vertex $v \in V \setminus \{a_i, a_n\}$ not in R_a. As above, since the graph is complete, then there is an edge $[a_i, v]$ and this edge does not intersect R_a. Then, in the circular ordering of curves around a_i, one and only one among c and c' is between $[a_i, a]$ and $[a_i, v]$. See Figure 15 which shows the possible situations. Then the circular ordering of $[a_i, a]$, $[a_i, v]$ and $e_{i,n}$ around a_i is not the same in D and D', which is in contradiction with the hypothesis D and D' have the same subsketch. So, all the vertices, except a_i and a_n, are drawn inside R_a.

Now, let us prove that $i = 1$. Assume $i > 1$. Then a_1 is inside R_a, as well as the edges $e_{1,i}$ and $e_{1,n}$ (except the endpoints a_i and a_n). Then $e_{1,i} \cup e_{1,n}$ separates R_a in two connected components. See the left and middle pictures of Figure 16. By hypothesis, the corner at the boundary of the unbounded region of the drawing is $(a_1, e_{1,2}, e_{1,3})$ in the two drawings D_{i+1} and D'_{i+1}. This is impossible as it implies that, for one drawing or the other, the unbounded region is contained in one of these two connected components of R_a. So we have $i = 1$.

Now, let us prove that a_n is in the unbounded region of the common drawing $D_1 = D'_1$ of K_{n-1}. Assume that a_n is not in this unbounded region. Since all drawn vertices of K_{n-1} except a_1 are in R_a (by the results above), then there is an edge uv of K_{n-1} with its two endpoints in R_a such that a_n is inside $[a_1, u, v]$, as shown on the right part of Figure 16. Then, it is impossible that the two drawings D and D' have the same corner $(a_1, e_{1,2}, e_{1,3})$ at the boundary of their unbounded region (as this would imply that this unbounded region is bounded by $c \cup c'$). This is a contradiction with the hypothesis of the proposition. So a_n is in the unbounded region of the common drawing $D_1 = D'_1$ of K_{n-1}.

Finally, we check that there is a proper homeomorphism of the plane transforming the curve c representing $e_{1,n}$ in D into a curve c'' such that all the vertices except a_1 and a_n are inside the unbounded component of the plane minus $c' \cup c''$. Briefly, one just has to make the vertex a_n turn around the drawing $D_1 = D'_1$ of K_{n-1} in a suitable way, as shown in Figure 17. Precisely, all vertices of D_1 except a_1 are in R_a (denoted by K_{n-2} and depicted by a dark grey circle in the left and middle pictures), and all edges having a portion outside R_a must turn around a_1 to cross both c and c' (depicted by light grey strips in the left and middle pictures). So, the transformations shown in the left and middle pictures ensure that all vertices of K_{n-2} are in the same region delimited by c'' and c'. Then, a_n can possibly have to be turned several times around D_1, by repeating at various places of the curve the transformation shown in the right picture (e.g., when c forms a spiral around a_1), so that all vertices of K_{n-2} are in the unbounded region delimited by c'' and c'.

We have proven that, up to a proper homeomorphism, we can assume that there is no drawn vertex inside R. We make this assumption for the rest of the proof.

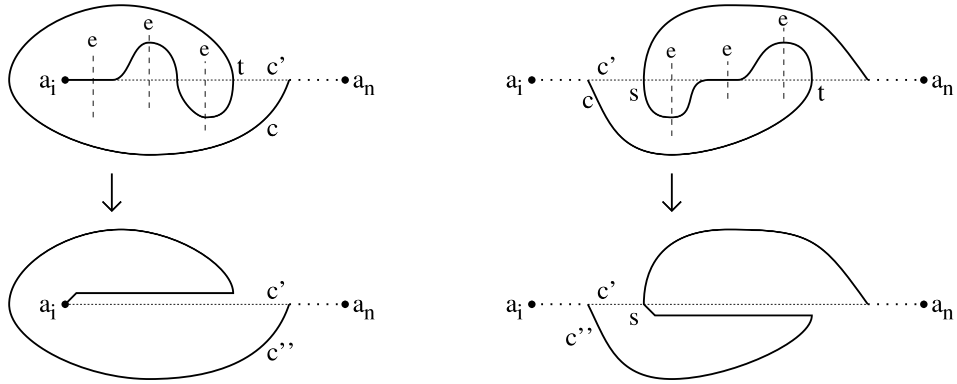


Figure 18: Proof of Proposition 3.8, part (b). The dashed edges denoted e cannot exist as they would intersect either a non-cutable component or a non-cutable point in $c \cap c'$. The arrows indicate a proper homeomorphism transforming c into c'' such that the number of endpoints of segments in $c'' \cap c'$ is strictly smaller than this number in $c \cap c'$.

• *Part (b), illustrated in Figure 18.*

The goal of this part is to show the following: *up to a proper homeomorphism, R consists in a sequence of components along c (or, equally, c'), such that the boundary of each component is formed by only one segment of c and only one segment of c', which are both exterior (that is, in the boundary of R_∞).*

Note that, as for Part (a), we mean that R could be redefined after performing a proper homeomorphism, so that the above property holds.

As the two drawings have the same subsketch, every edge crossing c crosses c' and conversely.

First, let us prove the following claim: *if a drawn edge e crosses c or c', then the intersection point is exterior.* Indeed, assume (without loss of generality) that e crosses c, and call R_1 and R_2 the components whose boundaries contain the crossing point. Assume R_1 and R_2 are both bounded. Since R_1 is bounded, no endpoint of e is inside R_1, by part (a) of the proof. Then e has to cross c' in the boundary of R_1, because e cannot cross c twice, by axiom (D2). Similarly, since R_2 is bounded, then e has to cross c' in the boundary of R_2. Since e cannot cross c' twice, by axiom (D2), the two above crossing points of e and c' are the same. Hence e joins the crossing point of e and c to the crossing point of e and c' both in R_1 and R_2, meaning that e is not homeomorphic to a segment, which is a contradiction.

Furthermore, we prove the following claim: *if a drawn edge meets c and c' at the same intersection point, then the intersection point has the property that removing it from the closure of R disconnects the closure of R (that is, equivalently, the edge minus this intersection point is contained in R_∞).* Indeed, assume that a portion of e is contained in a component (whose boundary contains the intersection point), then e has an endpoint in this component, which is a contradiction with part (a), or e has to cross c or c' twice, which is a contradiction with axiom (D2). An element of $c \cap c'$ with the above property is called *cutable*. Hence, the set of non-cutable points in $c \cap c'$ has an empty intersection with the drawn edges (distinct from $e_{i,n}$).

The two above claims directly imply the following: *if a component contains a portion of a drawn edge, then its boundary contains an exterior segment of c and an exterior segment of c'.* Indeed the intersection point cannot belong to $c \cap c'$ by the second claim, so it belongs to an exterior segment of c (or c') in the boundary of the component, by the first claim, and there must exist another intersection point with c' (or c), and hence an exterior segment of c' in the boundary. A component with the above property is called *cutable*. Hence a non-cutable component has an empty intersection with the drawing.

