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A note on deterministic zombies*

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Abstract

Zombies and Survivor is a variant of the well-studied game of Cops and Robbers where the zombies (cops) can only move closer to the survivor (robber). We consider the deterministic version of the game where a zombie can choose their path if multiple options are available. Similar to the cop number, the zombie number of a graph is the minimum number of zombies required to capture the survivor. In this short note, we solve a question by Fitzpatrick et al., proving that the zombie number of the Cartesian product of two graphs is at most the sum of their zombie numbers. We also give a simple graph family with cop number 2 and an arbitrarily large zombie number.

1 Introduction

Cops and Robbers is a pursuit-evasion two-player game: the *cops* and the *robber*. At the initial round each cop choose a starting vertex, then the robber choose its starting vertex; then at each round, each cop choose to move to an adjacent vertex, or to stay on its current vertex, then the robber has a similar choice. The cops win if after a finite number of rounds, a cop occupies the same vertex as the robber; and vice-versa, the robber wins if he can indefinitely avoid the cops. The *cop number* of a graph G , which is denoted by $c(G)$, is the minimum number of cops needed to guarantee that they have a winning strategy. Albeit being very simple, this game is related to fundamental questions regarding the structural properties of graphs (see [Bon12], [BP17] and [Bon11] for a survey, and additional background on this game).

In this paper, we consider a new variant of this game, namely *Zombies and Survivors*, defined in [FHMP16] as follows: zombies take the place of the cops and survivors take the place of the robber. The zombies, being of limited intelligence, have a very simple objective in each round – to move closer to a survivor. Therefore, each zombie must move along some shortest path, or geodesic, joining itself and a nearest survivor. We say that the zombies capture a survivor if one of the zombies moves onto the same vertex as a survivor. In this version, zombies may have a choice as to which shortest path to follow, if there are multiple ones. A

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different version of the game involves randomness in the choice of the shortest path. We refer the interested reader to [BMPGP16, Pra19] and do not consider the topic further.

Following [FHMP16], we only consider the case of a unique survivor in the graph, and assume all graphs throughout the paper to be connected. Similarly to cops and robbers, the *zombie number* of a graph G is the minimum number of zombies needed to ensure that the survivor will be eventually captured, and is denoted by $z(G)$.

We focus on the following two questions:

Question 1.1 (Question 10 in [FHMP16]). Is $z(G \square H) \leq z(G) + z(H)$ for all graphs G and H ?

Question 1.2 (Question 19 in [FHMP16]). Over all graphs G , how large can the ratio $\frac{z(G)}{c(G)}$ be?

Here, we answer Question 1.1 in the affirmative, improving upon Theorems 11, 13 and 14 in [FHMP16]. By noting that $z(Q_3) = 2$, we also obtain immediately that $z(Q_n) = \lceil \frac{2n}{3} \rceil$. This was the object of Conjecture 18 in [FHMP16], though it was since solved independently in [OO19] and [Fit18].

Theorem 1.3. *For all graphs G and H , we have $z(G \square H) \leq z(G) + z(H)$.*

We also argue that the ratio in Question 1.2 can be arbitrarily large. Note that a union G of cycles all sharing a vertex satisfies trivially $c(G) \leq 2$.

Theorem 1.4. *For every integer k , there is a graph G_k that is a union of cycles sharing a vertex such that $z(G_k) \geq k$.*

This was already argued in [OO19], but our construction and proof are arguably simpler. Additionally, the graphs we present are outerplanar graphs, and in fact cacti. Informally, this gives little hope for Question 1.2 to have a bounded answer in a meaningful graph class.

We prove Theorem 1.3 in Section 2, Theorem 1.4 in Section 3, and conclude in Section 4 with some open problems which seem of interest to us.

2 Cartesian products of graphs

Proof of Theorem 1.3. Given a vertex $u \in G \square H$, we denote its coordinates in G and H as (u_G, u_H) . Given two vertices $u, v \in G \square H$, we denote $d_G(u, v) = d_G(u_G, v_G)$ (respectively $d_H(u, v) = d_H(u_H, v_H)$) the distance between u and v in the projection of $G \square H$ on G (respectively H). A *copy* of G (respectively H) is the subgraph induced in $G \square H$ by all vertices u with $u_H = w$ (respectively $u_G = x$) where w is some vertex in H (respectively x is some vertex in G). Let \mathcal{S}_G be an optimal strategy for $z(G)$ zombies in G , and \mathcal{S}_H be an optimal strategy for $z(H)$ zombies in H . Throughout the proof, we denote by s the vertex occupied by the survivor.

We are now ready to describe a winning strategy (for zombies) involving $z(G) + z(H)$ zombies. We will distinguish two types of zombies: a set B of $z(G)$ blue zombies, which are placed according to \mathcal{S}_G in some copy of G , and a set R of $z(H)$ red zombies, which are placed according to \mathcal{S}_H in some copy of H . Note that for every $x, y \in B$, we have $d_H(x, s) = d_H(y, s)$. We maintain that property step after step, and denote the corresponding value d_H . Similarly, for every $x, y \in R$, we have $d_G(x, s) = d_G(y, s)$: we denote that value d_G .

