A note on deterministic zombies

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Abstract

Zombies and Survivor is a variant of the well-studied game of Cops and Robbers where the zombies (cops) can only move closer to the survivor (robber). We consider the deterministic version of the game where a zombie can choose their path if multiple options are available. Similar to the cop number, the zombie number of a graph is the minimum number of zombies required to capture the survivor. In this short note, we solve a question by Fitzpatrick et al., proving that the zombie number of the Cartesian product of two graphs is at most the sum of their zombie numbers. We also give a simple graph family with cop number 2 and an arbitrarily large zombie number.

1 Introduction

Cops and Robbers is a pursuit-evasion two-player game: the *cops* and the *robber*. At the initial *round* each cop choose a starting vertex, then the robber choose its starting vertex; then at each round, each cop choose to move to an adjacent vertex, or to stay on its current vertex, then the robber has a similar choice. The cops win if after a finite number of rounds, a cop occupies the same vertex as the robber; and vice-versa, the robber wins if he can indefinitely avoid the cops. The *cop number* of a graph G, which is denoted by $c(G)$, is the minimum number of cops needed to guarantee that they have a winning strategy. Albeit being very simple, this game is related to fundamental questions regarding the structural properties of graphs (see [Bon12], [BP17] and [Bon11] for a survey, and additional background on this game).

In this paper, we consider a new variant of this game, namely Zombies and Survivors, defined in [FHMP16] as follows: zombies take the place of the cops and survivors take the place of the robber. The zombies, being of limited intelligence, have a very simple objective in each round – to move closer to a survivor. Therefore, each zombie must move along some shortest path, or geodesic, joining itself and a nearest survivor. We say that the zombies capture a survivor if one of the zombies moves onto the same vertex as a survivor. In this version, zombies may have a choice as to which shortest path to follow, if there are multiple ones. A

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different version of the game involves randomness in the choice of the shortest path. We refer the interested reader to [BMPGP16, Pra19] and do not consider the topic further.

Following [FHMP16], we only consider the case of a unique survivor in the graph, and assume all graphs throughout the paper to be connected. Similarly to cops and robbers, the *zombie number* of a graph G is the minimum number of zombies needed to ensure that the survivor will be eventually captured, and is denoted by $z(G)$.

We focus on the following two questions:

Question 1.1 (Question 10 in [FHMP16]). Is $z(G\Box H) \leqslant z(G) + z(H)$ for all graphs G and H?

Question 1.2 (Question 19 in [FHMP16]). Over all graphs G, how large can the ratio $\frac{z(G)}{c(G)}$ be?

Here, we answer Question 1.1 in the affirmative, improving upon Theorems 11, 13 and 14 in [FHMP16]. By noting that $z(Q_3) = 2$, we also obtain immediately that $z(Q_n) = \lceil \frac{2n}{3} \rceil$. This was the object of Conjecture 18 in [FHMP16], though it was since solved independently in [OO19] and [Fit18].

Theorem 1.3. *For all graphs G and H, we have $z(G\Box H) \leqslant z(G) + z(H)$.*

We also argue that the ratio in Question 1.2 can be arbitrarily large. Note that a union G of cycles all sharing a vertex satisfies trivially $c(G) \leqslant 2$.

Theorem 1.4. *For every integer k, there is a graph G_k that is a union of cycles sharing a vertex such that $z(G_k) \geqslant k$.*

This was already argued in [OO19], but our construction and proof are arguably simpler. Additionally, the graphs we present are outerplanar graphs, and in fact cacti. Informally, this gives little hope for Question 1.2 to have a bounded answer in a meaningful graph class.

We prove Theorem 1.3 in Section 2, Theorem 1.4 in Section 3, and conclude in Section 4 with some open problems which seem of interest to us.

2 Cartesian products of graphs

Proof of Theorem 1.3. Given a vertex $u \in G\Box H$, we denote its coordinates in G and H as (u_G, u_H). Given two vertices u, v in $G\Box H$, we denote $d_G(u,v) = d_G(u_G, v_G)$ (respectively $d_H(u,v) = d_H(u_H, v_H)$) the distance between u and v in the projection of $G\Box H$ on G (respectively H). A *copy* of G (respectively H) is the subgraph induced in $G\Box H$ by all vertices u with $u_H = w$ (respectively $u_G = x$) where w is some vertex in H (respectively x is some vertex in G). Let $\mathcal{S}_G$ be an optimal strategy for $z(G)$ zombies in G, and $\mathcal{S}_H$ be an optimal strategy for $z(H)$ zombies in H. Throughout the proof, we denote by s the vertex occupied by the survivor.

We are now ready to describe a winning strategy (for zombies) involving $z(G) + z(H)$ zombies. We will distinguish two types of zombies: a set B of $z(G)$ blue zombies, which are placed according to $\mathcal{S}_G$ in some copy of G, and a set R of $z(H)$ red zombies, which are placed according to $\mathcal{S}_H$ in some copy of H. Note that for every $x, y \in B$, we have $d_H(x,s) = d_H(y,s)$. We maintain that property step after step, and denote the corresponding value d_H. Similarly, for every $x, y \in R$, we have $d_G(x,s) = d_G(y,s)$: we denote that value d_G.