Now we prove the goal of this part. Let us call S the set of endpoints of the segments in $c \cap c'$ (recall that $c \cap c'$ consists of a disjoint union of closed curved segments). For s and t in S, let us denote by $c_{s,t}$ and $c'_{s,t}$ the portions of the curves c and c' between s and t, respectively, and let us call $R_{s,t}$ the subset of R delimited by $c_{s,t}$ and $c'_{s,t}$.

For a first construction, let us assume that a_i (or, similarly, a_n) is not exterior. Such a situation is illustrated in the left part of Figure 18. In this situation, there exists $t \in S$, $t \neq a_i$, such that $R_{a_i,t} \cup c_{a_i,t} \cup c'_{a_i,t}$ is formed by components which are all non-cutable, their boundaries, and segments in $c \cap c'$ whose points are all non-cutable. By the above claims, $R_{a_i,t} \cup c_{a_i,t} \cup c'_{a_i,t}$ has an empty intersection with all the edges of the drawing (distinct from $e_{i,n}$). Then, as illustrated in the left part of Figure 18, there is a proper homeomorphism transforming c into c'' such that the number of endpoints in $c \cap c''$ is strictly smaller than the number of endpoints in S (the curve c'' is made parallel to c' in a suitable neighbourhood of $c'_{a_i,t}$, and joins a_i in a suitable neighbourhood of a_i, so that, at least, the intersection point t vanishes).

For a second construction, let us order the set S of endpoints of the segments in $c \cap c'$ with respect to their positions in the curve c directed from a_i to a_n. Assume that the ordering of S with respect to c' is not the same. Such a situation is illustrated in the right part of Figure 18. In this situation, there exist $s \in S$ and $t \in S$, $s \neq t$, such that $R_{s,t} \cup c_{s,t} \cup c'_{s,t}$ is formed by components which are all non-cutable, their boundaries, and segments in $c \cap c'$ whose points are all non-cutable. By the above claims, $R_{s,t} \cup c_{s,t} \cup c'_{s,t}$ has an empty intersection with all edges of the drawing (distinct from $e_{i,n}$). Then, as illustrated in the right part of Figure 18, there is a proper homeomorphism transforming c into c'' such that the number of endpoints in $c'' \cap c'$ is strictly smaller than the number of endpoints in S (the curve c'' is made parallel to c' in a suitable neighbourhood of $c'_{s,t}$, and joins s in a suitable neighbourhood of s, so that, at least, the intersection point t vanishes).

The two above constructions can be repeated (independently of each other) while a_i and a_n are not exterior, and while the ordering of S along the two curves representing $e_{i,n}$ is not the same. The process will end because the number of elements of S strictly decreases at each operation.

Finally, we have proven that, up to a proper homeomorphism, we can assume that both a_i and a_n are exterior, and that the ordering of S along c and c' is the same. This also directly implies that the boundary of each component is formed by one exterior segment of c and one exterior segment of c'. We make these assumptions for the rest of the proof.

• *Part (c), illustrated in Figure 19.*

Under the assumptions made in parts (a) and (b) of the proof following an orientation-preserving homeomorphism of the plane, we now proceed to a "sweeping" from c to c', yielding the result of the proposition. Note that the sweeping technique below is rather usual in pseudoline arrangements and related fields. Yet, we give a complete proof in the context of this paper.

Consider a drawn edge of $D_i = D'_i$ intersecting R. By the assumptions made in parts (a) and (b) of the proof, we have that the edge intersects exactly one component, its two endpoints are in R_∞, and the edge is formed by three segments: one contained in R_∞ and containing the first endpoint, one contained in R_∞ and containing the second endpoint, and one contained in the component, between the intersections with c and c' (recall that the two drawings have the same subksetch, hence the edge crosses c and c').

Let C be the set of pairs of edges $\{f, g\}$ whose intersection is inside R. Since R does not contain a drawn vertex and since f and g cross $e_{i,n}$, we have that $[[e_{i,n}, f, g]]$ is a tame triangle in D and D'. By definition, the set of $[[e_{i,n}, f, g]]$, $\{f, g\} \in C$, is the set of permuted triangles between D_{i+1} and D'_{i+1}. Consider the graph $H = (V_H, E_H)$ whose vertices are all intersection points of drawn edges inside R, and whose edges

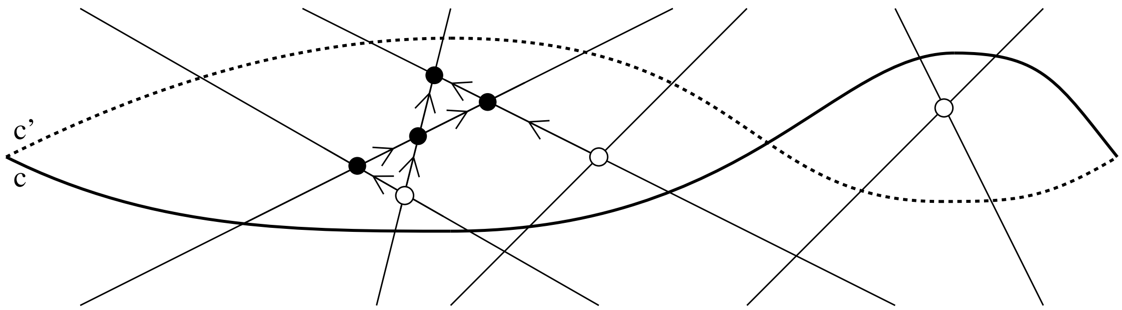


Figure 19: Proof of Proposition 3.8, part (c). The sweeping from the curve c to the curve c' consists in passing through each of the intersection vertices which are between the two curves, by means of successive triangle mutations. The ordering follows the arrows between those vertices. At any step, the vertices with no incoming edges can be used for a mutation. Any of the three white vertices can thus be used first.

are given by all portions of drawn edges of $D_i = D'_i$ joining these vertices. See Figure 19 for an example. Observe that H is not necessarily connected, and can even have isolated vertices. We direct every edge in E_H from c' to c (it is well-defined since each edge in E_H is contained in a drawn edge of $D_i = D'_i$ intersecting R, as considered above).

We prove that the directed graph H has no directed cycle. Assume there is a directed cycle formed by successive vertices represented in the plane by the points $p_1, \dots, p_n, p_{n+1}$, with $p_{n+1} = p_1$. We may assume that two segments $[p_j, p_{j+1}]$ and $[p_k, p_{k+1}]$ intersect if and only if $|k - j| = 1$. Otherwise we can restrict the cycle to a smaller one. Then the union of these segments is homeomorphic to a circle, and these segments are all directed in the same way (clockwise or counterclockwise) by the directions of edges in the directed cycle of H. Let q_j, $1 \leq j \leq n$, be the intersection point of c with the drawn edge containing $[p_j, p_{j+1}]$. The directions of edges (p_j, p_{j+1}) of H, $1 \leq j \leq n$, induce directions along c for the ordered pairs (q_j, q_{j+1}), $1 \leq j < n$, as well as for (q_n, q_1), such that these directions are all the same along c, which is impossible as c is homeomorphic to a segment.

