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On the chromatic numbers of signed triangular and hexagonal grids ^{*}

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Abstract

A signed graph is a simple graph with two types of edges. Switching a vertex v of a signed graph corresponds to changing the type of each edge incident to v .

A homomorphism from a signed graph G to another signed graph H is a mapping $\varphi : V(G) \rightarrow V(H)$ such that, after switching any number of the vertices of G , φ maps every edge of G to an edge of the same type in H . The chromatic number $\chi_s(G)$ of a signed graph G is the order of a smallest signed graph H such that there is a homomorphism from G to H .

We show that the chromatic number of signed triangular grids is at most 10 and the chromatic number of signed hexagonal grids is at most 4.

Keywords: Homomorphisms, Chromatic number, Signed Graphs, Hexagonal Grids, Triangular Grids.
2000 MSC: 05C15, 05C60

1. Homomorphisms of signed and 2-edge-colored graphs

A *2-edge-colored graph* or a *signed graph* $G = (V, E, s)$ is a simple graph (V, E) with two kinds of edges: positive and negative edges. We do not allow parallel edges nor loops. The signature $s : E(G) \rightarrow \{-1, +1\}$ assigns to each edge its sign. For the concepts discussed in this article, 2-edge-colored graphs and signed graphs only differ on the notion of homomorphisms.

Given two 2-edge-colored graphs G and H , the mapping $\varphi : V(G) \rightarrow V(H)$ is a *homomorphism* if φ maps every edge of G to an edge of H with the same sign. This can be seen as coloring the graph G by using the vertices of H as colors. The target graph H gives us the rules that this coloring must follow. If vertices 1 and 2 of H are adjacent with a positive (resp. negative) edge, then every pair of adjacent vertices in G colored with 1 and 2 must be adjacent with a positive (resp. negative) edge.

Switching a vertex v of a 2-edge-colored or signed graph corresponds to reversing the signs of all the edges that are incident to v .

Given two signed graphs G and H , the mapping $\varphi : V(G) \rightarrow V(H)$ is a *homomorphism* if there is a homomorphism from G to H after switching some subset of the vertices of G and/or switching some subset of the vertices of H . However, switching in H is unnecessary (as explained in Section 3.3 of [1]).

The *chromatic number* $\chi_s(G)$ of a signed graph G is the order of a smallest signed graph H such that G admits a homomorphism to H . The chromatic number $\chi_s(\mathcal{C})$ of a class of signed graphs \mathcal{C} is the maximum of the chromatic numbers of the graphs in the class.

Homomorphisms of signed graphs were introduced by Naserasr, Rollová and Sopena [1]. This type of homomorphism allows us to generalize several classical problems such as Hadwiger's conjecture [1, 2] and have therefore been studied by many researchers. Here are several known results on the chromatic number of some classes of signed graphs that are related to the classes we study in this article.

Theorem 1. *The chromatic number of signed planar graphs is at most 40 [3].*

Theorem 2. *The chromatic number of signed planar graphs with girth at least 6 is at most 6 [4].*

Theorem 3. *The chromatic number of connected signed graphs with maximum degree 3 is at most 6 [5].*

Theorem 4. *The chromatic number of signed graphs with maximum average degree less than 3 is at most 6 [5].*

Theorem 5. *The chromatic number of signed square grids is at most 6 [6].*

Theorem 6. *The chromatic number of 2-edge-colored square grids is at most 9 [7].*

In Section 2 we present our results on the chromatic number of hexagonal and triangular grids and in Section 3 we introduce the target graphs that we use in Sections 4 and 5 to prove these results.

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2. Results

A square (resp. triangular, hexagonal) grid is a finite induced subgraph of the graph associated with the tiling of the plane with squares (resp. equilateral triangles, equilateral hexagons). See Figures 4 and 5. Since signed hexagonal grids have maximum degree 3 and therefore maximum average degree at most 3, we already know that their chromatic number is at most 6 by Theorem 3 or by Theorem 4. Theorem 2 allows us to reach the same conclusion. Moreover, signed triangular grids are planar and have therefore chromatic number at most 40 by Theorem 1. We improve these bounds as follows.