The set B applies the following strategy: as long as d_H is positive, all the zombies in B move towards s in H (choosing to keep the same coordinate in G). Note that this is a valid move, as there is a shortest path to s going through the corresponding vertex. Once $d_H = 0$, all zombies in B either follow \mathcal{S}_G (if s_H is unchanged) or move toward s in H to remain in

the same copy of G as s (if s_H changed). Note that either way, we maintain $d_H = 0$. The set R applies the same strategy, symmetrically with H and G instead of G and H .

We observe that neither d_H nor d_G increases. Additionally, at every step, either s_H or s_G is unchanged. Assume s_H is unchanged. Then d_H , if positive, decreases. If $d_H = 0$, then all zombies in B follow \mathcal{S}_G . Since $d_H = 0$ for the rest of the game, B is one step closer to catching the survivor. Meanwhile, if s_H is changed, then d_H does not change, and B is not further away from capturing the survivor according to \mathcal{S}_G . Since the winning strategy \mathcal{S}_G terminates in a finite number of steps, and the same analysis holds for \mathcal{S}_H , the process for $G \square H$ terminates and the survivor is eventually captured and eaten.

□

3 Simple graphs with large ratio

Proof of Theorem 1.4. For $k \in \mathbb{N}^*$, let G_k be the graph obtained by taking k disjoint copies of $C_5, C_{13}, \dots, C_{2^{k+2}-3}$, for a total of k^2 cycles, and merging all of them on one vertex u (see Figure 1). Note that $|V(G_k)| \sim k \cdot 2^{k+3}$ as $k \rightarrow \infty$. We will argue that $z(G_k) \geq k$. We define a direction for all cycles, which we will refer to as *clockwise*.

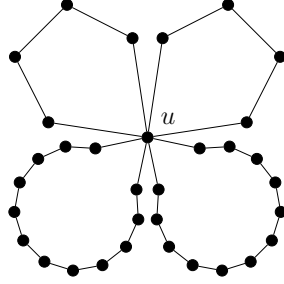


Figure 1: The graph G_2

Assume for a contradiction that $z(G_k) \leq k - 1$, and consider a starting position for $k - 1$ zombies in G_k . Since there are k copies of cycles $C_5, C_{13}, \dots, C_{2^{k+2}-3}$, and only $k - 1$ zombies, by the pigeon-hole principle there is one copy which contains no zombie except possibly on the vertex u . We will focus on u and the vertices of that copy, and ignore from now on the rest of the graph. The goal, perhaps somewhat counter-intuitively, is to gather zombies closely behind the survivor, so that eventually the survivor can safely circle around the cycle of length $2^{k+2} - 3$ forever without encountering any zombie. *Circling around* a cycle means walking around the cycle clockwise until reaching u .

The survivor use the following winning strategy: For i from 1 to k , while the the i^{th} closest zombie is at distance at least $2^{i+2} - 1$, the survivor choose the second vertex in the cycle of length $2^{i+2} - 3$ if they have not chosen a starting point yet. Then the survivor circle around the cycle of length $2^{i+2} - 3$.

The strategy for the survivor is elementary. By circling around in an appropriate way, the survivor makes sure that at some point, the first i zombies are within distance $2^{i+2} - 2$ behind. Since there are only $k - 1$ zombies, this guarantees that circling around the cycle of length $2^{k+2} - 3$ is eventually safe and leads to a surviving strategy for the survivor. The only crucial property about the behaviour of zombies is that the distance between the survivor and a given zombie never increases. Note that since all cycles are odd, free will has in fact no impact for zombies.

1 In our algorithm, zombies are ranked by increasing distance to the survivor, with ties
2 broken arbitrarily. The k^{th} zombie, which does not exist, is considered to be at infinite distance.
3 When the survivor has not chosen a starting point yet, distance is considered as 2 more than
4 the distance to u . Note that u itself might not be a suitable starting point as there could be a
5 zombie on it.

6 To argue that the strategy is safe for the survivor, it suffices to point out that when the
7 survivor enters the cycle of length $2^{i+2} - 3$ (for some i), all zombies are either at distance at
8 most $2^{i+1} - 2$ or at least $2^{i+2} - 1$. In the first case, the shortest path to the survivors makes them
9 circle around the cycle clockwise (since $2^{i+1} - 2 < \frac{2^{i+2}-3}{2}$). In the second case, they do not
10 reach u before the survivor has finished circling around the cycle (since $2^{i+2} - 1 > 2^{i+2} - 3 + 1$).
11
12 \square

13 4 Conclusion

14 To conclude, we offer two open questions. While not of obvious depth, we believe that both
15 touch at the heart of what it means for a graph G to require $z(G)$ zombies. In particular, if
16 a survivor plays so as to survive for as long as possible, are all $z(G)$ zombies within short
17 distance at time of death?

18 *Question 4.1.* For every graph G , and for a graph G' obtained from G by successively adding
19 vertices of degree 1, does it always hold that $z(G') = z(G)$?

20 *Question 4.1* can be interpreted as: is there any advantage for zombies to individually wait
21 for some pre-announced time at the beginning of the game (and then activate and follow the
22 standard rules)?

23 *Question 4.2.* For any graph G , is there an integer k such that, for G'_k the graph obtained from
24 G by subdividing all edges k times then adding the original edges, $z(G'_k) \geq z(G) + 1$?

25 Concerning *Question 4.2*, it suffices to consider $G = C_4$ to observe that $k = 4$ is not enough,
26 however, we have no reason to think that k should be large.

27 **Addendum.** After submission of this manuscript, an independent proof of [Theorem 1.3](#) was
28 published [[KB21](#)].

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