Hence, in the directed graph H, there exists a vertex with no incoming edge. This vertex is the intersection of two drawn edges f and g, and, by definition of E_H and its directions, $[[e_{i,n}, f, g]]$ is a free triangle in D_{i+1}, permuted between D_{i+1} and D'_{i+1}. Let D''_{i+1} be the mutation of this triangle from D_{i+1}. There is one less permuted triangle between D'_{i+1} and D''_{i+1} than between D_{i+1} and D'_{i+1}, and common permuted triangles are the same. Iterating this construction for D''_{i+1} and D'_{i+1} readily gives a sequence of triangle mutations of permuted triangles between D_{i+1} and D'_{i+1}, where each permuted triangle is used exactly once. □

We now combine the previous results to prove the main result of the paper, by means of an inductive construction. Briefly, in order to build a sequence of mutations from a drawing D to another drawing D' of the complete graph with the same subsketch and corner, we use a sequence of mutations, with a suitable property, from $D - e$ to $D' - e$, which are the two drawings obtained by removing some edge e. In the first stage, this sequence is updated, yielding a sequence of mutations starting from D and transforming the subdrawing $D - e$ of D into the subdrawing $D' - e$ of D' (the suitable property, along with Proposition 3.3, ensures that the involved triangles do not contain a vertex of the graph, allowing to use the Cutting Lemma 2.1 with respect to e). In the second stage, in order to get D', we complete the sequence with mutations all involving the edge e (using the sweeping process from Proposition 3.8 with respect to e). We thus obtain a sequence of mutations from D to D', also with the suitable property. In order to allow the induction, this construction actually works for subdrawings obtained by removing several edges having all the same endpoint from the initial complete graph drawings (precisely, for drawings denoted D_i and D'_i in Notation 3.7).

Theorem 3.10. *Let D and D' be two complete graph drawings in the plane with the same corner and subsketch. There exists a sequence of mutations from the sketch of D to the sketch of D' (or, in topological terms, a sequence of mutations and orientation-preserving homeomorphisms from the drawing D to the*

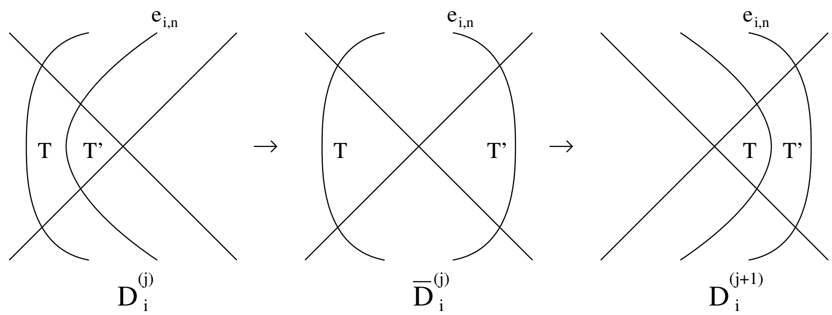


Figure 20: First stage of Theorem 3.10 or Algorithm 1: triangle mutations when T is cut by $e_{i,n}$.

drawing D').

Moreover, denoting such a sequence of mutations by $D = D^{(0)} \to D^{(1)} \to \ldots \to D^{(k-1)} \to D^{(k)} = D'$, this sequence can be chosen so that, for any intermediate sketch $D^{(j)}$, $0 \leq j \leq k-1$, the mutated triangle from $D^{(j)}$ to $D^{(j+1)}$ is contained in a permuted tame triangle between $D^{(j)}$ and D'.

Such a sequence $S(D, D')$ is built in an inductive way by Algorithm 1, or, equivalently, in a recursive way by Algorithm 2 (where we use Notation 3.7, in particular: $D_n = D$, as well as $D'_n = D'$, is a drawing of K_n; and D_1, as well as D'_1, is a drawing of K_{n-1}).

Proof. We prove the result by induction on n and $1 < i \leq n$, and we use Notation 3.7. The construction is performed by Algorithm 1, and we can check its validity simultaneously. In order to lighten notations, the induction on the number of vertices n does not appear as an index in the notations, but it is underlying as D_1 and D'_1 are drawings of the complete graph with $n-1$ vertices, as precised below.

If $n = 3$, then $D = D'$ since the two drawings have the same corner by assumption, and Algorithm 1 correctly returns $S(D, D') = D$. We assume for the rest of the proof that $n > 3$.

Computation of the first mutated triangle $T(D_i, D'_i)$ in the sequence from D_i to D'_i, for $1 < i \leq n$
(if it exists, otherwise the value $\emptyset$ is returned)

    if $n \leq 3$ or $D_i = D'_i$ then $T(D_i, D'_i) := \emptyset$
    if $n > 3$ then let $T = T(D_{i-1}, D'_{i-1})$
        if $T \neq \emptyset$ then
            if T is free in D_i then $T(D_i, D'_i) := T$
            otherwise T is cut by $e_{i,n}$ in D_i then there exists (by Cutting Lemma 2.1) a unique T'
                contained in T, free in D_i, with $e_{i,n} \in T'$, and $T(D_i, D'_i) := T'$
        if $T = \emptyset$ then there exists (by Sweeping Proposition 3.8) a triangle T', free in D_i, with $e_{i,n} \in T'$,
                permuted between D_i and D'_i, and $T(D_i, D'_i) := T'$ (arbitrary choice)

Computation of $S(D, D') = S(D_n, D'_n)$

    if $T(D_n, D'_n) = \emptyset$ then $S(D, D') := D$
    otherwise D'' being obtained by mutation of $T(D_n, D'_n)$ from D
                $S(D, D') := D \to S(D'', D')$

Algorithm 2: Recursive algorithm for Theorem 3.10, using Notation 3.7.

According to the induction hypothesis, we assume that the sequence

$$S(D_i - e_{i,n}, D'_i - e_{i,n}) \ = \ S(D_{i-1}, D'_{i-1}) \ = \ D_{i-1}^{(0)} \to \ldots \to D_{i-1}^{(l)}$$

Computation of a sequence of mutations $S(D, D')$ from D to D'

If $n \leq 3$ then $S(D, D') := D$.