Theorem 7. *The chromatic number of signed hexagonal grids is 4.*

Theorem 8. *The chromatic number of signed triangular grids is at most 10.*

In order to prove these theorems, we will show that every signed hexagonal grid admits a homomorphism to a target graph of order 4 we call T_4 (see Figure 1) and that every signed triangular grid admits a homomorphism to a target graph of order 10 called SP_9^+ . Constructions of T_4 and SP_9^+ are explained in Section 3. Note that it is conjectured that every signed planar graph admits a homomorphism to SP_9^+ [8]. Theorem 8 brings further evidence toward this conjecture.

3. Target Graphs

A 2-edge-colored graph (V, E, s) is said to be *antiautomorphic* if it is isomorphic to $(V, E, -s)$.

A 2-edge-colored graph $G = (V, E, s)$ is said to be K_n -*transitive* if for every pair of cliques $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ in G such that $s(u_i u_j) = s(v_i v_j)$ for all $i \neq j$, there exists an automorphism that maps u_i to v_i for all i . For $n = 1$ or 2 , we say that the graph is *vertex-transitive* or *edge-transitive*, respectively.

A 2-edge-colored graph G has *Property $P_{k,n}$* if for every sequence of k distinct vertices (v_1, v_2, \dots, v_k) that induces a clique in G and for every sign vector $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \{-1, +1\}^k$ there exist at least n distinct vertices $\{u_1, u_2, \dots, u_n\}$ such that $s(v_i u_j) = \alpha_i$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

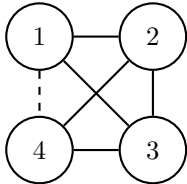


Figure 1: The graph T_4 .

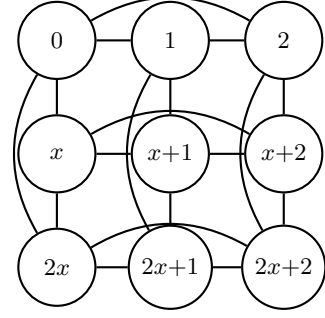


Figure 2: The graph SP_9 , non-edges are negative edges.

The 2-edge-colored Paley graph SP_9 has vertex set $V(SP_9) = \mathbb{F}_9$, the field of order 9. Two vertices u and $v \in V(SP_9)$, $u \neq v$, are connected with a positive edge if $u - v$ is a square in \mathbb{F}_9 and with a negative edge otherwise. Notice that this definition is consistent because $9 \equiv 1 \pmod{4}$ so -1 is a square in \mathbb{F}_9 and if $u - v$ is a square then $v - u$ is also a square.

Given a 2-edge-colored graph G with signature s_G , we create the *antitwinned graph* of G denoted by $\rho(G)$ as follows:

Let G^{+1}, G^{-1} be two copies of G . The vertex corresponding to $v \in V(G)$ in G^i is denoted by v^i .

- $V(\rho(G)) = V(G^{+1}) \cup V(G^{-1})$
- $E(\rho(G)) = \{u^i v^j : uv \in E(G), i, j \in \{-1, +1\}\}$
- $s_{\rho(G)}(u^i v^j) = i \times j \times s_G(u, v)$

By construction, for every vertex v of G , v^{-1} and v^{+1} are *antitwins*, the positive neighbors of v^{-1} are the negative neighbors of v^{+1} and vice versa. A 2-edge-colored graph is *antitwinned* if every vertex has a unique antitwin.

When coloring a 2-edge-colored graph with an antitwinned graph, we say that two vertices have the same *identity* if they are mapped to the same vertex or vertices that are antitwinned. In an antitwinned 2-edge-colored graph we denote the antitwin of v with \bar{v} .