The set B applies the following strategy: as long as d_H is positive, all the zombies in B move towards s in H (choosing to keep the same coordinate in G). Note that this is a valid move, as there is a shortest path to s going through the corresponding vertex. Once $d_H = 0$, all zombies in B either follow $\mathcal{S}_G$ (if s_H is unchanged) or move toward s in H to remain in the same copy of G as s (if s_H changed). Note that either way, we maintain $d_H = 0$. The set R applies the same strategy, symmetrically with H and G instead of G and H.

We observe that neither d_H nor d_G increases. Additionally, at every step, either s_H or s_G is unchanged. Assume s_H is unchanged. Then d_H, if positive, decreases. If $d_H = 0$, then all zombies in B follow $\mathcal{S}_G$. Since $d_H = 0$ for the rest of the game, B is one step closer to catching the survivor. Meanwhile, if s_H is changed, then d_H does not change, and B is not further away from capturing the survivor according to $\mathcal{S}_G$. Since the winning strategy $\mathcal{S}_G$ terminates in a finite number of steps, and the same analysis holds for $\mathcal{S}_H$, the process for $G \square H$ terminates and the survivor is eventually captured and eaten.

□

3 Simple graphs with large ratio

Proof of Theorem 1.4. For $k \in \mathbb{N}^*$, let G_k be the graph obtained by taking k disjoint copies of $C_5, C_{13}, \ldots, C_{2^{k+2}-3}$, for a total of k^2 cycles, and merging all of them on one vertex u (see Figure 1). Note that $|V(G_k)| \sim k \cdot 2^{k+3}$ as $k \to \infty$. We will argue that $z(G_k) \geqslant k$. We define a direction for all cycles, which we will refer to as *clockwise*.

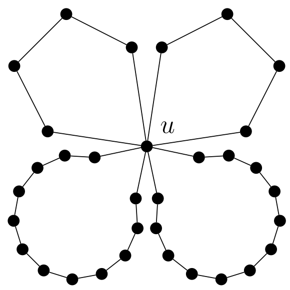

Figure 1: The graph G_2

Assume for a contradiction that $z(G_k) \leqslant k-1$, and consider a starting position for $k-1$ zombies in G_k. Since there are k copies of cycles $C_5, C_{13}, \ldots, C_{2^{k+2}-3}$, and only $k-1$ zombies, by the pigeon-hole principle there is one copy which contains no zombie except possibly on the vertex u. We will focus on u and the vertices of that copy, and ignore from now on the rest of the graph. The goal, perhaps somewhat counter-intuitively, is to gather zombies closely behind the survivor, so that eventually the survivor can safely circle around the cycle of length $2^{k+2}-3$ forever without encountering any zombie. *Circling around* a cycle means walking around the cycle clockwise until reaching u.

The survivor use the following winning strategy: For i from 1 to k, while the the i^{th} closest zombie is at distance at least $2^{i+2}-1$, the survivor choose the second vertex in the cycle of length $2^{i+2}-3$ if they have not chosen a starting point yet. Then the survivor circle around the cycle of length $2^{i+2}-3$.

The strategy for the survivor is elementary. By circling around in an appropriate way, the survivor makes sure that at some point, the first i zombies are within distance $2^{i+2}-2$ behind. Since there are only $k-1$ zombies, this guarantees that circling around the cycle of length $2^{k+2}-3$ is eventually safe and leads to a surviving strategy for the survivor. The only crucial property about the behaviour of zombies is that the distance between the survivor and a given zombie never increases. Note that since all cycles are odd, free will has in fact no impact for zombies.

In our algorithm, zombies are ranked by increasing distance to the survivor, with ties broken arbitrarily. The k^{th} zombie, which does not exist, is considered to be at infinite distance. When the survivor has not chosen a starting point yet, distance is considered as 2 more than the distance to u. Note that u itself might not be a suitable starting point as there could be a zombie on it.

To argue that the strategy is safe for the survivor, it suffices to point out that when the survivor enters the cycle of length $2^{i+2}-3$ (for some i), all zombies are either at distance at most $2^{i+1}-2$ or at least $2^{i+2}-1$. In the first case, the shortest path to the survivors makes them circle around the cycle clockwise (since $2^{i+1}-2 < \frac{2^{i+2}-3}{2}$). In the second case, they do not reach u before the survivor has finished circling around the cycle (since $2^{i+2}-1 > 2^{i+2}-3+1$). □

4 Conclusion

To conclude, we offer two open questions. While not of obvious depth, we believe that both touch at the heart of what it means for a graph G to require $z(G)$ zombies. In particular, if a survivor plays so as to survive for as long as possible, are all $z(G)$ zombies within short distance at time of death?

Question 4.1. For every graph G, and for a graph G' obtained from G by successively adding vertices of degree 1, does it always hold that $z(G') = z(G)$?

Question 4.1 can be interpreted as: is there any advantage for zombies to individually wait for some pre-announced time at the beginning of the game (and then activate and follow the standard rules)?

Question 4.2. For any graph G, is there an integer k such that, for G'_k the graph obtained from G by subdividing all edges k times then adding the original edges, $z(G'_k) \geqslant z(G)+1$?

Concerning Question 4.2, it suffices to consider $G = C_4$ to observe that $k = 4$ is not enough, however, we have no reason to think that k should be large.

Addendum. After submission of this manuscript, an independent proof of Theorem 1.3 was published [KB21].

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