If $n > 3$ and $1 < i \leq n$ then

denote $S(D_{i-1}, D'_{i-1}) = D^{(0)}_{i-1} \to \ldots \to D^{(l)}_{i-1}$

Comment: it is a sequence of l mutations from $D_{i-1} = D^{(0)}_{i-1}$ to $D'_{i-1} = D^{(l)}_{i-1}$; if $i = 2$ then D_1 and D'_1 are drawings of the complete graph with $n-1$ vertices, and the sequence is obtained by the computation for $n-1$ instead of n and for $i = n-1$; furthermore, this sequence has the property that, for any $0 \leq j \leq l-1$, the mutated triangle from $D^{(j)}_{i-1}$ to $D^{(j+1)}_{i-1}$ is contained in a permuted tame triangle between $D^{(j)}_{i-1}$ and D'_{i-1}, and hence this triangle does not contain a_n (by Proposition 3.3).

let $D^{(0)}_i := D_i$, let $S := D^{(0)}_i$

Comment: we build $S(D_i, D'_i)$ as a variable S that is extended step by step from D_i.

for j from 0 to $l-1$ do

if the triangle T mutated in $D^{(j)}_{i-1}$ is free in $D^{(j)}_i$ then

let $D^{(j+1)}_i$ be obtained by mutation of T from $D^{(j)}_i$

let $S := S \to D^{(j+1)}_i$

if the triangle T mutated in $D^{(j)}_{i-1}$ is cut by $e_{i,n}$ in $D^{(j)}_i$ then

let T' be the unique free triangle contained in T in $D^{(j)}_i$ (Cutting Lemma 2.1)

let $\overline{D}^{(j)}_i$ be obtained by mutation of T' from $D^{(j)}_i$ (Figure 20)

let $D^{(j+1)}_i$ be obtained by mutation of T from $\overline{D}^{(j)}_i$

let $S := S \to \overline{D}^{(j)}_i \to D^{(j+1)}_i$

Comment: at this step, we have $S = D^{(0)}_i \to \ldots \to D^{(l)}_i$ (with a length possibly greater than $l+1$) and we have $D^{(l)}_{i-1} = D'_{i-1}$, that is $D^{(l)}_i$ and D'_i are equal except for before relations involving $e_{i,n}$.

let $D^{(l)}_i \to \ldots \to D^{(m)}_i$ be a sequence of mutations from $D^{(l)}_i$ to $D'_i = D^{(m)}_i$ (Sweeping Proposition 3.8)

let $S(D_i, D'_i) := S \to D^{(l+1)}_i \to \ldots \to D^{(m)}_i$

Algorithm 1: Inductive algorithm for Theorem 3.10, using Notation 3.7.

has been built, and that every mutated triangle used along the sequence is contained in a tame triangle which is permuted between the sketch to which the mutation is applied and the final sketch D'. Observe that if $i = 2$ then D_1 and D'_1 are drawings of the complete graph on $n-1$ vertices $a_1, \ldots, a_{n-1}$. In this case, the sequence has been obtained by the construction applied to the parameter $n-1$ instead of n and the parameter i with value $n-1$.

• *First stage.*

Let us assume that the above sequence is not the trivial one $S(D_{i-1}, D'_{i-1}) = D_{i-1}$. Let T be the first mutated triangle in this sequence. By induction hypothesis, T is contained in a tame permuted triangle between D_{i-1} and D' (which is of course also a triangle of D_i). So, by Proposition 3.3, T does not contain the vertex a_n. So, the triangle T is either a free tame triangle in D_i, or it is a tame triangle cut by $e_{i,n}$ in D_i.

In the first case, T is free in D_i. Then it can be mutated from D_i, in exactly the same way as it is mutated from D_{i-1}. Obviously, we preserve the property that T is contained in a tame permuted triangle between D_i and D'. And next, we can proceed with the same construction applied to the sequence $D^{(1)}_{i-1} \to \ldots \to D^{(l)}_{i-1}$.

In the second case, illustrated in Figure 20, the triangle T is cut by $e_{i,n}$ in D_i, and it does not contain a drawn vertex. Then, by Lemma 2.1, there exists a unique free triangle T' of D_i, contained in T, with $e_{i,n} \in T'$. Then T' can be mutated from $D_i = D^{(0)}_i$, yielding a drawing $\overline{D}^{(0)}_i$. Obviously, we have that T' is contained in a tame permuted triangle between D_i and D', since T' is contained in T which is contained in a tame permuted triangle between D_{i-1} and D'. Then T is free in the resulting drawing $\overline{D}_i^{(0)}$, and it can be mutated from the resulting drawing $\overline{D}_i^{(0)}$, in exactly the same way as it is mutated from D_{i-1}. Obviously, T is contained in a tame permuted triangle between $\overline{D}_i^{(0)}$ and D'. And next, we can proceed with the same construction applied to the sequence $D_{i-1}^{(1)} \to \ldots \to D_{i-1}^{(l)}$.

Successively applying these two cases yields a sequence S of mutations starting from $D_i = D_i^{(0)}$ and resulting in a drawing $D_i^{(l)}$ such that $D_i^{(l)} - e_{i,n} = D_{i-1}^{(l)} = D'_i - e_{i,n}$. It is also the sequence S resulting from Algorithm 1. Note that if we have $S(D_{i-1}, D'_{i-1}) = D_{i-1}$ above, then we directly consider the next stage.

• *Second stage.*

As seen above, all the triangles not containing $e_{i,n}$ and involved in mutations in S are exactly the same as in the sequence $S(D_{i-1}, D'_{i-1})$. We effectively have $D_{i-1}^{(l)} = D'_{i-1}$, where $D_{i-1}^{(l)}$ is obtained by removing $e_{i,n}$ from $D_i^{(l)}$. Hence $D_i^{(l)}$ and D'_i are the same drawings except for the before relations involving $e_{i,n}$.

So now, we can proceed with building the second part of the sequence $S(D, D')$ by using the sweeping process of Proposition 3.8. We obtain a sequence from $D_i^{(l)}$ to D'_i using only mutations of triangles containing $e_{i,n}$, and permuted between $D_i^{(l)}$ and D'_i. Moreover, each triangle permuted between $D_i^{(l)}$ and D'_i is used exactly once in the sequence from $D_i^{(l)}$ to D'_i, hence each mutated triangle is tame (by definition of a mutation) and permuted between the drawing to which the mutation is applied and D'_i, or equivalently D'.

So, finally, we have built a sequence of mutations from D_i to D'_i. Furthermore, we have checked that every mutated triangle is contained in a tame permuted triangle between the drawing to which the mutation is applied and D'. Also, we have checked that Algorithm 1 performs the construction.

Finally, the equivalence between Algorithm 1 and the recursive Algorithm 2 is easy to check using the above discussion. Obviously, at each level i of the construction, the first part of the sequence built in Algorithm 1 is addressed in Algorithm 2 when $T \neq \emptyset$, and the second part of the sequence built in Algorithm 1 is addressed in Algorithm 2 when $T = \emptyset$. We leave the details. ☐

Corollary 3.11. *Let D and D' be two complete graph drawings in the plane with the same corner and subsketch. We have $D \neq D'$ if and only if there exists a tame permuted triangle between D and D'. More precisely, for every triple of edges e, f, g, if f is before g along e in D but not in D', then the triangle $[[e, f, g]]$ is tame and permuted between D and D'.*

Proof. If $D \neq D'$, then the first triangle in a sequence of mutations from D to D' is contained in a tame permuted triangle by Theorem 3.10. If f is before g along e in D but not in D', then the triangle $[[e, f, g]]$ has to be mutated at some step in the sequence of mutations from D to D' (it is the only way to change the before relation of f and g along e). Hence $[[e, f, g]]$ has to be tame and permuted between D and D'. ☐