Lemma 9 ([9]). *Let G and H be 2-edge-colored graphs. The two following propositions are equivalent:*

- *The 2-edge-colored graph G admits a homomorphism to $\rho(H)$.*
- *The signed graph G' defined by the 2-edge-colored graph G admits a homomorphism to the signed graph H .*

Given a signed graph H , we define H^+ to be H with an added universal vertex ∞ that is positively connected to all the other vertices (See Figure 3). We will use this construction to create the target graph SP_9^+ used in Section 5.

We will use the following properties of our target graphs to prove our theorems.

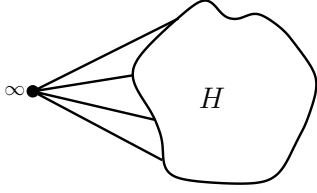


Figure 3: The construction of H^+ .

Lemma 10 ([10]). *The graph $\rho(SP_9^+)$ is vertex-transitive, edge-transitive, antiautomorphic and has Properties $P_{1,9}$, $P_{2,4}$ and $P_{3,1}$.*

Lemma 11. *The graph $\rho(T_4)$ has Property $P_{1,3}$ and the following Property that we call $P_{2,1}^*$:*

Let $u, v \in V(\rho(T_4))$ such that $u \neq v$, $u \neq \bar{v}$ and $\{u, v\} \neq \{\bar{1}, 4\}, \{1, \bar{4}\}, \{2, 3\}, \{2, \bar{3}\}$. There exist at least one vertex in $\rho(T_4)$ that is a positive neighbor of both u and v .

The last lemma can be checked by going through every pair of adjacent vertices in $\rho(T_4)$.

4. Proof of Theorem 7

In this section, we prove that the chromatic number of signed hexagonal grids equals 4. To get this result, we first prove that 2-edge-colored hexagonal grids admit a homomorphism to the 2-edge-colored graph $\rho(T_4)$. Lemma 9 will allow us to conclude.

Lemma 12. *Every 2-edge-colored hexagonal grid admits a homomorphism to the 2-edge-colored graph $\rho(T_4)$.*

Proof. Let G be a 2-edge-colored hexagonal grid and s be its signature. We want to show that G admits a homomorphism to $\rho(T_4)$. We give two coordinates to each vertex of G as follows. Find a planar embedding of G such that every face of size 6 is a regular hexagon and there are horizontal edges. Let l_1, l_2, \dots, l_n be the horizontal lines that have at least one vertex on them such that l_i is above l_j if and only if $i < j$. The leftmost vertex on each line l_i gets coordinates $(i, 1)$, the second leftmost $(i, 2)$ and so on. We denote the vertex of coordinate (i, j) with $v_{i,j}$. See Figure 4. These coordinates allow us to create an order on the vertex set of G by saying that $v_{i,j} < v_{k,l}$ if $i < k$ or $i = k, j < l$.

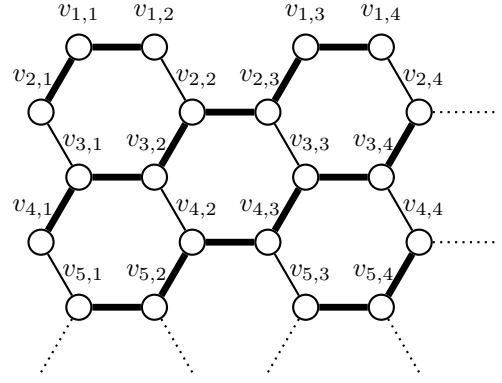


Figure 4: A hexagonal grid.

Let A be a 2-edge-colored graph that admits a homomorphism φ to an antitwinned graph B and let A' be the graph obtained after switching A at a vertex v . By Lemma 9, A' admits a homomorphism φ' to B . It suffices to take φ' as follows:

$$\varphi'(u) = \begin{cases} \varphi(u) & \text{if } u \neq v \\ \overline{\varphi(v)} & \text{otherwise} \end{cases}$$

Therefore we can, without loss of generality, start by switching G at several vertices such that every edge $v_{i,j}v_{k,l}$ with $i = k$ and $j + 1 = l$ or $i + 1 = k$ and $j = l$ are positive (these edges are thicker in Figure 4).