Corollary 3.12 (completing Lemmas 3.1 and 3.2)**.** *Let D be a complete graph drawing in the plane. The subsketch and a corner of D determine, for every triplet of edges $\{e, f, g\}$:*

- *the before relation of f and g along e, when f and g do not cross each other;*
- *if $[[e, f, g]]$ is a wild triangle, and, if so, the three before relations involving e, f and g;*
- *if the surface of $[[e, f, g]]$ contains a drawn vertex, and, if so, the three before relations involving e, f and g.*

Proof. A sequence of mutations from D to D' affects only tame triangles containing no vertex of the graph. Mutations do not affect the fact that f and g have a common endpoint, nor that f and g do not cross each other, nor that $[[e, f, g]]$ is a wild triangle, nor that the surface of $[[e, f, g]]$ contains a drawn vertex. Hence the involved *before* relations must be the same in the two drawings, that is determined by the subsketch. ☐

Corollary 3.13. *Two complete graph drawings in the sphere with the same subsketch are the same up to homeomorphism and a sequence of triangle mutations.* ☐

Remark 3.14. Let us observe that the trick of the proof is to maintain the particular property of the sequence stated in the theorem, so that Proposition 3.3 ensures that the sequence of mutations can be used again when a new vertex is added. Beware that it is not possible to demand that the sequence uses only permuted triangles (which would imply by the way a minimum number of mutations, since every permuted triangle has to be mutated at some step). Indeed two drawings may have no free permuted triangle between each other. See Section 4.1 and Figure 21.

4 Examples and applications

4.1 Triangle mutations in pseudoline arrangements

A *pseudoline arrangement* may be defined as a finite set of curves in the affine plane, each one being the image of a line under an homeomorphism of the plane, and such that any two pseudolines cross each other exactly once. We will always consider *uniform* pseudoline arrangements, i. e. no three pseudolines can meet at the same point. We consider that a pseudoline arrangement is labelled and given with the *circular ordering* of the pseudolines at infinity, and is defined up to an orientation-preserving homeomorphism. Pseudoline arrangements (equivalent to rank 3 oriented matroids) are well studied objects, see [10, Chapter 4]. They satisfy simple axiomatics with the *before* relation [10], and even first order axiomatics [13].

Here, a pseudoline arrangement can be considered as a structure similar to a sketch in which: the *inc* and *circ* relations are not useful; the crossing relation is trivial (each element crosses each other element once); and the *before* relations determine the arrangement (each pseudoline is directed, the arrangement is determined by the linear ordering of the crossings on each pseudoline). Hence all definitions about triangles and mutations can be given in exactly the same way in pseudoline arrangements.

So, the previous result and algorithm apply naturally. For an arrangement A on $E = \{e_1, \ldots, e_n\}$, we denote by A_k, $1 \leq k \leq n$, the arrangement on $E_k = \{e_1, \ldots, e_k\}$ obtained by restriction from A, and we replace D_i with A_i and $e_{i,n}$ with e_i. Theorem 3.10 yields a sequence of mutations from any arrangement A to any other arrangement A' with the same number of pseudolines and the same circular ordering at infinity.

The well-known Ringel's theorem on pseudoline arrangements [25] states that if A and A' are two uniform pseudoline arrangements with the same number of elements and the same circular ordering at infinity then there exists a sequence of mutations from A to A'. In Theorem 3.10, we generalize Ringel's theorem to complete graph drawings. Furthermore, Theorem 3.10 also gives the following slight strengthening of Ringel's theorem (since in the generalization to graph drawings we avoid mutations of the non-permuted triangles that could contain drawn vertices, see Remark 3.14). This result, as already mentioned in [15], can be seen as an easy case of the construction of Theorem 3.10: the first stage is trivial, and the second stage consists of a sweeping, which is folklore in pseudoline arrangements. Let us mention that a similar inductive construction for a sequence of mutations in a pseudoline arrangement was used in [26], and that the following result can also be derived from [23] where regions of pseudoline arrangements are labelled and common labels are preserved in a sequence of mutations from an arrangement to another one.

Theorem 4.1. *If A and A' are two uniform pseudoline arrangements with the same number of elements and the same circular ordering at infinity then there exists a sequence of mutations from A to A' avoiding mutations of triangles not contained in a permuted triangle between A and A'.*

An important remark is that it is not possible in general to transform a configuration into another one using only mutations of permuted free triangles. Indeed, it would mean that there is always a permuted free triangle between two different configurations, which is false, as shown by an example from [14], from which Figure 21 is taken and made straight. Let us mention that this example was relevant also in [4], and that one of these two arrangements was also a significant example for another problem in [10, Figure 1.11.2].

A sequence of mutations between these two arrangements is built in Example 4.2. It shows two pseudoline arrangements having all their free triangles (123, 145, 356 and 246) in the same position. Then a sequence of mutations from one to the other must begin with the mutation of a non-permuted triangle. Hence the minimal number of mutations needed in the sequence may be strictly larger than the number of permuted triangles. For instance in the below sequence, we use twice the mutation of 356.

To our knowledge, the problem of building a minimal sequence of mutations between two pseudoline arrangements (or complete graph drawings) is open.

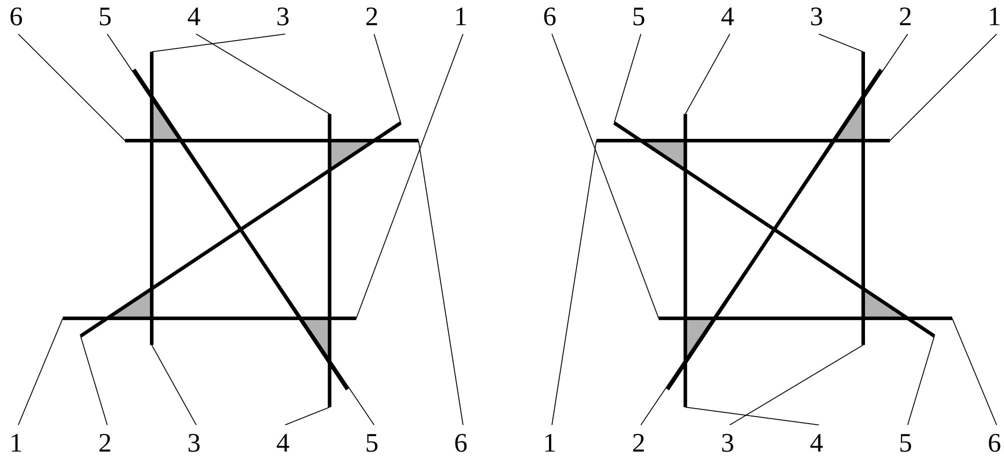


Figure 21: Two arrangements with no permuted free triangle (using an example from [14]).

Example 4.2. The sequences of triangles built by the algorithm of Theorem 3.10 applied to the arrangements of Figure 21 are the following. We separate the two built subsequences: the first one (corresponding to $D_i = D_i^{(0)} \to \ldots \to D_i^{(l)}$ in Theorem 3.10) built from the previous level, and the second one when only the last pseudoline has to be moved (corresponding to $D_i^{(l)} \to \ldots \to D_i^{(m)} = D'_i$ in Theorem 3.10).

- At level 3: $\emptyset$ (triangles 123 are the same in both arrangements)
- At level 4: $(\emptyset) \to (234 \to 134 \to 124)$ (only 4 has to be moved)
- At level 5: $(235 \to 234 \to 135 \to 134 \to 125 \to 124) \to (\emptyset)$ (the first sequence is sufficient)
- At level 6: $(\mathbf{356} \to 235 \to 346 \to 234 \to 135 \to 134 \to 125 \to 124)$
$\to (236 \to 136 \to 126 \to 146 \to 156 \to 456 \to 256 \to \mathbf{356})$

4.2 Plane visualization of geometric spatial complete graphs given with oriented matroid data

Consider a set E of $n+1$ points in the 3-dimensional real (or rational) space in general position, a plane in general position with this configuration, and $a \in E$ the extremal point in E with respect to the plane (i. e. the distance from a to the plane is maximal). Then the projections, from a to the plane, of the segments formed by all pairs of vertices is a complete (geometric) graph drawing of K_n in the plane (see Figure 22).

Theorem 4.3. *The rank 4 oriented matroid defined by E determines a corner and the subsketch of the complete graph drawing in the plane obtained by projection from the extremal point $a \in E$. Hence it determines the drawing up to a sequence of triangle mutations.*

Proof. With the oriented matroid, we know for each triplet in E, and for each pair of other points, if these two points are on the same side or the opposite sides w.r.t. the plane spanned by the triplet. In oriented matroid terms: we know the relative signs of elements in a cocircuit defined by the triplet. Since all edges are straight, then we easily get:

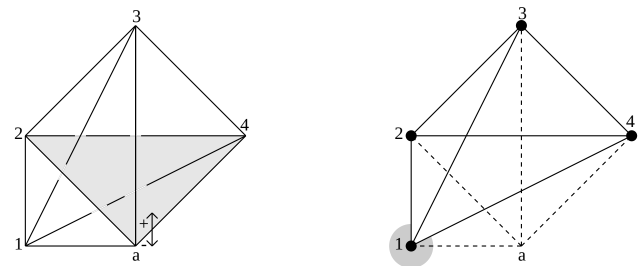


Figure 22: Plane projection (on the right) of a geometric spatial complete graph (on the left).

- the *cross* relations: for vertices e_1, e_2, f_1, f_2, the drawn edge $[e_1, e_2]$ crosses the drawn edge $[f_1, f_2]$ if and only if e_1 and e_2 are in opposite sides w.r.t. the plane defined by a, f_1, f_2, and f_1 and f_2 are in opposite sides w.r.t. the plane defined by a, e_1, e_2.
- the *circ* relations: for vertices x, y, z, the edges $[x, y]$ and $[x, z]$ are consecutive in the circular ordering around x if and only if no vertex is at the same time in the same side as y w.r.t. the plane defined by a, x, z and in the same side as z w.r.t. the plane defined by a, x, y.
- a corner of the drawing: it is given by a vertex P and two edges $[P, x]$ and $[P, y]$ such that $(P, [P, x], [P, y]) \in$ *circ*, and all vertices not in $\{a, P, x\}$ are in the same side w.r.t. the plane defined by a, P, x, and all vertices not in $\{a, P, y\}$ are in the same side w.r.t. the plane defined by a, P, y.

Hence, the subsketch is determined (but not all the drawing: the linear orderings of crossings along edges of triangles are not determined in general by the oriented matroid). We finish the proof using Theorem 3.10. □

With Theorem 4.3, we know that if two configurations of points define the same oriented matroid up to a bijection of the ground set, then the projections of the segments joining them, from extremal points being in bijection, are the same up to a sequence of triangle mutations and orientation-preserving homeomorphisms.

Hence, finally, we obtain a certain modelling of plane visualization of geometric spatial complete graphs, based on two structural levels. Indeed, the point a plays the role of a point of view for the visualization. When a moves in a region delimited by the planes formed by other points of the configuration, then the oriented matroid data, and hence the subsketch, are unchanged, while the plane drawing and its sketch can change with a sequence of triangle mutations. When a crosses a plane, then the oriented matroid data changes (a sign changes in some cocircuits).

Let us mention that using a non-extremal point a and a projection onto the sphere at infinity from a yields a complete graph drawing in the sphere. Theorem 4.3 can be extended to this case, not taking into account the corner, and consistently adapting Theorem 3.10 for drawings in the sphere. We leave the details.

Let us also point out that the obtained result is not trivial since it is impossible in general to transform a point configuration into another one with the same oriented matroid structure, by an isotopy of the space, while preserving the oriented matroid structure along the transformation (which would have been, if true, an immediate way to build the required sequence of mutations). This fact is known in oriented matroid theory as the *Universality Theorem of Mnëv* [21][10], stating that realization spaces of oriented matroids are not necessarily connected, and in fact are in some sense equivalent to semi-algebraic varieties.

5 Logical reductions for complete graph drawings

This section is independent from previous results. We show that the crossing relations and a corner of a complete graph drawing are sufficient to determine its map, subsketch, and further information on the drawing, using explicit first order logic formulas.

By this way, the subsketch plus the corner of a complete graph drawing can be reduced to the crossing relation only plus the corner. In [24, Section 5], another reduction of the subsketch was given: it was shown that the map of a complete graph drawing determines the crossing relations, and hence the subsketch (one just has to check this property for K_4). Both reductions (to the map or to the crossing relations) are also proved in [18, Proposition 6]. Let us mention that other information could be used to derive the subsketch, for instance, as noticed also in [24, Section 5], order type on the vertices of K_n (i.e. the orientation of its triples) determines the set of crossing pairs of edges.

Obviously, thanks to these reductions, the assumptions in Theorem 3.10 can be also reduced.

As regards logic formulas, let us recall that first order logic only allows individual quantifications over variables, which are here the vertices and edges of the graph (but not over sets of variables, as in monadic second order logic, nor over relations, as in second order logic, that is, for instance, logic formulas can use "for any $a \in V$", but not "for any $A \subseteq V$" nor "for any ordering of V"). Broadly speaking, in the theoretical computer science field that studies the logic of graphs, the fact that a certain graph property or construction can be expressed in one type of logic or another can have impacts in terms of computational complexity, model checking, satisfiability, data compression...

For the construction of the next theorem, we consider an intermediate piece of information given by the cyclic ordering of three edges e, f, g with a common endpoint x around this endpoint. For a formal convenience, we encode it by a relation called $between_D \subseteq V_G \times E_G \times E_G \times E_G$ and defined by: $(x, e, f, g) \in between_D$ if the edges e, f, g all have endpoint x, and f is between e and g in the circular order of the edges around x (note that the order is essential in the sentence: f is not between g and e; and note that $(x, e, f, g) \in between_D$ is equivalent to $(x, f, g, e) \in between_D$ and to $(x, g, e, f) \in between_D$). See Figure 23.