To do that, for every vertex $v_{i,j}$ such that $i + j = 1 \pmod{2}$ and $i \geq 2$ in the order defined earlier we do the following:

- If $s(v_{i,j}v_{i-1,j}) = s(v_{i-1,j}v_{i-1,j+1}) = -1$, we switch G at $v_{i-1,j}$.
- If $s(v_{i,j}v_{i-1,j}) = 1$ and $s(v_{i-1,j}v_{i-1,j+1}) = -1$, we switch G at $v_{i,j}$ and $v_{i-1,j}$.
- If $s(v_{i,j}v_{i-1,j}) = -1$ and $s(v_{i-1,j}v_{i-1,j+1}) = 1$, we switch G at $v_{i,j}$.

We now create a homomorphism φ from G to $\rho(T_4)$ by coloring each vertex in the order defined earlier. We partition the vertices of G into two sets V_1 and V_2 . In V_1 we put every vertex $v_{i,j}$ such that $i + j \equiv 0 \pmod{2}$. In V_2 we put all the other vertices.

When coloring a vertex $v_{i,j}$ in V_1 , such a vertex is adjacent to one already colored vertex (unless $i = 1$ in which case it is trivial to color $v_{i,j}$). Therefore, Property $P_{1,3}$ of $\rho(T_4)$ tells us that there are at least 3 possible colors for $v_{i,j}$ with respect to its already colored neighbor. Note that among these 3 available colors, there is no antitwins. Thanks to this remark, it is always possible to choose one color as follows:

- If $\varphi(v_{i-1,j+1}) = 1$ or $\bar{1}$, $\varphi(v_{i,j}) \notin \{1, \bar{1}, 4, \bar{4}\}$;
- If $\varphi(v_{i-1,j+1}) = 4$ or $\bar{4}$, $\varphi(v_{i,j}) \notin \{1, \bar{1}, 4, \bar{4}\}$;

- If $\varphi(v_{i-1,j+1}) = 2$ or $\bar{2}$, $\varphi(v_{i,j}) \notin \{2, \bar{2}, 3, \bar{3}\}$;
- If $\varphi(v_{i-1,j+1}) = 3$ or $\bar{3}$, $\varphi(v_{i,j}) \notin \{2, \bar{2}, 3, \bar{3}\}$;

When coloring a vertex $v_{i,j}$ in V_2 , such a vertex is adjacent to two already colored vertex (unless $i = 1$ or $j = 1$ in which cases it is trivial to color $v_{i,j}$). Vertex $v_{i-1,j}$ belongs to V_1 and we can therefore use $P_{2,1}^*$, thanks to the restrictions on $\varphi(v_{i-1,j})$ defined earlier, to find a color for each vertex in V_2 . \square

We use Lemmas 9 and 12 to prove that the chromatic number of signed hexagonal grids is at most 4. Note that a cycle on 6 vertices with exactly one negative edge needs at least 4 colors to be colored [11]. Therefore, the chromatic number of signed hexagonal grids is 4.

5. Proof of Theorem 8

In this section, we prove that the chromatic number of signed triangular grids is at most 10. To get this result, we first prove that 2-edge-colored triangular grids admit a homomorphism to the 2-edge-colored graph $\rho(SP_9^+)$. Lemma 9 will allow us to conclude.

Lemma 13. *Every 2-edge-colored triangular grid admits a homomorphism to the 2-edge-colored graph $\rho(SP_9^+)$.*

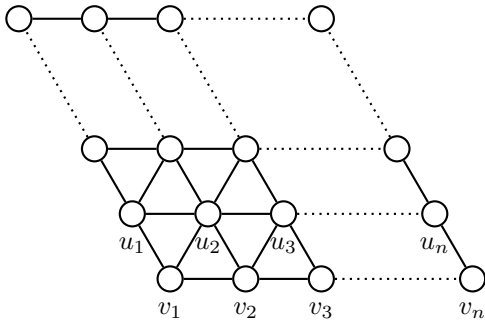


Figure 5: A triangular grid.