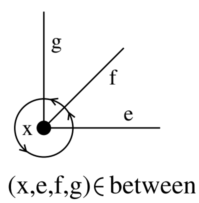


Figure 23: Between relation.

Theorem 5.1. *The map, the subsketch, the circ relations, the between relations, the inside relations, and the dcross relations of a complete graph drawing in the plane are determined, through first order logic formulas, by its cross relations and a corner.*

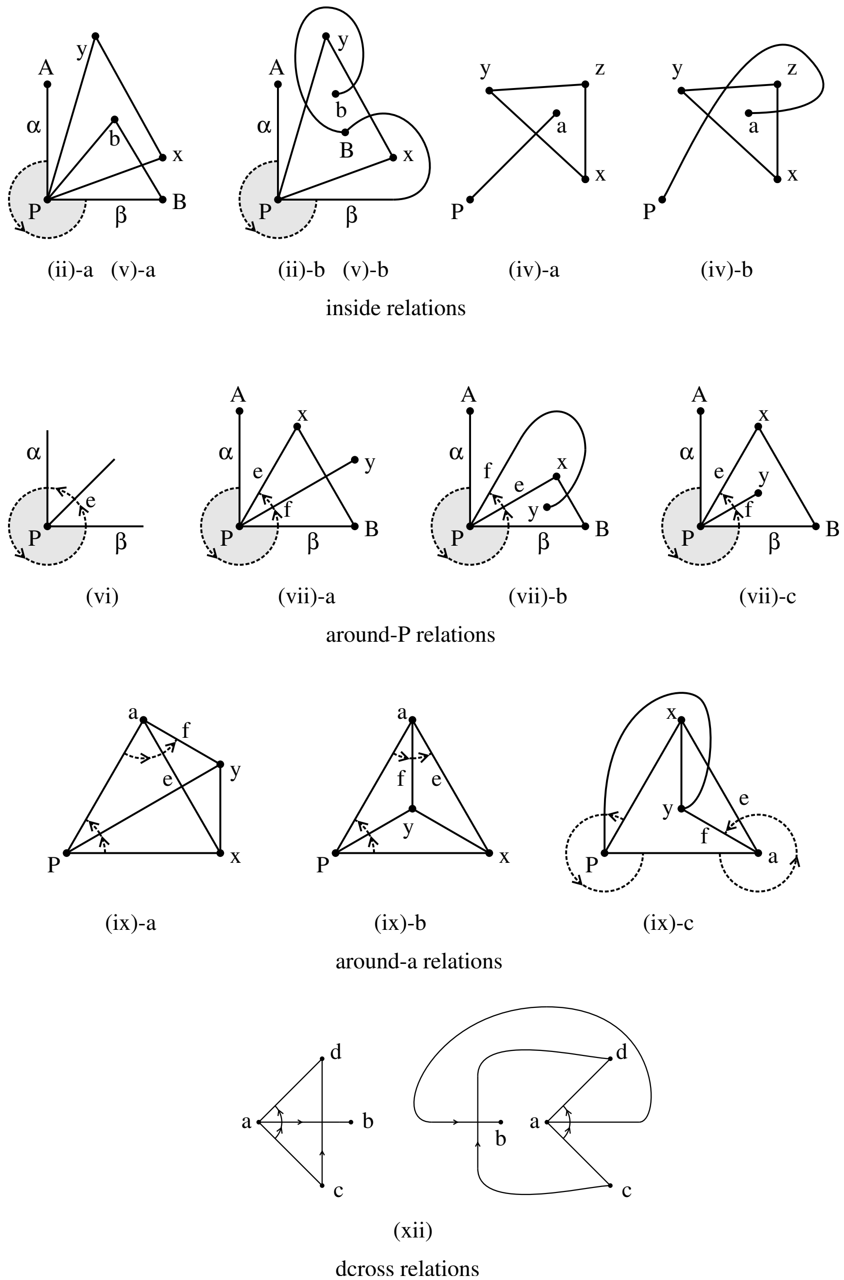


Figure 24: Proof of Theorem 5.1.

Proof. The construction is step by step and extensively uses the topological definition of the corner, along with axioms (D1), (D2), and (D3) of a topological drawing. The ordering of the steps is important. Illustrations are given in Figure 24. Let D be a complete graph drawing, with corner (P, α, β). The vertex P is an endpoint of α and β, and the other endpoints of α and β are denoted by A and B, respectively.

- *Inside relations.*

(i) *For any $x, y, z \in V_G$, we have: P is not inside $[x, y, z]$.* This is direct by definition of the corner (P, α, β), otherwise P would be contained in a bounded region. Note that A or B may be inside $[x, y, z]$.

(ii) *For any $x, y \in V_G$, we have: A is inside $[P, x, y]$ if and only if $[P, A]$ crosses $[x, y]$, and B is* *inside* $[P,x,y]$ *if and only if* $[P,B]$ *crosses* $[x,y]$. First, if $[P,B]$ does not cross $[x,y]$ then B is not inside $[P,x,y]$, otherwise $B \notin \{x,y\}$ and the drawn edge α is contained in the bounded region $[P,x,y]$, which is a contradiction with the definition of the corner (P,α,β), see Figure 24 (ii)-a. Conversely if $[P,B]$ crosses $[x,y]$, then, by definition of a corner, the part of $[P,B]$ between P and the intersection point between $[P,B]$ and $[x,y]$ is contained in the boundary of the unbounded region of the plane delimited by the drawing. So B is inside $[P,x,y]$, see Figure 24 (ii)-b. The same reasoning holds for A.

(iii) *For any* $a \in V_G$*, we have:* a *is inside* $[P,A,B]$ *if and only if* $[P,a]$ *does not cross* $[A,B]$. If a is inside $[P,A,B]$ then $[P,a]$ does not cross $[A,B]$ otherwise $[P,A]$ or $[P,B]$ (respectively), would be contained in a bounded region of the plane delimited by $[P,a]$, $[A,B]$, and $[P,B]$ or $[P,A]$ (respectively). Conversely, since $[P,a]$ does not cross $[P,A]$ nor $[P,B]$, if $[P,a]$ does not cross $[A,B]$ and a is not inside $[P,A,B]$ then either $[P,a,B]$ contains $[PA]$, which is forbidden, or $[P,B]$ is contained in the region delimited by $[P,a]$,$[a,B]$, $[B,A]$, and $[A,P]$, which is forbidden too.

(iv) *For* $a,x,y,z \in V_G - \{P\}$*, we have:* a *is inside* $[x,y,z]$ *if and only if* $[P,a]$ *crosses one or three edges within* $[x,y]$*,* $[x,z]$*,* $[y,z]$. With (i) the vertex P is not inside $[x,y,z]$. And, by axiom (D2) of a drawing, two vertices are in the same region delimited by three other vertices if and only if the edge between them meet the sides of the region an even number of times. For example, in Figure 24 (iv)-a, $[P,a]$ crosses $[x,y]$ and only $[x,y]$, and on Figure 24 (iv)-b, $[P,a]$ crosses $[x,y]$, $[x,z]$ and $[y,z]$.

(v)-a. *For* $b,x,y \in V_G - \{P,B\}$*, we have: if* B *is not inside* $[P,x,y]$ *then* b *is inside* $[P,x,y]$ *if and only if* $[B,b]$ *crosses one or three edges within* $[P,x]$*,* $[P,y]$*,* $[x,y]$. This is obvious for the same reason as in (iv), see Figure 24 (v)-a. The other case to handle is analogous by replacing B with A.

(v)-b. *Under the same hypothesis as above, we have: if* B *is inside* $[P,x,y]$ *then* b *is inside* $[P,x,y]$ *if and only if* $[B,b]$ *crosses zero or two edges within* $[P,x]$*,* $[P,y]$*,* $[x,y]$. This is obvious for the same reason as in (iv), see Figure 24 (v)-b. The other case to handle is analogous by replacing B with A.

- *Between relations around* P*.*

(vi) *For all* $e \in E_G$ *with endpoint* P*, we have:* $(P,\beta,e,\alpha) \in between_D$. This is direct by definition of the corner (P,α,β). See Figure 24 (vi).

(vii) For two edges e and f with endpoint P distinct from β, let us consider the expression of the relations $(P,\beta,e,f) \in between_D$ or $(P,\beta,f,e) \in between_D$. We denote by x and y the other endpoint of e and f, respectively. We directly have the following.

(vii)-a. *If* f *crosses* $[x,B]$ *and* y *is not inside* $[P,x,B]$ *then* f *is between* β *and* e. See Figure 24 (vii)-a. *Similarly if* e *crosses* $[y,B]$ *and* x *is not inside* $[P,y,B]$ *then* e *is between* β *and* f.

(vii)-b. *If* f *crosses* $[x,B]$ *and* y *is inside* $[P,x,B]$ *then* e *is between* β *and* f. See Figure 24 (vii)-b, since β cannot be between f and e by definition of the corner. *Similarly if* e *crosses* $[y,B]$ *and* x *is inside* $[P,y,B]$ *then* f *is between* β *and* e.

(vii)-c. *If* f *does not cross* $[x,B]$ *and* e *does not cross* $[y,B]$*, then we have either* x *inside* $[P,B,y]$ *and then* e *between* β *and* f*, or* y *inside* $[P,B,x]$ *and then* f *between* β *and* e. See Figure 24 (vii)-c.

(viii) At last, the between relations involving $\{e,f,\beta\}$ around the vertex P, for any edges $e,f \in E_G$, induce immediately the between relations around P for any edges $e,f,g \in E_G$ with endpoint P since we have: $between(P,e,f,g) = \big(between(P,\beta,e,f) \wedge between(P,\beta,f,g)\big) \vee \big(between(P,\beta,g,e) \wedge between(P,\beta,e,f)\big) \vee \big(between(P,\beta,f,g) \wedge between(P,\beta,g,e)\big)$, where $between(x,y,z,t)$ means $(x,y,z,t) \in between$.

- *Between relations around* $a \in V_G \setminus P$*.*

(ix) Let $a \in V_G - \{P\}$ and $e,f \in E_G$ having endpoint a, with x and y the other endpoints of e and f, respectively. Since the relations *between* around P are known from (viii), up to exchange of e and f, we assume that $[P,y]$ is between $[P,x]$ and $[P,a]$ around P. Let us check that the relations *between* around a in the subdrawing of the complete graph formed by $\{x,y,a,P\}$ are directly determined by the *inside* and *cross* relations restricted to this subdrawing. Precisely, we have the following cases (note that P cannot be inside $[a,x,y]$ since it is the vertex at the corner).

(ix)-a. *If* y *is not inside* $[P,a,x]$ *and* $[P,y]$ *crosses* $[a,x]$ *then the drawing is given (up to an orientation-preserving homeomorphism) by Figure 24 (ix)-a, and* e *is between* $[a,P]$ *and* f *around* a*. Similarly, if* x *is not inside* $[P,a,y]$ *and* $[P,x]$ *and* $[a,y]$ *cross then* e *is between* $[a,P]$ *and* f *around* a*. And if* a *is not inside* $[P,x,y]$ *and* $[P,a]$ *and* $[x,y]$ *cross then* f *is between* $[a,P]$ *and* e *around* a*.*

(ix)-b. *If* y *is inside* $[P,a,x]$ *and* $[P,y]$ *does not cross* $[a,x]$*, then the sub-drawing is given (up to an orientation-preserving homeomorphism) by Figure 24 (ix)-b, and* f *is between* $[a,P]$ *and* e *around* a*. Similarly, if* x *is inside* $[P,a,y]$ *and* $[P,x]$ *does not cross* $[a,y]$*, then* e *is between* f *and* $[a,P]$ *around* a*. Similarly, if* a *is inside* $[P,x,y]$ *and* $[P,a]$ *does not cross* $[x,y]$ *then* e *is between* f *and* $[a,P]$ *around* a*.*

(ix)-c. *If* y *is inside* $[P,a,x]$ *and* $[P,y]$ *crosses* $[a,x]$ *then the sub-drawing is given (up to an orientation-preserving homeomorphism) by Figure 24 (ix)-c, and* e *is between* $[a,P]$ *and* f *around* a*. Similarly if* x *is inside* $[P,a,y]$ *and* $[P,x]$ *crosses* $[a,y]$ *then* e *is between* $[a,P]$ *and* f *around* a*. And if* a *is inside* $[P,a,y]$ *and* $[P,x]$ *crosses* $[a,y]$ *then* f *is between* $[a,P]$ *and* e *around* a*.*

(x) The between relations, involving $\{e,f,[a,P]\}$ around a given vertex a, for any edges $e,f \in E_G$, induce immediately the between relations around a for any edges $\{e,f,g\}$ just as in (viii).

- *Circular ordering relations around vertices (completing the map and subsketch).*

(xi) Finally, all required *between* and *inside* relations have been expressed from the crossing, graph and corner. At each step the formula is first order, using formulas built at previous steps. We now check that the *circ* relations (and hence the map and subsketch) are expressed from the *between* relations by a first order formula. It is true since $(x,e,f) \in circ$ if and only if, for every edge g with endpoint x, $(x,e,f,g) \in between$.

- *Directed crossing relations*

(xii) Since the *between* relations are determined, we immediately get the *dcross* relations by using the restrictions to 4 vertices subdrawings. Indeed: *if* $[a,b]$ *crosses* $[c,d]$ *with* $(a,b) \in \overrightarrow{E}_G$*,* $(c,d) \in \overrightarrow{E}_G$*,* $(a,[a,c],[a,b],[a,d]) \in between_D$ *then we have* $((a,b),(c,d)) \in dcross_D$*.* See Figure 24 (xii). □

Remark 5.2. It was claimed in [15] that relations addressed in Corollary 3.12 could be checked in first order logic from the subsketch and corner. Corollary 3.12 proves that they are effectively determined by the subsketch and corner. Then, the use of first order logic is possible by considering all possible drawings of K_n for $n \leq 7$. We leave the details.

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