Proof. Remember that SP_9^+ is SP_9 with an added vertex ∞ that is positively adjacent to every other vertex. Let G be a 2-edge-colored triangular grid and s be the signature of G . We proceed by induction on the horizontal rows of G as depicted in Figure 5. Note that the first row of G is trivial to color. Let G' be G without the last row. By the induction hypothesis, there is a homomorphism φ' from G' to $\rho(SP_9^+)$. We now show that we can extend this homomorphism to a homomorphism φ from the whole graph G to $\rho(SP_9^+)$.

Let v_1, v_2, \dots, v_n be the vertices of the last row of G and u_1, u_2, \dots, u_n be the vertices of the second to last row of G (the last row of G'). See Figure 5.

By Property $P_{2,4}$ of $\rho(SP_9^+)$, we can find two colors (or even four but we only need two) to color v_1 with respect

to its already colored neighbors (u_1 and u_2). Note that we do not take the color of u_3 into account (yet).

Without loss of generality, we can assume that u_2u_3 is a positive edge, $\varphi'(u_2) = 0$ and $\varphi'(u_3) = 1$ because $\rho(SP_9^+)$ is edge-transitive and antiautomorphic.

Suppose $s(u_2v_1) = s(u_2v_2) = s(u_3v_2) = s(v_1v_2) = +1$. The four colors available for v_2 by Property $P_{2,4}$ of $\rho(SP_9^+)$ with respect to the colors of u_2 and u_3 are $2, \infty, \overline{x+2}$ and $\overline{2x+2}$.

Since $\varphi'(u_2) = 0$ and $s(u_2v_1) = +1$, the two colors available for v_1 belong to the set:

$$\{1, 2, x, \overline{x+1}, \overline{x+2}, 2x, \overline{2x+1}, \overline{2x+2}, \infty\}$$

If v_1 is colored in 1, v_2 can be colored in $2, \infty, \overline{x+2}$ or $\overline{2x+2}$.

If v_1 is colored in 2, v_2 can be colored in ∞ .

If v_1 is colored in x , v_2 can be colored in ∞ or $\overline{x+2}$.

If v_1 is colored in $\overline{x+1}$, v_2 can be colored in 2 or $\overline{x+2}$.

If v_1 is colored in $\overline{x+2}$, v_2 can be colored in $\overline{2x+2}$.

If v_1 is colored in $2x$, v_2 can be colored in ∞ or $\overline{x+2}$.

If v_1 is colored in $\overline{2x+1}$, v_2 can be colored in 2 or $\overline{2x+2}$.

If v_1 is colored in $\overline{2x+2}$, v_2 can be colored in $\overline{x+2}$.

If v_1 is colored in ∞ , v_2 can be colored in 2.

We can now see that any pair of vertices in $\{1, 2, x, \overline{x+1}, \overline{x+2}, 2x, \overline{2x+1}, \overline{2x+2}, \infty\}$ allows v_2 to be colored in at least two colors.

We can proceed in a similar manner with the following three cases and arrive to the same conclusion:

- $s(u_2v_1) = -1, s(u_2v_2) = s(u_3v_2) = s(v_1v_2) = +1,$
- $s(u_2v_1) = +1, s(u_2v_2) = -1, s(u_3v_2) = s(v_1v_2) = +1,$
- $s(u_2v_1) = s(u_2v_2) = +1, s(u_3v_2) = -1, s(v_1v_2) = +1.$

By Lemma 9, each of these four cases also accounts for 3 other cases: the signature obtained after switching at v_1, v_2 and both v_1 and v_2 . We have therefore covered all 16 (2^4) possible signatures of u_2v_1, u_2v_2, u_3v_2 and v_1v_2 .

Therefore, v_2 can be colored in at least 2 colors. Similarly, we can find at least two colors for v_3 and so on until v_n . Finally, we can arbitrarily choose one of these two colors for v_n , accordingly choose a color for v_{n-1} and so on to get a homomorphism φ from G to $\rho(SP_9^+)$. \square

We conclude the proof of Theorem 8 by using Lemmas 13 and 9.

We say that a cycle is unbalanced if it has an odd number of negative edges. Every C_4 in the signed triangular grid from Figure 6 is unbalanced and it can be colored with 6 colors (note that the resulting target graph is SP_5^+). We

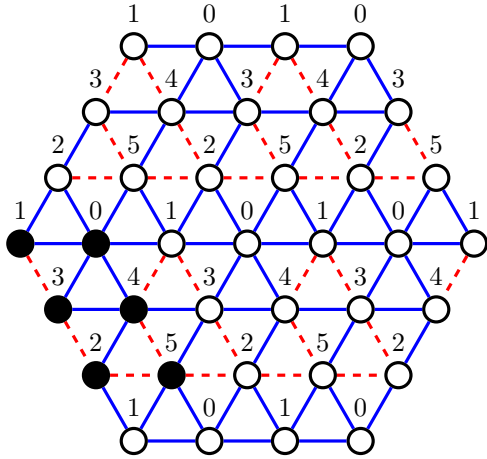


Figure 6: A signed triangular grid in which every C_4 is unbalanced colored with 6 colors.

can easily extend this construction to create a signed triangular grid of any size such that every C_4 is unbalanced and it can be colored with 6 colors. To do so, we can repeat a motif made of six vertices in all directions (the black vertices in Figure 6 represent this motif). By Proposition 3.2 of [12], this means that every signed triangular grid such that every C_4 is unbalanced can be colored with 6 colors. This is of particular interest because when coloring an unbalanced C_4 , every vertex in the cycle must have different identities (this is not the case with a C_4 that is not unbalanced) and therefore a graph in which every C_4 is unbalanced is especially hard to color with few colors. Therefore, we conjecture the following:

Conjecture 14. *The chromatic number of signed triangular grids is 6.*

Note that the chromatic number of signed triangular grids is at least 6 since a wheel on 7 vertices such that every C_4 is unbalanced cannot be colored with 5 colors. To prove it, let G be a 2-edge-colored wheel with vertex set $\{u, u_1, \dots, u_6\}$ and center u , and let T be an antitwinned graph of vertex set $\{0, \bar{0}, 1, \bar{1}, \dots, 4, \bar{4}\}$ where \bar{i} is the antitwin of i such that G admits a homomorphism φ to T . By Proposition 3.2 of [12], it is possible to switch some of the vertices of G such that the outer cycle alternates between positive and negative edges and every edge incident to u is positive. Assume w.l.o.g. that $\varphi(u) = 0$ (note that we can relabel the vertices of T if needed). We now show that we cannot color a pair of vertices (v_1, v_2) of G with antitwins. Suppose v_1 and v_2 have colors that are antitwinned. If v_1 or $v_2 = u$ then we have a contradiction because v_1 and v_2 are adjacent. Otherwise, v_1 and v_2 are both positively adjacent to u which also gives us a contradiction. Therefore we can assume w.l.o.g. that we do not need to use colors $\bar{0}, \bar{1}, \dots, \bar{4}$. W.l.o.g. let $\varphi(u_1) = 1$, $\varphi(u_2) = 2$ and $\varphi(u_3) = 3$. We can color u_4 in either 1 or 4. Suppose we color u_4 with 1. We have to color u_5 in 4 and we then cannot color u_6 . Suppose we color u_4 with

4. We can color u_5 with 1 or 2 but both possibilities do not allow us to color u_6 . Therefore G does not admit a homomorphism to T . We conclude by using Lemma 9.

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